Instructor's Solutions Manual

to accompany

Elementary Number Theory and Its Applications

Fifth Edition

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CHAPTER 1

The Integers

1.1. Numbers, Sequences, and Sums

1.1.1. a. The set of integers greater than 3 is well-ordered. Every subset of this set is also a subset of the set of positive integers, and hence must have a least element.

b. The set of even positive integers is well-ordered. Every subset of this set is also a subset of the set of positive integers, and hence must have a least element.

c. The set of positive rational numbers is not well-ordered. This set does not have a least element. If a/b were the least positive rational number then $a/(b+a)$ would be a smaller positive rational number, which is a contradiction.

d. The set of positive rational numbers of the form $a/2$ is well-ordered. Consider a subset of numbers of this form. The set of numerators of the numbers in this subset is a subset of the set of positive integers, so it must have a least element b. Then $b/2$ is the least element of the subset.

e. The set of nonnegative rational numbers is not well-ordered. The set of positive rational numbers is a subset with no least element, as shown in part c.

1.1.2. Let S be the set of all positive integers of the form $a - bk$. S is not empty since $a - b(-1) = a + b$ is a positive integer. Then the well-ordering principle implies that S has a least element, which is the number we're looking for.

1.1.3. Suppose that x and y are rational numbers. Then $x = a/b$ and $y = c/d$, where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$. Then $xy = (a/b) \cdot (c/d) = ac/bd$ and $x + y = a/b + c/d = (ad + bc)/bd$ where $bd \neq 0$. Since both $x + y$ and xy are ratios of integers, they are both rational.

1.1.4. a. Suppose that x is rational and y is irrational. Then there exist integers a and b such that $x = \frac{a}{b}$ where a and b are integers with $b \neq 0$. Suppose that $x + y$ is rational. Then there exist integers c and d with $d \neq 0$ such that $x + y = \frac{c}{d}$. This implies that $y = (x + y) - x = (a/b) - (c/d) = (ad - bc)/bd$, which means that y is rational, a contradiction. Hence $x + y$ is irrational.

b. This is false. A counterexample is given by $\sqrt{2} + (-\sqrt{2}) = 0$.

c. This is false. A counterexample is given by $0 \cdot \sqrt{2} = 0$.

d. This is false. A counterexample is given by $\sqrt{2} \cdot \sqrt{2} = 2$.

1.1.5. Suppose that $\sqrt{3}$ were rational. Then there would exist positive integers a and b with $\sqrt{3} = a/b$. Consequently, the set $S = \{k\sqrt{3} \mid k \text{ and } k\sqrt{3} \text{ are positive integers}\}$ is nonempty since $a = b\sqrt{3}$. Therefore, by the well-ordering property, S has a smallest element, say $s = t\sqrt{3}$. We have $s\sqrt{3} - s = s\sqrt{3} - t\sqrt{3} = (s - t)\sqrt{3}$. Since $s\sqrt{3} = 3t$ and s are both integers, $s\sqrt{3} - s = (s - t)\sqrt{3}$ must also be an integer. Furthermore, it is positive, since $s\sqrt{3} - s = s(\sqrt{3} - 1)$ and $\sqrt{3} > 1$. It is less than s since $s = t\sqrt{3}$, $s\sqrt{3} = 3t$, and $\sqrt{3} < 3$. This contradicts the choice of s as the smallest positive integer in S. It follows that $\sqrt{3}$ is irrational.

1.1.6. Let S be a set of negative integers. Then the set $T = \{-s : s \in S\}$ is a set of positive integers. By the well-ordering principle, T has a least element t_0. We prove that $-t_0$ is a greatest element of S. First note

that since $t_0 \in S$, then $t_0 = -s_0$ for some $s_0 \in S$. Then $-t_0 = s_0 \in S$. Second, if $s \in S$, then $-s \in T$, so $t_0 \leq -s$. Multiplying by -1 yields $s \leq -t_0$. Since the choice of s was arbitrary, we see that $-t_0$ is greater than or equal to every element of S.

1.1.7. a. Since $0 \leq 1/4 < 1$, we have $[1/4] = 0$.

b. Since $-1 \leq -3/4 < 0$, we have $[-3/4] = -1$.

c. Since $3 \leq 22/7 < 4$, we have $[22/7] = 3$.

d. Since $-2 \leq -2 < -1$, we have $[-2] = -2$.

e. We compute $[1/2 + [1/2]] = [1/2 + 0] = [1/2] = 0$.

f. We compute $[-3 + [-1/2]] = [-3 - 1] = [-4] = -4$.

1.1.8. a. Since $-1 \leq -1/4 < 0$, we have $[-1/4] = -1$.

b. Since $-4 \leq -22/7 < -3$, we have $[-22/7] = -4$.

c. Since $1 \leq 5/4 < 2$, we have $[5/4] = 1$.

d. We compute $[[1/2]] = [0] = 0$.

e. We compute $[[3/2] + [-3/2]] = [1 + (-2)] = [-1] = -1$.

f. We compute $[3 - [1/2]] = [3 - 0] = [3] = 3$.

1.1.9. a. Since $[8/5] = 1$, we have $\{8/5\} = 8/5 - [8/5] = 8/5 - 1 = 3/5$.

b. Since $[1/7] = 0$, we have $\{1/7\} = 1/7 - [1/7] = 1/7 - 0 = 1/7$.

c. Since $[-11/4] = -3$, we have $\{-11/4\} = -11/4 - [-11/4] = -11/4 - (-3) = 1/4$.

d. Since $[7] = 7$, we have $\{7\} = 7 - [7] = 7 - 7 = 0$.

1.1.10. a. Since $[-8/5] = -2$, we have $\{-8/5\} = -8/5 - [-8/5] = -8/5 - (-2) = 2/5$.

b. Since $[22/7] = 3$, we have $\{22/7\} = 22/7 - [22/7] = 22/7 - 3 = 1/7$.

c. Since $[-1] = -1$, we have $\{-1\} = -1 - [-1] = -1 - 1 = 0$.

d. Since $[-1/3] = -1$, we have $\{-1/3\} = -1/3 - [-1/3] = -1/3 - (-1) = 2/3$.

1.1.11. If x is an integer, then $[x] + [-x] = x - x = 0$. Otherwise, $x = z + r$, where z is an integer and r is a real number with $0 < r < 1$. In this case, $[x] + [-x] = [z + r] + [-z - r] = z + (-z - 1) = -1$.

1.1.12. Let $x = [x] + r$ where $0 \leq r < 1$. We consider two cases. First suppose that $r < \frac{1}{2}$. Then $x + \frac{1}{2} = [x] + (r + \frac{1}{2}) < [x] + 1$ since $r + \frac{1}{2} < 1$. It follows that $[x + \frac{1}{2}] = [x]$. Also $2x = 2[x] + 2r < 2[x] + 1$ since $2r < 1$. Hence $[2x] = 2[x]$. It follows that $[x] + [x + \frac{1}{2}] = [2x]$. Next suppose that $\frac{1}{2} \leq r < 1$. Then $[x] + 1 \leq x + (r + \frac{1}{2}) < [x] + 2$, so that $[x + \frac{1}{2}] = [x] + 1$. Also $2[x] + 1 \leq 2[x] + 2r = 2([x] + r) = 2x < 2[x] + 2$ so that $[2x] = 2[x] + 1$. It follows that $[x] + [x + \frac{1}{2}] = [x] + [x] + 1 = 2[x] + 1 = [2x]$.

1.1.13. We have $[x] \leq x$ and $[y] \leq y$. Adding these two inequalities gives $[x] + [y] \leq x + y$. Hence $[x + y] \geq [[x] + [y]] = [x] + [y]$.

1.1.14. Let $x = a+r$ and $y = b+s$, where a and b are integers and r and s are real numbers such that $0 \le r, s < 1$. By Exercise 14, $[2x] + [2y] = [x] + [x + \frac{1}{2}] + [y] + [y + \frac{1}{2}]$. We now need to show that $[x + \frac{1}{2}] + [y + \frac{1}{2}] \ge [x + y]$. Suppose $0 \le r, s < \frac{1}{2}$. Then $[x + \frac{1}{2}] + [y + \frac{1}{2}] = a + b + [r + \frac{1}{2}] + [s + \frac{1}{2}] = a + b$, and $[x + y] = a+b+[r+s] = a+b$, as desired. Suppose that $\frac{1}{2} \le r, s < 1$. Then $[x+\frac{1}{2}]+[y+\frac{1}{2}] = a+b+[r+\frac{1}{2}]+[s+\frac{1}{2}] = a + b + 2$, and $[x + y] = a + b + [r + s] = a + b + 1$, as desired. Suppose that $0 \le r < \frac{1}{2} \le s < 1$. Then $[x + \frac{1}{2}] + [y + \frac{1}{2}] = a + b + 1$, and $[x + y] \le a + b + 1$.

1.1.15. Let $x = a + r$ and $y = b + s$, where a and b are integers and r and s are real numbers such that $0 \le r, s < 1$. Then $[xy] = [ab + as + br + sr] = ab + [as + br + sr]$, whereas $[x][y] = ab$. Thus we have $[xy] \ge [x][y]$. If x and y are both negative, then $[xy] \le [x][y]$. If one of x and y is positive and the other negative, then the inequality could go either direction. For examples take $x = -1.5, y = 5$ and $x = -1, y = 5.5$. In the first case we have $[-1.5 \cdot 5] = [-7.5] = -8 > [-1.5][5] = -2 \cdot 5 = -10$. In the second case we have $[-1 \cdot 5.5] = [-5.5] = -6 < [-1][5.5] = -1 \cdot 5 = -5$.

1.1.16. If x is an integer then $-[-x] = -(-x) = x$, which certainly is the least integer greater than or equal to x. Let $x = a + r$, where a is an integer and $0 < r < 1$. Then $-[-x] = -[-a - r] = -(-a + [-r]) = a - [-r] = a + 1$, as desired.

1.1.17. Let $x = [x] + r$. Since $0 \le r < 1$, $x + \frac{1}{2} = [x] + r + \frac{1}{2}$. If $r < \frac{1}{2}$, then $[x]$ is the integer nearest to x and $[x+\frac{1}{2}] = [x]$ since $[x] \le x+\frac{1}{2} = [x]+r+\frac{1}{2} < [x]+1$. If $r \ge \frac{1}{2}$, then $[x]+1$ is the integer nearest to x (choosing this integer if x is midway between $[x]$ and $[x+1]$) and $[x+\frac{1}{2}] = [x]+1$ since $[x]+1 \le x+r+\frac{1}{2} < [x]+2$.

1.1.18. Let $y = x + n$. Then $[y] = [x] + n$, since n is an integer. Therefore the problem is equivalent to proving that $[y/m] = [[y]/m]$ which was done in Example 1.34.

1.1.19. Let $x = k + \epsilon$ where k is an integer and $0 \le \epsilon < 1$. Further, let $k = a^2 + b$, where a is the largest integer such that $a^2 \le k$. Then $a^2 \le k = a^2 + b \le x = a^2 + b + \epsilon < (a + 1)^2$. Then $[\sqrt{x}] = a$ and $[\sqrt{[x]}] = [\sqrt{k}] = a$ also, proving the theorem.

1.1.20. Let $x = k + \epsilon$ where k is an integer and $0 \le \epsilon < 1$. Choose w from $0, 1, 2, \ldots, m - 1$ such that $w/m \le \epsilon < (w + 1)/m$. Then $w \le m\epsilon < w + 1$. Then $[mx] = [mk + m\epsilon] = mk + [m\epsilon] = mk + w$. On the other hand, the same inequality gives us $(w + j)/m \le \epsilon + j/m < (w + 1 + j)/m$, for any integer $j = 0, 1, 2, \ldots, m - 1$. Note that this implies $[\epsilon + j/m] = [(w + j)/m]$ which is either 0 or 1 for j in this range. Indeed, it equals 1 precisely when $w+j \ge m$, which happens for exactly w values of j in this range. Now we compute $\sum_{j=0}^{m-1}[x + j/m] = \sum_{j=0}^{m-1}[k + \epsilon + j/m] = \sum_{j=0}^{m-1} k + [\epsilon + j/m] = mk + \sum_{j=0}^{m-1}[(w + j)/m] = mk + \sum_{j=m-w}^{m-1} 1 = mk + w$ which is the same as the value above.

1.1.21. a. Since the difference between any two consecutive terms of this sequence is 8, we may compute the nth term by adding 8 to the first term $n - 1$ times. That is, $a_n = 3 + (n - 1)8 = 8n - 5$.

b. For each n, we have $a_n - a_{n-1} = 2^{n-1}$, so we may compute the nth term of this sequence by adding all the powers of 2, up to the $(n - 1)$th, to the first term. Hence $a_n = 5 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 5 + 2^n - 2 = 2^n + 3$.

c. The nth term of this sequence appears to be zero, unless n is a perfect square, in which case the term is 1. If n is not a perfect square, then $[\sqrt{n}] < \sqrt{n}$, where $[x]$ represents the greatest integer function. If n is a perfect square, then $[\sqrt{n}] = \sqrt{n}$. Therefore, $[[\sqrt{n}]/\sqrt{n}]$ equals 1 if n is a perfect square and 0 otherwise, as desired.

d. This is a Fibonacci-like sequence, with $a_n = a_{n-1} + a_{n-2}$, for $n \ge 3$, and $a_1 = 1$, and $a_2 = 3$.

1.1.22. a. Each term given is 3 times the preceding term, so we conjecture that the nth term is the first term multiplied by 3, $n - 1$ times. So $a_n = 2 \cdot 3^{n-1}$.

b. In this sequence, $a_n = 0$ if n is a multiple of 3, and equals 1 otherwise. Let $[x]$ represent the greatest integer function. Since $[n/3] < n/3$ when n is not a multiple of 3 and $[n/3] = n/3$ when n is a multiple of 3, we have that $a_n = 1 - [[n/3]/(n/3)]$.

c. If we look at the difference of successive terms, we have the sequence $1, 1, 2, 2, 3, 3, \ldots$. So if n is odd, say $n = 2k+1$, then a_n is obtained by adding $1+1+2+2+3+3+\cdots+k+k = 2t_k$ to the first term, which is 1. (Here t_k stands for the kth triangular number.) So if n is odd, then $a_n = 1 + 2t_k$ where $k = (n-1)/2$. If n is even, say $n = 2k$, then $a_n = a_{2k+1} - k = 1 - k + 2t_k$.

d. This is a Fibonacci-like sequence, with $a_n = a_{n-1} + 2a_{n-2}$, for $n \geq 3$, and $a_1 = 3$, and $a_2 = 5$.

1.1.23. Three possible answers are $a_n = 2^{n-1}$, $a_n = (n^2 - n + 2)/2$, and $a_n = a_{n-1} + 2a_{n-2}$.

1.1.24. Three possible answers are $a_n = a_{n-1}a_{n-2}$, $a_n = a_{n-1} + 2n - 3$, and $a_n =$ the number of letters in the nth word of the sentence "If our answer is correct we will join the Antidisestablishmentarianism Society and boldly state that 'If our answer is correct we will join the Antidisestablishmentarianism Society and boldly state....' "

1.1.25. This set is exactly the sequence $a_n = n - 100$, and hence is countable.

1.1.26. The function $f(n) = 5n$ is a one-to-one correspondence between this set and the set of integers, which is known to be countable.

1.1.27. One way to show this is to imitate the proof that the set of rational numbers is countable, replacing a/b with $a + b\sqrt{2}$. Another way is to consider the function $f(a + b\sqrt{2}) = 2^a 3^b$ which is a one-to-one map of this set into the rational numbers, which is known to be countable.

1.1.28. Let A and B be two countable sets. If one or both of the sets are finite, say A is finite, then the listing $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots$, where any b_i which is also in A is deleted from the list, demonstrates the countability of $A \cup B$. If both sets are infinite, then each can be represented as a sequence: $A = \{a_1, a_2, \ldots\}$, and $B = \{b_1, b_2, \ldots\}$. Consider the listing $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ and form a new sequence c_i as follows. Let $c_1 = a_1$. Given that c_n is determined, let c_{n+1} be the next element in the listing which is different from each c_i with $i = 1, 2, \ldots, n$. Then this sequence is exactly the elements of $A \cup B$, which is therefore countable.

1.1.29. Suppose $\{A_i\}$ is a countable collection of countable sets. Then each A_i can be represented by a sequence, as follows:

$$\begin{array}{rcccccc} A_1 & = & a_{11} & a_{12} & a_{13} & \ldots \\ A_2 & = & a_{21} & a_{22} & a_{23} & \ldots \\ A_3 & = & a_{31} & a_{32} & a_{33} & \ldots \\ & \vdots & & & & \end{array}$$

Consider the listing $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots$, in which we first list the elements with subscripts adding to 2, then the elements with subscripts adding to 3 and so on. Further, we order the elements with subscripts adding to k in order of the first subscript. Form a new sequence c_i as follows. Let $c_1 = a_1$. Given that c_n is determined, let c_{n+1} be the next element in the listing which is different from each c_i with $i = 1, 2, \ldots, n$. Then this sequence is exactly the elements of $\bigcup_{i=1}^{\infty} A_i$, which is therefore countable.

1.1.30. a. Note that $\sqrt{2} \approx 1.4 = 7/5$, so we might guess that $\sqrt{2} - 7/5 \approx 0$. If we multiply through by 5 we expect that $5\sqrt{2} - 7$ should be small, and its value is approximately 0.071 which is much less than $1/8 = 0.125$. So we may take $a = 5 \leq 8$ and $b = 7$.

b. As in part a., note that $\sqrt[3]{2} = 1.2599\ldots \approx 1.25 = 5/4$, so we investigate $4\sqrt[3]{2} - 5 = 0.039\ldots \leq 1/8$. So we may take $a = 4 \leq 8$ and $b = 5$.

c. Since we know that $\pi \approx 22/7$ we investigate $|7\pi - 22| = 0.0088\ldots \leq 1/8$. So we may take $a = 7 \leq 8$ and $b = 22$.

d. Since $e \approx 2.75 = 11/4$ we investigate $|4e - 11| = 0.126\ldots$, which is too large. A closer approximation to e is 2.718. We consider the decimal expansions of the multiples of $1/7$ and find that $5/7 = .714\ldots$, so $e \approx 19/7$. Therefore we investigate $|7e - 19| = 0.027 \leq 1/8$. So we may take $a = 7 \leq 8$ and $b = 19$.

1.1.31. a. Note that $\sqrt{3} = 1.73 \approx 7/4$, so we might guess that $\sqrt{3} - 7/4 \approx 0$. If we multiply through by 4 we find that $|4\sqrt{3} - 7| = 0.07\ldots < 1/10$. So we may take $a = 4 \leq 10$ and $b = 7$.

b. It is helpful to keep the decimal expansions of the multiples of $1/7$ in mind in these exercises. Here $\sqrt[3]{3} = 1.442\ldots$ and $3/7 = 0.428\ldots$ so that we have $\sqrt[3]{3} \approx 10/7$. Then as in part a., we investigate $|7\sqrt[3]{3} - 10| = 0.095\ldots < 1/10$. So we may take $a = 7 \leq 10$ and $b = 10$.

c. Since $\pi^2 = 9.869\ldots$ and $6/7 = 0.857\ldots$, we have that $\pi^2 \approx 69/7$, so we compute $|7\pi^2 - 69| = 0.087\ldots < 1/10$. So we may take $a = 7 \leq 10$ and $b = 69$.

d. Since $e^3 = 20.0855\ldots$ we may take $a = 1$ and $b = 20$ to get $|1e^3 - 20| = 0.855\ldots < 1/10$.

1.1.32. For $j = 0, 1, 2, \ldots, n+1$, consider the $n+2$ numbers $\{j\alpha\}$, which all lie in the interval $0 \leq \{j\alpha\} < 1$. We can partition this interval into the $n+1$ subintervals $(k-1)/(n+1) \leq x < k/(n+1)$ for $k = 1, \ldots, n+1$. Since we have $n+2$ numbers and only $n+1$ intervals, by the pigeonhole principle, some interval must contain at least two of the numbers. So there exist integers r and s such that $0 \leq r < s \leq n+1$ and $|\{r\alpha\} - \{s\alpha\}| \leq 1/(n+1)$. Let $a = s - r$ and $b = [s\alpha] - [r\alpha]$. Since $0 \leq r < s \leq n+1$, we have $1 \leq a \leq n$. Also, $|a\alpha - b| = |(s-r)\alpha - ([s\alpha] - [r\alpha])| = |(s\alpha - [s\alpha]) - (r\alpha - [r\alpha]a)| = |\{s\alpha\} - \{r\alpha\}| < 1/(n+1)$. Therefore, a and b have the desired properties.

1.1.33. The number α must lie in some interval of the form $r/k \leq \alpha < (r+1)/k$. If we divide this interval into equal halves, then α must lie in one of the halves, so either $r/k \leq \alpha < (2r+1)/2k$ or $(2r+1)/2k \leq \alpha < (r+1)/k$. In the first case we have $|\alpha - r/k| < 1/2k$, so we take $u = r$. In the second case we have $|\alpha - (r+1)/k| < 1/2k$, so we take $u = r + 1$.

1.1.34. Suppose that there are only finitely many positive integers $q_1, q_2, \ldots, q_n$ with corresponding integers $p_1, p_2, \ldots, p_n$ such that $|\alpha - p_i/q_i| < 1/q_i^2$. Since α is irrational, $|\alpha - p_i/q_i|$ is positive for every i, and so is $|q_i\alpha - p_i|$ so we may choose an integer N so large that $|q_i\alpha - p_i| > 1/N$ for all i. By Dirichlet's Approximation Theorem, there exist integers r and s with $1 \leq s \leq N$ such that $|s\alpha - r| < 1/N < 1/s$, so that $|\alpha - r/s| < 1/s^2$, and s is not one of the q_i. Therefore, we have another solution to the inequality. So no finite list of solutions can be complete, and we conclude that there must be an infinite number of solutions.

1.1.35. First we have $|\sqrt{2} - 1/1| = 0.414\ldots < 1/1^2$. Second, Exercise 30, part a., gives us $|\sqrt{2} - 7/5| < 1/50 < 1/5^2$. Third, observing that $3/7 = 0.428\ldots$ leads us to try $|\sqrt{2} - 10/7| = 0.014\ldots < 1/7^2 = 0.0204\ldots$. Fourth, observing that $5/12 = 0.4166\ldots$ leads us to try $|\sqrt{2} - 17/12| = 0.00245\ldots < 1/12^2 = 0.00694\ldots$.

1.1.36. First we have $|\sqrt[3]{5} - 1/1| = 0.7099\ldots < 1/1^2$. Second, $|\sqrt[3]{5} - 5/3| = 0.04\ldots < 1/3^2$. Third, since $\sqrt[3]{5} = 1.7099\ldots$, we try $|\sqrt[3]{5} - 17/10| = 0.0099\ldots < 1/10^2$. Likewise, we get a fourth rational number with $|\sqrt[3]{5} - 171/100| = 0.000024\ldots < 1/100^2$. Fifth, consideration of multiples of $1/7$ leads to $|\sqrt[3]{5} - 12/7| = 0.0043\ldots < 1/7^2$.

1.1.37. We may assume that b and q are positive. Note that if $q > b$, we have $|p/q - a/b| = |pb - aq|/qb \geq 1/qb > 1/q^2$. Therefore, solutions to the inequality must have $1 \leq q \leq b$. For a given q, there can be only finitely many p such that the distance between the rational numbers a/b and p/q is less than $1/q^2$ (indeed there is at most one.) Therefore there are only finitely many p/q satisfying the inequality.

1.1.38. a. Since $n2$ is an integer for all n, so is $[n2]$, so the first ten terms of the spectrum sequence are 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

b. The sequence for $n\sqrt{2}$, rounded, is 1.414, 2.828, 4.242, 5.656, 7.071, 8.485, 9.899, 11.314, 12.728, 14.142. When we apply the floor function to these numbers we get 1, 2, 4, 5, 7, 8, 9, 11, 12, 14 for the spectrum sequence.

c. The sequence for $n(2+\sqrt{2})$, rounded, is 3.414, 6.828, 10.24, 13.66, 17.07, 20.48, 23.90, 27.31, 30.73, 34.14. When we apply the floor function to these numbers we get 3, 6, 10, 13, 17, 20, 23, 27, 30, 34, for the spectrum sequence.

d. The sequence for ne, rounded is 2.718, 5.436, 8.155, 10.87, 13.59, 16.31, 19.03, 21.75, 24.46, 27.18. When we apply the floor function to these numbers we get 2, 5, 8, 10, 13, 16, 19, 21, 24, 27, for the spectrum sequence.

e. The sequence for $n(1+\sqrt{5})/2$, rounded, is 1.618, 3.236, 4.854, 6.472, 8.090, 9.708, 11.33, 12.94, 14.56, 16.18. When we apply the floor function to these numbers we get 1, 3, 4, 6, 8, 9, 11, 12, 14, 16 for the spectrum sequence.

1.1.39. a. Since $n3$ is an integer for all n, so is $[n3]$, so the first ten terms of the spectrum sequence are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30.

b. The sequence for $n\sqrt{3}$, rounded, is 1.732, 3.464, 5.196, 6.928, 8.660, 10.39, 12.12, 13.86, 15.59, 17.32. When we apply the floor function to these numbers we get 1, 3, 5, 6, 8, 10, 12, 13, 15, 17 for the spectrum sequence.

c. The sequence for $n(3+\sqrt{3})/2$, rounded, is 2.366, 4.732, 7.098, 9.464, 11.83, 14.20, 16.56, 18.93, 21.29, 23.66. When we apply the floor function to these numbers we get 2, 4, 7, 9, 11, 14, 16, 18, 21, 23 for the spectrum sequence.

d. The sequence for $n\pi$, rounded is 3.142, 6.283, 9.425, 12.57, 15.71, 18.85, 21.99, 25.13, 28.27, 31.42. When we apply the floor function to these numbers we get 3, 6, 9, 12, 15, 18, 21, 25, 28, 31, for the spectrum sequence.

1.1.40. Since $\alpha \neq \beta$, their decimal expansions must be different. If they differ in digits that are to the left of the decimal point, then $[\alpha] \neq [\beta]$, so certainly the spectrum sequences are different. Otherwise, suppose that they differ in the kth position to the right of the decimal. Then $[10^k\alpha] \neq [10^k\beta]$, and so the spectrum sequences will again differ.

1.1.41. Assume that $1/\alpha + 1/\beta = 1$. Note first that for all integers n and m, $m\alpha \neq n\beta$, for otherwise, we solve the equations $m\alpha = n\beta$ and $1/\alpha + 1/\beta = 1$ and get rational solutions for α and β, a contradiction. Therefore the sequences $m\alpha$ and $n\beta$ are disjoint.

For an integer k, define $N(k)$ to be the number of elements of the sequences $m\alpha$ and $n\beta$ which are less than k. Now $m\alpha < k$ if and only if $m < k/\alpha$, so there are exactly $[k/\alpha]$ members of the sequence $m\alpha$ less than k. Likewise, there are exactly $[k/\beta]$ members of the sequence $n\beta$ less than k. So we have $N(k) = [k/\alpha] + [k/\beta]$. By definition of the greatest integer function, we have $k/\alpha - 1 < [k/\alpha] < k/\alpha$ and $k/\beta - 1 < [k/\beta] < k/\beta$, where the inequalities are strict because the numbers are irrational. If we add these inequalities we get $k/\alpha + k/\beta - 2 < N(k) < k/\alpha + k/\beta$ which simplifies to $k - 2 < N(k) < k$. Since $N(k)$ is an integer, we conclude that $N(k) = k - 1$. This shows that there is exactly one member of the union of the sequences $m\alpha$ and $n\beta$ in each interval of the form $k - 1 \leq x < k$, and therefore, when we apply the floor function to each member, exactly one will take on the value k.

Conversely, suppose that α and β are irrational numbers such that $1/\alpha + 1/\beta \neq 1$. If $1/\alpha + 1/\gamma = 1$ then we know from the first part of the theorem that the spectrum sequences for α and γ partition the positive integers. By Exercise 40., we know that the spectrum sequences for β and γ are different, so the sequences for α and β can not partition the positive integers.

1.1.42. The first two Ulam numbers are 1 and 2. Since $3 = 1 + 2$, it is the third Ulam number and since $4 = 1 + 3$, it is the fourth Ulam number. Note that 5 is not an Ulam number since $5 = 1 + 4 = 2 + 3$. The fifth Ulam number is 6 since $6 = 4 + 2$ and no other two Ulam numbers have 6 as their sum. We have $7 = 4 + 3 = 6 + 1$, so 7 is not an Ulam number. The sixth Ulam number is $8 = 6 + 2$. Note that $9 = 8 + 1 = 6 + 3$ and $10 = 8 + 2 = 4 + 6$ so neither 9 nor 10 is an Ulam number. The seventh Ulam number is 11 since $11 = 8 + 3$ is the unique way to write 11 as the sum of two distinct Ulam numbers. Next note that $12 = 8 + 4 = 1 + 11$ so that 12 is not an Ulam number. Note that $13 = 11 + 2$ is the unique way to write 13 as the eighth Ulam number. We see that $14 = 13 + 1 = 11 + 3$ and $15 = 2 + 13 = 4 + 11$, so that neither 14 nor 15 are Ulam numbers. We note that $16 = 3 + 13$ is the unique way to write 16 as the sum of two Ulam numbers, so that the ninth Ulam number is 16. Note that $17 = 1 + 16 = 4 + 13$ so that 17 is not an Ulam number. Note that $18 = 2 + 16$ is the unique way to write 18 as the sum of two Ulam numbers so that 18 is the tenth Ulam number. In summary, the first ten Ulam numbers are: 1, 2, 3, 4, 6, 8, 11, 13, 16, 18.

1.1.43. Assume that there are only finitely many Ulam numbers. Let the two largest Ulam numbers be u_{n-1} and u_n. Then the integer $u_n + u_{n-1}$ is an Ulam number larger than u_n. It is the unique sum of two Ulam numbers since $u_i + u_j < u_n + u_{n-1}$ if $j < n$ or $j = n$ and $i < n - 1$.

1.1.44. Suppose that e is rational so that $e = a/b$ where a and b are integers and $b \neq 0$. Let $k \geq b$ be an integer and set $c = k!(e - 1 - 1/1! - 1/2! - 1/3! - \cdots - 1/k!)$. Since every denominator in the expression divides evenly into $k!$, we see that c is an integer. Since $e = 1 + 1/1! + 1/2! + \cdots$, we have $0 < c = k!(1/(k+1)! + 1/(k+2)! + \cdots) = 1/(k+1) + 1/(k+1)(k+2) + \cdots < 1/(k+1) + 1/(k+1)^2 + \cdots$. This last geometric series is equal to $1/k$, so we have that $0 < c < 1/k$, which is impossible since c is an integer. Therefore e must be irrational.

1.2. Sums and Products

1.2.1. a. We have $\displaystyle\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$.

b. We have $\displaystyle\sum_{j=1}^{5} (-3) = (-3) + (-3) + (-3) + (-3) + (-3) = -15$.

c. We have $\displaystyle\sum_{j=1}^{5} 1/(j+1) = 1/2 + 1/3 + 1/4 + 1/5 + 1/6 = 29/20$.

1.2.2. a. We have $\displaystyle\sum_{j=0}^{4} 3 = 3 + 3 + 3 + 3 + 3 = 15$.

b. We have $\displaystyle\sum_{j=0}^{4} (j-3) = (-3) + (-2) + (-1) + 0 + 1 = -5$.

c. We have $\displaystyle\sum_{j=0}^{4} (j+1)/(j+2) = 1/2 + 2/3 + 3/4 + 4/5 + 5/6 = 71/20$.

1.2.3. a. We use the formula from Example 1.15 as follows. We evaluate the sum $\displaystyle\sum_{j=0^8 2^j} = 2^9 - 1 = 511$ as in Example 1.17. Then we have $\displaystyle\sum_{j=1^8 2^j} = \sum_{j=0^8 2^j} -2^0 = 510$.

b. We could proceed as in part (a), or we may do the following: $\displaystyle\sum_{j=1}^{8} 5(-3)^j = \sum_{j=0}^{7} 5(-3)^{j+1} = \sum_{j=0}^{7} -15(-3)^j$. We may apply the formula in Example 1.15 to this last sum, with $a = -15, n = 7$ and $r = -3$, to get the sum equal to $\dfrac{-15(-3)^8 - (-15)}{-3 - 1} = 24600$.

c. We manipulate the sum as in part b., so we can apply the formula from Example 1.15. $\sum_{j=1}^{8} 3(-1/2)^j = \sum_{j=0}^{7} 3(-1/2)^{j+1} = \sum_{j=0}^{7}(-3/2)(-1/2)^j = \dfrac{(-3/2)(-1/2)^8 - (-3/2)}{-1/2-1} = -\dfrac{255}{256}$.

1.2.4. a. We have $\sum_{j=0}^{10} 8 \cdot 3^j = \dfrac{8 \cdot 3^{11} - 8}{3-1} = 708584$, using the formula from Example 1.15 with $a = 8, n = 10$ and $r = 3$.

b. We have $\sum_{j=0}^{10}(-2)^{j+1} = \sum_{j=0}^{10}(-2)(-2)^j = \dfrac{(-2)\cdot(-2)^{11} - (-2)}{(-2)-1} = -1366$, using the formula from Example 1.15 with $a = -2, n = 10$ and $r = -2$.

c. We have $\sum_{j=0}^{10}(1/3)^j = \dfrac{(1/3)^{11} - 1}{(1/3)-1} = \dfrac{88573}{59049}$, using the formula from Example 1.15 with $a = 1, n = 10$ and $r = (1/3)$.

1.2.5. The sum $\sum_{k=1}^{n}[\sqrt{k}]$ counts 1 for every value of k with $\sqrt{k} \geq 1$. There are n such values of k in the range $k = 1, 2, 3, \ldots, n$. It counts another 1 for every value of k with $\sqrt{k} \geq 2$. There are $n-3$ such values in the range. The sum counts another 1 for each value of k with $\sqrt{k} \geq 3$. There are $n-8$ such values in the range. In general, for $m = 1, 2, 3, \ldots, [\sqrt{n}]$ the sum counts a 1 for each value of k with $\sqrt{k} \geq m$, and there are $n - (m^2 - 1)$ values in the range. Therefore $\sum_{k=1}^{n}[\sqrt{k}] = \sum_{m=1}^{[\sqrt{n}]} n - (m^2-1) = [\sqrt{n}](n+1) - \sum_{m=1}^{[\sqrt{n}]} m^2 = [\sqrt{n}](n+1) - ([\sqrt{n}]([\sqrt{n}]+1)(2[\sqrt{n}]+1))/6$.

1.2.6. We see that $t_n = \sum_{j=1}^{n} j$, and $t_{n-1} = \sum_{j=1}^{n-1} j = \sum_{j=1}^{n-1}(n-j)$. Now, $t_{n-1} + t_n = \sum_{j=1}^{n-1}(n-j+j) + n = n(n-1) + n = n^2$.

1.2.7. We see that $t_n = \sum_{j=1}^{n} j = \sum_{j=1}^{n}(n-j+1)$. Thus, $2t_n = \sum_{j=1}^{n} j + \sum_{j=1}^{n}(n-j+1) = \sum_{j=1}^{n}(n+1) = n(n+1)$.

1.2.8. It is clear that $p_1 = 1$. Suppose we know p_{k-1}. To compute p_k we consider k nested pentagons as in the figure. Note that $p_k - p_{k-1}$ counts the number of dots on three sides of the outer pentagon. Each side consists of k dots, but two of the dots belong to two sides. Therefore $p_k - p_{k-1} = 3k - 2$, which is the formula desired. Then $p_n = 3n - 2 + p_{n-1} = 3n - 2 + 3(n-1) - 2 + p_{n-2} = 3n - 2 + 3(n-1) - 2 + 3(n-2) - 2 + p_{n-3} = \cdots = 3n - 2 + 3(n-1) - 2 + \cdots + 3(1) - 2 = \sum_{k=1}^{n}(3k-2)$.

1.2.9. From Exercise 8, we have $p_n = \sum_{k=1}^{n}(3k-2) = 3\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} 1 = 3n(n+1)/2 - 2n = (3n^2-n)/2$. On the other hand, $t_{n-1} + n^2 = (n-1)n/2 + n^2 = (3n^2-n)/2$, which is the same as above.

1.2.10. a. Consider a regular hexagon which we border successively by hexagons with $3, 4, 5, \ldots$ on each side. Define the *hexagonal number* h_k to be the number of dots contained in the k nested hexagons.

b. First note that $h_1 = 1$. To get a recursive relationship we consider $h_k - h_{k-1}$, which counts the dots added to the $(k-1)$st hexagon to obtain the kth hexagon. To do this, we must add 4 sides of k dots each, but 3 of the dots belong to two sides. Therefore $h_k - h_{k-1} = 4k - 3$. A closed formula is then given by adding these differences together: $h_k = \sum_{i=1}^{k}(4i-3) = 4t_k - 3k = 4k(k+1)/2 - 3k = 2k^2 - k$.

1.2.11. a. Consider a regular heptagon which we border successively by heptagons with $3, 4, 5, \ldots$ on each side. Define the *heptagonal numbers* $s_1, s_2, s_3, \ldots, s_k, \ldots$ to be the number of dots contained in the k nested heptagons.

b. First note that $s_1 = 1$. To get a recursive relationship we consider $s_k - s_{k-1}$, which counts the dots added to the $(k-1)$st heptagon to obtain the kth heptagon. To do this, we must add 5 sides of k dots each, but 4 of the dots belong to two sides. Therefore $s_k - s_{k-1} = 5k - 4$. A closed formula is then given by adding these differences together: $s_k = \sum_{i=1}^{k}(5i-4) = 5t_k - 4k = 5k(k+1)/2 - 4k = (5k^2 - 3k)/2$.

1.2.12. First consider the difference $T_k - T_{k-1}$. This counts the number of dots on one face of the kth tetrahedron. But this is simply the kth nested triangle used to define the triangular numbers. Therefore, $T_k - T_{k-1} = t_k$. Hence, since $T_1 = t_1 = 1$, it follows that $T_n = \sum_{k=1}^{n} t_k$.

1.2.13. We continue with the formula from Exercise 12. $T_n = \sum_{k=1}^{n} t_k = \sum_{k=1}^{n} k(k+1)/2$. Exploiting the same technique as in Example 1.19, we consider $(k+1)^3 - k^3 = 3k^2 + 3k + 1 = 3(k^2 + k) + 1$ and solve for $k^2 + k$ to get $k^2 + k = (k+1)^3 - k^3/3 - (1/3)$. Then $T_n = (1/2)\sum_{k=1}^{n} k(k+1) = (1/6)\sum_{k=1}^{n}((k+1)^3 - k^3) - (1/6)\sum_{k=1}^{n} 1$. The first sum is telescoping and the second sum is trivial, so we have $T_n = (1/6)((n+1)^3 - 1^3) - (1/6) = (n^3 + 3n^2 + 2n)/6$.

1.2.14. Using the fact $n! = n \cdot (n-1)!$, we find that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, $8! = 40320$, $9! = 362880$, and $10! = 3628800$.

1.2.15. Each of these four quantities are products of 100 integers. The largest product is 100^{100}, since it is the product of 100 factors of 100. The second largest is 100! which is the product of the integers $1, 2, \ldots, 100$, and each of these terms is less or equal to 100. The third largest is $(50!)^2$ which is the product of $1^2, 2^2, \ldots, 50^2$, and each of these factors j^2 is less than $j(50+j)$, whose product is 100!. The smallest is 2^{100} which is the product of 100 2's.

1.2.16. a. $\displaystyle\prod_{i=1}^{n} ka_i = k^n \prod_{i=1}^{n} a_i$.

b. $\displaystyle\prod_{i=1}^{n} ia_i = (a_1)(2a_2)\cdots(na_n) = (1 \cdot 2 \cdots n)(a_1 a_2 \cdots a_n) = n! \prod_{i=1}^{n} a_i$.

c. $\displaystyle\prod_{i=1}^{n} a_i^k = \left(\prod_{i=1}^{n} a_i\right)^k$.

1.2.17. $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right) = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$. Let $a_j = 1/(j+1)$. Notice that this is a telescoping sum, and using the notation in the text preceding Example 1.15, we have $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k(k+1)}\right) = \sum_{j=1}^{n} a_{j-1} - a_j) = -(a_n - a_0) = 1 - 1/(n+1) = n/(n+1)$.

1.2.18. $\displaystyle\sum_{k=2}^{n}\frac{1}{k^2-1} = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{n}\left(\left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right)\right) =$
$\displaystyle\frac{1}{2}\sum_{k=2}^{n}(\frac{1}{k-1} - \frac{1}{k}) + \frac{1}{2}\sum_{k=2}^{n}(\frac{1}{k} - \frac{1}{k+1}) = \frac{1}{2}(1 - \frac{1}{n}) + \frac{1}{2}(\frac{1}{2} - \frac{1}{n+1}) = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$.

1.2.19. We sum both sides of the identity $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ from $k = 1$ to $k = n$. $\sum_{k=1}^{n}((k+1)^3 - k^3) = (n+1)^3 - 1$, since the sum is telescoping. $\sum_{k=1}^{n}(3k^2 + 3k + 1) = 3(\sum_{k=1}^{n} k^2) + 3(\sum_{k=1}^{n} k) + \sum_{k=1}^{n} 1 = 3(\sum_{k=1}^{n} k^2) + 3n(n+1)/2 + n$. As these two expressions are equal, solving for $\sum_{k=1}^{n} k^2$, we find that $\sum_{k=1}^{n} k^2 = (n/6)(2n+1)(n+1)$.

1.2.20. We sum both sides of the identity $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ from $k = 1$ to $k = n$. Using Exercise 19 we find that $\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$.

1.2.21. a. $10! = (7!)(8 \cdot 9 \cdot 10) = (7!)(720) = (7!)(6!)$.

b. $10! = (7!)(6!) = (7!)(5!) \cdot 6 = (7!)(5!)(3!)$.

c. $16! = (14!)(15 \cdot 16) = (14!)(240) = (14!)(5!)(2!)$.

d. $9! = (7!)(8 \cdot 9) = (7!)(6 \cdot 6 \cdot 2) = (7!)(3!)(3!)(2!)$

1.2.22. Since $c = a_1!a_2! \cdots a_n!$ and $b = (a_1!a_2! \cdots a_n!) - 1$, it follows that $c! = c \cdot (c-1)! = c \cdot b! = a_1!a_2! \cdots a_n! \cdot b!$.

1.2.23. Assume that $x \leq y$. Then $z! = x! + y! \leq y! + y! = 2(y!)$. Since $z > y$ we have $z! \geq (y+1)y!$. This implies that $y + 1 \leq 2$. Hence the only solution with x, y, and z positive integers is $x = y = 1$ and $z = 2$.

1.2.24. a. $\displaystyle\prod_{j=2}^{n}(1 - \frac{1}{j}) = (1 - 1/2)(1 - 1/3) \cdots (1 - 1/n) = \frac{1}{2}\frac{2}{3}\frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n}$.

b. $\displaystyle\prod_{j=2}^{n}(1 - \frac{1}{j^2}) = \prod_{j=2}^{n}(1 - 1/j)\prod_{j=2}^{n}(1 + 1/j) = \left(\frac{1}{n}\right)\left(\frac{3}{2}\frac{4}{3}\frac{5}{4} \cdots \frac{n+1}{n}\right) = \frac{n+1}{2n}$.

1.3. Mathematical Induction

1.3.1. For $n = 1$ we have $1 < 2^1 = 2$. This is the basis case. Now assume $n < 2^n$. We then have $n + 1 < 2^n + 1 < 2^n + 2^n = 2^{n+1}$. This completes the inductive step and the proof by mathematical induction.

1.3.2. We have $2 = 2$, $2 + 4 = 6$, $2 + 4 + 6 = 12$, $2 + 4 + 6 + 8 = 20$, and $2 + 4 + 6 + 8 + 10 = 30$. We conjecture that $\sum_{j=1}^{n} 2j = n(n+1)$ since this formula holds for small values of n. To prove this by mathematical induction we have $\sum_{j=1}^{1} 2j = 2 = 2 \cdot (1+1)$ so the result is true for 1. Now assume that the formula holds for n. Then $\sum_{j=1}^{n+1} 2j = (\sum_{j=1}^{n} 2j) + 2(n+1) = n(n+1) + 2(n+1) = (n+1)(n+2)$. This completes the proof.

1.3.3. For the basis step we have $\displaystyle\sum_{k=1}^{1}\frac{1}{k^2} = 1 \leq 2 - \frac{1}{1} = 1$. For the inductive step, we assume that $\displaystyle\sum_{k=1}^{n}\frac{1}{k^2} \leq 2 - \frac{1}{n}$. Then, $\displaystyle\sum_{k=1}^{n+1}\frac{1}{k^2} = \sum_{k=1}^{n}\frac{1}{k^2} + \frac{1}{(n+1)^2} \leq 2 + \frac{1}{(n+1)^2}$ by the induction hypothesis. This is less than $2 - \frac{1}{n+1} + \frac{1}{(n+1)^2} = 2 - \frac{1}{n+1}(1 - \frac{1}{n+1}) \leq 2 - \frac{1}{n+1}$, as desired.

1.3.4. For the basis step, we have $\displaystyle\sum_{k=1}^{1}\frac{1}{k(k+1)} = \frac{1}{2}$. For the inductive step, we assume that $\displaystyle\sum_{k+1}^{n}\frac{1}{k(k+1)} = \frac{n}{n+1}$. Then, $\displaystyle\sum_{k=1}^{n+1}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$, as desired.

1.3.5. We see that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and so on. We conjecture that $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. To prove this by mathematical induction we first note that the basis step follows since $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Next, we assume that $\mathbf{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$.

1.3.6. The basis step holds since $1 = 1 \cdot (1+1)/2$. For the inductive step assume that $\sum_{j=1}^{n} j = n(n+1)/2$. It follows that

$$\sum_{j=1}^{n+1} j = \sum_{j=1}^{n} j + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)(\frac{n}{2} + 1) = \frac{(n+1)(n+2)}{2}.$$

This finishes the inductive proof.

1.3.7. For the basis step, we have $\sum_{j=1}^{1} j^2 = 1 = 1(1+1)(2\cdot 1+1)/6$. For the inductive step, we assume that $\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$. Then, $\sum_{j=1}^{n+1} j^2 = \sum_{j=1}^{n} j^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2 = (n+1)\left(n(2n+1)/6 + n + 1\right) = (n+1)(2n^2+7n+6)/6 = (n+1)(n+2)[2(n+1)+1]/6$, as desired.

1.3.8. For the basis step, we have $\sum_{j=1}^{1} j^3 = 1$, and $(1(1+1)/2)^2 = 1$ also. For the inductive step, we assume that $\sum_{j=1}^{n} j^3 = (n(n+1)/2)^2$. Then, $\sum_{j=1}^{n+1} j^3 = \sum_{j=1}^{n} j^3 + (n+1)^3 = (n(n+1)/2)^2 + n^3 + 3n^2 + 3n + 1 = ((n+1)(n+2)/2)^2$, as desired.

1.3.9. For the basis step, we have $\sum_{j=1}^{1} j(j+1) = 2 = 1(2)(3)/3$. Assume it is true for n. Then $\sum_{j=1}^{n+1} j(j+1) = n(n+1)(n+2)/3 + (n+1)(n+2) = (n+1)(n+2)(n/3+1) = (n+1)(n+2)(n+3)/3$.

1.3.10. For the basis step, we have $\sum_{j=1}^{1} (-1)^{j-1} j^2 = 1 = (-1)^{1-1}1(1+1)/2$. For the inductive step, we assume that $\sum_{j=1}^{n} (-1)^{j-1} j^2 = (-1)^{n-1} n(n+1)/2$. Then, $\sum_{j=1}^{n+1} (-1)^{j-1} j^2 = \sum_{j=1}^{n} (-1)^{j-1} j^2 + (-1)^n (n+1)^2 = (-1)^{n-1} n(n+1)/2 + (-1)^n (n+1)^2 = (-1)^n \frac{1}{2}(n+1)[2(n+1) - n] = (-1)^{(n+1)-1}(n+1)(n+2)/2$, as desired.

1.3.11. We have $\displaystyle\prod_{j=1}^{n} 2^j = 2^{\sum_{j=1}^{n} j} = 2^{n(n+1)/2}$ since $\displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

1.3.12. We use mathematical induction. For $n = 1$ we have $\sum_{j=1}^{1} j \cdot j! = 1 \cdot 1! = 1 = (1+1)! - 1 = 1$. Now assume that $\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$. Then $\sum_{j=1}^{n+1} j \cdot j! = (n+1)! - 1 + (n+1) \cdot (n+1)! = (n+1)!(1 + n + 1) - 1 = (n+2)! - 1$. This completes the proof.

1.3.13. We will prove this using mathematical induction. We see that $12 = 4 \cdot 3$. Now assume that postage of n cents can be formed, with $n = 4a + 5b$, where a and b are nonnegative integers. To form $n+1$ cents postage, if $a > 0$ we can replace a 4-cent stamp with a 5-cent stamp; that is, $n + 1 = 4(a-1) + 5(b+1)$. If no 4-cent stamps are present, then all 5-cent stamps were used. It follows that there must be at least three 5-cent stamps and these can be replaced by four 4-cent stamps; that is, $n + 1 = 4(a+4) + 5(b-3)$.

1.3.14. We prove this using mathematical induction. We see that $54 = 4 \cdot 10 + 2 \cdot 7$. Now assume that postage of n cents can be formed, with $n = 10a + 7b$, where a and b are positive integers. To form $n+1$ cents postage, if $a > 1$ we can replace 2 ten-cent stamps with 3 seven-cent stamps, that is, $n + 1 = 10(a-2) + 7(b+3)$. If $a < 2$, then notice that $b \geq 7$. We can replace 7 seven-cent stamps with 5 ten-cent stamps, that is, $n + 1 = 10(a+5) + 7(b-7)$.

1.3.15. We use mathematical induction. The inequality is true for $n = 0$ since $H_{2^0} = H_1 = 1 \geq 1 = 1 + 0/2$. Now assume that the inequality is true for n, that is, $H_{2^n} \geq 1 + n/2$. Then $H_{2^{n+1}} = \sum_{j=1}^{2^n} 1/j + \sum_{j=2^n+1}^{2^{n+1}} 1/j \geq H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} 1/2^{n+1} \geq 1 + n/2 + 2^n \cdot 1/2^{n+1} = 1 + n/2 + 1/2 = 1 + (n+1)/2$. This completes the inductive proof.

1.3.16. For the basis step, we have $H_{2^0} = H_1 = 1 \leq 1 + 0 = 1$. For the inductive step, we assume that $H_{2^n} \leq 1 + n$. Then,

$$H_{2^{n+1}} = H_{2^n} + \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{j} < 1 + n + 2^n \frac{1}{2^n} = 1 + (n+1),$$

as desired.

1.3.17. For the basis step, we have $(2 \cdot 1)! = 2 < 2^{2 \cdot 1}(1!)^2 = 4$. For the inductive step, we assume that $(2n)! < 2^{2n}(n!)^2$. Then $[2(n+1)]! = (2n)!(2n+1)(2n+2) < 2^{2n}(n!)^2(2n+1)(2n+2) < 2^{2n}(n!)^2(2n+2)^2 = 2^{2(n+1)}[(n+1)!]^2$, as desired.

1.3.18. We will use the second principle of mathematical induction to prove this. For the basis step, we have $x - y$ is a factor of $x^1 - y^1$. For the inductive step, we assume that $x - y$ is a factor of $x^n - y^n$ and $x^{n-1} - y^{n-1}$. Then, $x^{n+1} - y^{n+1} = (x^n - y^n)(x + y) + xy(x^{n-1} - y^{n-1})$. Since $x - y$ is a factor of both $(x^n - y^n)(x + y)$ and $xy(x^{n-1} - y^{n-1})$, it is a factor of $x^{n+1} - y^{n+1}$.

1.3.19. Let A be such a set. Define B as $B = \{x - k + 1 \mid x \in A \text{ and } x \geq k\}$. Since $x \geq k$, B is a set of positive integers. Since $k \in A$ and $k \geq k$, $k - k + 1 = 1$ is in B. Since $n + 1$ is in A whenever n is, $n + 1 - k + 1$ is in B whenever $n - k + 1$ is. Thus B satisfies the hypothesis for mathematical induction, i.e. B is the set of positive integers. Mapping B back to A in the natural manner, we find that A contains the set of integers greater than or equal to k.

1.3.20. The basis step holds since $2^4 = 16 < 4! = 24$. Now assume that $2^n < n!$. Then $2^{n+1} = 2 \cdot 2^n < 2 \cdot n! < (n + 1) \cdot n! = (n + 1)!$.

1.3.21. For the basis step, we have $4^2 = 16 < 24 = 4!$. For the inductive step, we assume that $n^2 < n!$. Then, $(n + 1)^2 = n^2 + 2n + 1 < n! + 2n + 1 < n! + 3n < n! + n! = 2n! < (n + 1)n! = (n + 1)!$, as desired.

1.3.22. The basis step is clear when $n = 0$. For the inductive step, we assume that $1 + hn \leq (1 + h)^n$. Then, $(1 + h)^{n+1} = (1 + h)^n(1 + h) \geq (1 + hn)(1 + h) = 1 + nh + h + nh^2 \geq 1 + h(n + 1)$ since nh^2 is positive. This last inequality proves the induction hypothesis.

1.3.23. We use the second principle of mathematical induction. For the basis step, if the puzzle has only one piece, then it is assembled with exactly 0 moves. For the induction step, assume that all puzzles with $k \leq n$ pieces require $k - 1$ moves to assemble. Suppose it takes m moves to assemble a puzzle with $n + 1$ pieces. Then the m move consists of joining two blocks of size a and b, respectively, with $a + b = n + 1$. But by the induction hypothesis, it requires exactly $a - 1$ and $b - 1$ moves to assemble each of these blocks. Thus, $m = (a - 1) + (b - 1) + 1 = a + b + 1 = n + 1$. This completes the induction.

1.3.24. The $n = 2$ case does not follow from the $n = 1$ case, since, when $n = 2$, the set of horses labelled 1 to $n - 1$ (which is just the set containing horse 1) does not have any common elements with the set of horses labelled from 2 to n (which is just the set containing horse 2.)

1.3.25. Suppose that $f(n)$ is defined recursively by specifying the value of $f(1)$ and a rule for finding $f(n+1)$ from $f(n)$. We will prove by mathematical induction that such a function is well-defined. First, note that $f(1)$ is well-defined since this value is explicitly stated. Now assume that $f(n)$ is well-defined. Then $f(n + 1)$ also is well-defined since a rule is given for determining this value from $f(n)$.

1.3.26. The function is $f(n) = 2^n$. For the basis step, we have $f(1) = 2 = 2^1$. For the inductive step, we assume that $f(n) = 2^n$. Then, $f(n + 1) = 2f(n) = 2 \cdot 2^n = 2^{n+1}$, as desired.

1.3.27. We have $g(1) = 2, g(2) = 2^{g(1)} = 4, g(3) = 2^{g(2)} = 2^4 = 16$, and $g(4) = 2^{g(3)} = 2^{16} = 65536$.

1.3.28. The basis step is given. For the inductive step, we assume that the value of f at the first n positive integers are uniquely determined. Then $f(n + 1)$ is uniquely determined from the rule. Therefore, by mathematical induction, $f(n)$ is determined for every positive integer n.

1.3.29. We use the second principle of mathematical induction. The basis step consists of verifying the formula for $n = 1$ and $n = 2$. For $n = 1$ we have $f(1) = 1 = 2^1 + (-1)^1$ and for $n = 2$ we have $f(2) = 5 = 2^2 + (-1)^2$. Now assume that $f(k) = 2^k + (-1)^k$ for all positive integers k with $k < n$ where $n > 2$. By the induction hypothesis it follows that $f(n) = f(n - 1) + 2f(n - 2) = (2^{n-1} + (-1)^{n-1}) + 2(2^{n-2} + (-1)^{n-2}) = (2^{n-1} + 2^{n-1}) + (-1)^{n-2}(-1 + 2) = 2^n + (-1)^n$. This finishes the proof.

1.3.30. Since $2^5 = 32 > 25 = 5^2$, the basis step holds. Assume that $2^n > n^2$. Note that for $n > 4$, $2n^2 = n^2 + n^2 > n^2 + 3n = n^2 + 2n + n > n^2 + 2n + 1 = (n + 1)^2$. Then we have $(n + 1)^2 < 2n^2 < 2 \cdot 2^n = 2^{n+1}$, which completes the induction.

1.3.31. We use the second principle of mathematical induction. We see that $a_0 = 1 \leq 3^0 = 1$, $a_1 = 3 \leq 3^i = 3$, and $a_2 = 9 \leq 3^2 = 9$. These are the basis cases. Now assume that $a_k \leq 3^k$ for all integers k with $0 \leq k < n$. It follows that $a_n = a_{n-1} + a_{n-2} + a_{n-3} \leq 3^{n-1} + 3^{n-2} + 3^{n-3} = 3^{n-3}(1 + 3 + 9) = 13 \cdot 3^{n-3} < 27 \cdot 3^{n-3} = 3^n$. The induction argument is complete.

1.3.32. a. For the basis step notice that for 1 ring only, $1 = 2^1 - 1$ moves are needed. For the inductive step we assume that it takes $2^n - 1$ steps to transfer n rings. To make the inductive step, first transfer n of $n + 1$ rings to the third peg. This takes $2^n - 1$ steps. Now transfer the bottom ring to the second peg. This is one step. Then transfer the n rings on the third peg to the second peg. This is $2^n - 1$ more steps. Altogether, this takes $2^n - 1 + 1 + 2^n - 1 = 2^{n+1} - 1$ steps.

b. The world will last, according to this legend, $2^{64} - 1 = 18,446,744,073,709,551,615$ seconds $= 3.07445 \cdot 10^{17}$ minutes $= 5.12409 \cdot 10^{15}$ hours $= 2.13503 \cdot 10^{14}$ days $= 5.84942 \cdot 10^{11}$ years, that is more than 580 billion years.

1.3.33. Let P_n be the statement for n. Then P_2 is true, since we have $(a_1+a_2)/2)^2 - a_1a_2 = (a_1 - a_2/2)^2 \geq 0$. Assume P_n is true. Then by P_2, for $2n$ positive real numbers $a_1, \ldots, a_{2n}$ we have $a_1 + \cdots + a_{2n} \geq 2(\sqrt{a_1a_2} + \sqrt{a_3a_4} + \cdots + \sqrt{a_{2n-1}a_{2n}})$. Apply P_n to this last expression to get $a_1 + \cdots + a_{2n} \geq 2n(a_1a_2 \cdots a_{2n})^{1/2n}$ which establishes P_n for $n = 2^k$ for all k. Again, assume P_n is true. Let $g = (a_1a_2 \cdots a_{n-1})^{1/(n-1)}$. Applying P_n, we have $a_1 + a_2 + \cdots + a_{n-1} + g \geq n(a_1a_2 \cdots a_{n-1}g)^{1/n} = n(g^{n-1}g)^{1/n} = ng$. Therefore, $a_1 + a_2 + \cdots + a_{n-1} \geq (n-1)g$ which establishes P_{n-1}. Thus P_{2^k} is true and P_n implies P_{n-1}. This establishes P_n for all n.

1.3.34. There are four 2×2 chess boards with one square missing. Each can be covered with exactly one L-shaped piece. This is the basis step. Now assume that any $2^n \times 2^n$ chess board can be covered with L-shaped pieces. Consider a $2^{n+1} \times 2^{n+1}$ chess board with one square missing. Split this into four $2^n \times 2^n$ chess boards three of which contain every square and the fourth has one square missing. By the inductive hypothesis we can cover the fourth $2^n \times 2^n$ chess board because it is missing one square. Now use one L-shaped piece to cover the three squares in the other three chess boards that touch at the center of the larger $2^{n+1} \times 2^{n+1}$ chess board. What is left to cover is all the rest of the squares in each of the three $2^n \times 2^n$ chess boards. The inductive hypothesis says that we can cover all the remaining squares in each of these chess boards. This completes the proof.

1.3.35. Note that since $0 < p < q$ we have $0 < p/q < 1$. The proposition is trivially true if $p = 1$. We proceed by strong induction on p. Let p and q be given and assume the proposition is true for all rational numbers between 0 and 1 with numerators less than p. To apply the algorithm, we find the unit fraction $1/s$ such that $1/(s-1) > p/q > 1/s$. When we subtract, the remaining fraction is $p/q - 1/s = (ps - q)/qs$. On the other hand, if we multiply the first inequality by $q(s-1)$ we have $q > p(s-1)$ which leads to $p > ps - q$, which shows that the numerator of p/q is strictly greater than the numerator of the remainder $(ps - q)/qs$ after one step of the algorithm. By the induction hypothesis, this remainder is expressible as a sum of unit fractions, $1/u_1 + \cdots + 1/u_k$. Therefore $p/q = 1/s + 1/u_1 + \cdots + 1/u_k$ which completes the induction step.

1.3.36. a. Since $1/2 < 2/3$, we subtract to get $2/3 = 1/2 + 1/6$.

b. Since $1/2 < 5/8$, we subtract to get $5/8 = 1/2 + 1/8$.

c. Since $1/2 < 11/17$ we subtract to get $11/17 = 1/2 + 5/34$. The largest unit fraction less than $5/34$ is $1/7$ so we subtract to get $11/17 = 1/2 + 1/7 + 1/238$.

d. The largest unit fraction less than $44/101$ is $1/3$ so we subtract and get $44/101 = 1/3 + 31/303$. The largest unit fraction less than $31/303$ is $1/10$, so we subtract to get $44/101 = 1/3 + 1/10 + 7/3030$. The largest unit fraction less than $7/3030$ is $1/433$, so we subtract to get $44/101 = 1/3 + 1/10 + 1/433 + 1/1311990$. (Note that this is the result of the "greedy algorithm." Other representations are possible, such as $44/101 = 1/3 + 1/10 + 1/440 + 1/26664$.)

1.4. The Fibonacci Numbers

1.4.1. a. We have $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Hence $f_3 = f_2 + f_1 = 1 + 1 = 2$, $f_4 = f_3 + f_2 = 2 + 1 = 3$, $f_5 = 3 + 2 = 5$, $f_6 = 5 + 3 = 8$, $f_7 = 8 + 5 = 13$, $f_8 = 13 + 8 = 21$, $f_9 = 21 + 13 = 34$, and $f_{10} = 34 + 21 = 55$.

b. We continue beyond part (a) finding that $f_{11} = f_{10} + f_9 = 55 + 34 = 89$, $f_{12} = 89 + 55 = 144$, and $f_{13} = 144 + 89 = 233$.

c. We continue beyond part (b) finding that $f_{14} = f_{13} + f_{12} = 233 + 144 = 377$, and $f_{15} = 377 + 233 = 610$.

d. We continue beyond part (c) finding that $f_{16} = 610 + 377 = 987$, $f_{17} = 987 + 610 = 1597$, and $f_{18} = 1597 + 987 = 2584$.

e. We continue beyond part (d) finding that $f_{19} = 2584 + 1597 = 4181$, $f_{20} = 4181 + 2584 = 6765$.

f. We continue beyond part (e) finding that $f_{21} = 6765 + 4181 = 10946$, $f_{22} = 10946 + 6765 = 17711$, $f_{23} = 17711 + 10946 = 28657$, $f_{24} = 28657 + 17711 = 46368$, and $f_{25} = 46368 + 28657 = 75025$.

1.4.2. a. We continue from Exercise 1 part (a), finding that $f_{11} = 55 + 34 = 89$ and $f_{12} = 89 + 55 = 144$.

b. We continue from Exercise 1 part (c), finding that $f_{16} = 610 + 377 = 987$.

c. We computed $f_{24} = 46368$ in Exercise 1 part (f).

d. We continue from Exercise 1 part (f), finding that $f_{26} = 75025 + 46368 = 121393$, $f_{27} = 121393 + 75025 = 196418$, $f_{28} = 196418 + 121393 = 317811$, $f_{29} = 317811 + 196418 = 514229$, and $f_{30} = 514229 + 317811 = 832040$.

e. We continue from part (d), finding $f_{31} = 832040 + 514229 = 1346269$, and $f_{32} = 1346269 + 832040 = 2178309$.

f. We continue from part (e), finding $f_{33} = 2178309 + 1346269 = 3524578$, $f_{34} = 3524578 + 2178309 = 5702887$, $f_{35} = 5702887 + 3524578 = 9227465$ and $f_{36} = 9227465 + 5702887 = 14930352$.

1.4.3. Note that from the Fibonacci identity, whenever n is a positive integer, $f_{n+2} - f_n = f_{n+1}$. Then we have $2f_{n+2} - f_n = f_{n+2} + (f_{n+2} - f_n) = f_{n+2} + f_{n+1} = f_{n+3}$. If we add f_n to both sides of this equation, we have the desired identity.

1.4.4. Assuming n is a positive integer, we have compute $2f_{n+1} + f_n = f_{n+1} + (f_{n+1} + f_n) = f_{n+1} + f_{n+2} = f_{n+3}$. If we subtract f_n from both sides of this equation, we have the desired identity.

1.4.5. For $n = 1$ we have $f_{2\cdot 1} = 1 = 1^2 + 2 \cdot 1 \cdot 0 = f_1^2 + 2f_0f_1$, and for $n = 2$, we have $f_{2\cdot 2} = 3 = 1^2 + 2 \cdot 1 \cdot 1 = f_2^2 + 2f_1f_2$. So the basis step holds for strong induction. Assume, then that $f_{2n-4} = f_{n-2}^2 + 2f_{n-3}f_{n-2}$ and $f_{2n-2} = f_{n-1}^2 + 2f_{n-2}f_{n-1}$. Now compute $f_{2n} = f_{2n-1} + f_{2n-2} = 2f_{2n-2} + f_{2n-3} = 3f_{2n-2} - f_{2n-4}$. Now we may substitute in our induction hypotheses to set this last expression equal to $3f_{n-1}^2 + 6f_{n-2}f_{n-1} - f_{n-2}^2 - 2f_{n-3}f_{n-2} = 3f_{n-1}^2 + 6(f_n - f_{n-1})f_{n-1} - (f_n - f_{n-1})^2 - 2(f_{n-1} - f_{n-2})(f_n - f_{n-1}) = -2f_{n-1}^2 + 6f_nf_{n-1} - f_n^2 + 2f_n(f_n - f_{n-1}) - 2f_{n-1}(f_n - f_{n-1}) = f_n^2 + 2f_{n-1}f_n$ which completes the induction step.

1.4.6. For n a positive integer greater than 1, we have $f_{n+2} = f_{n+1} + f_n = (f_n + f_{n-1}) + f_n = (f_n + (f_n - f_{n-2})) + f_n = 3f_n - f_{n-2}$. Adding f_{n-2} to both sides yields the desired identity.

1.4.7. Note that $f_1 = 1 = f_2$, $f_1 + f_3 = 3 = f_4$, and $f_1 + f_3 + f_5 = 8 = f_6$ so we conjecture that $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$. We prove this by induction. The basis step is checked above. Assume that our formula is true for n, and consider $f_1 + f_3 + f_5 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2}$, which is

the induction step. Therefore the formula is correct.

1.4.8. Note that $f_2 = 1 = f_3 - 1$, $f_2 + f_4 = 4 = f_5 - 1$, and $f_2 + f_4 + f_6 = 12 = f_7 - 1$, so we conjecture that $f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$. We prove this by induction. The basis step is checked above. Assume that our formula is true for n, and consider $f_2 + f_4 + f_6 + \cdots + f_{2n} + f_{2n+2} = f_{2n+1} - 1 + f_{2n+2} = f_{2n+3} - 1$, which is the induction step. Therefore the formula is correct. Another solution is to subtract the formula in Exercise 7 from the formula in Example 1.27, as follows: $\sum_{i=1}^{n} f_{2i} = \sum_{i=1}^{2n} f_i - \sum_{i=1}^{n} f_{2i-1} = (f_{2n+2} - 1) - f_{2n} = f_{2n+1} - 1$.

1.4.9. First suppose $n = 2k$ is even. Then $f_n - f_{n-1} + \cdots + (-1)^{n+1} f_1 = (f_{2k} + f_{2k-1} + \cdots + f_1) - 2(f_{2k-1} + f_{2k-3} + \cdots + f_1) = (f_{2k+2} - 1) - 2(f_{2k})$ by the formulas in Example 1.23 and Exercise 3. This last equals $(f_{2k+2} - f_{2k}) - f_{2k} - 1 = f_{2k+1} - f_{2k} - 1 = f_{2k-1} - 1 = f_{n-1} - 1$. Now suppose $n = 2k + 1$ is odd. Then $f_n - f_{n-1} + \cdots + (-1)^{n+1} = f_{2k+1} - (f_{2k} - f_{2k-1} + \cdots - (-1)^{n+1} f_1) = f_{2k+1} - (f_{2k-1} - 1)$ by the formula just proved for the even case. This last equals $(f_{2k+1} - f_{2k-1}) + 1 = f_{2k} + 1 = f_{n-1} + 1$. We can unite the formulas for the odd and even cases by writing the formula as $f_{n-1} - (-1)^n$.

1.4.10. For $n = 1$ we have $f_3 = 2 = f_2^2 + f_1^2 = 1^2 + 1^2$. And when $n = 2$ we have $f_5 = 5 = 2^2 + 1^2 = f_3^2 + f_2^2$, so the basis steps hold for mathematical induction. Now assume, for the strong form of induction, that the identity holds for all values of n up to $n = k$. Then $f_{2k-3} = f_{k-1}^2 + f_{k-2}^2$ and $f_{2k-1} = f_k^2 + f_{k-1}^2$. Now we calculate $f_{2k+1} = f_{2k} + f_{2k-1} = f_{2k-1} + f_{2k-2} + f_{2k-1} = 2f_{2k-1} + (f_{2k-1} - f_{2k-3}) = 3f_{2k-1} - f_{2k-3}$. Now substituting in the induction hypothesis, makes this last expression equal to $3(f_k^2 + f_{k-1}^2) - f_{k-1}^2 - f_{k-2}^2 = 3f_k^2 + 2f_{k-2}^2 - (f_k - f_{k-1})^2 = 2f_k^2 + f_{k-1}^2 + 2f_k f_{k-1} = 2f_k^2 + (f_{k+1} - f_k)^2 + 2f_k(f_{k+1} - f_k) = f_{k+1}^2 + f_k^2$, which completes the induction step.

1.4.11. We can construct an induction proof similar to the ones in Exercises 5 and 10, or we may proceed as follows. From Exercise 5, we have $f_{2n} = f_n^2 + 2f_{n-1}f_n = f_n(f_n + f_{n-1} + f_{n-1}) = (f_{n+1} - f_{n-1})(f_{n+1} + f_{n-1}) = f_{n+1}^2 - f_{n-1}^2$, which is the desired identity.

1.4.12. Let $S_n = f_n + f_{n-1} + f_{n-2} + 2f_{n-3} + \cdots + 2^{n-4} f_2 + 2^{n-3} f_1$. We proceed by induction. If $n = 3$ we have $S_3 = f_3 + f_2 + f_1 = 2 + 1 + 1 = 4 = 2^{3-1}$, and when $n = 4$ we have $S_4 = f_4 + f_3 + f_2 + 2f_1 = 3 + 2 + 1 + 2 \cdot 1 = 8 = 2^{4-1}$, so the basis steps hold. Now assume the identity holds for all values less or equal to n and consider $S_{n+1} = f_{n+1} + f_n + f_{n-1} + 2f_{n-2} + 4f_{n-3} + \cdots + 2^{n-4} f_3 + 2^{n-3} f_2 + 2^{n-2} f_1$. We use the Fibonacci identity to expand every term except the last two to get $S_{n+1} = (f_n + f_{n-1}) + (f_{n-1} + f_{n-2}) + (f_{n-2} + f_{n-3}) + 2(f_{n-3} + f_{n-4}) + 4(f_{n-4} + f_{n-5}) + \cdots + 2^{n-4}(f_2 + f_1) + 2^{n-3} f_2 + 2^{n-2} f_1$. Next we regroup, taking the first term from each set of parentheses, plus the second last term together in one group, the last term from each set of parentheses together in another group, and leaving the last term by itself to get $S_{n+1} = (f_n + f_{n-1} + f_{n-2} + 2f_{n-3} + 4f_{n-4} + \cdots + 2^{n-4} f_2 + 2^{n-3} f_2) + (f_{n-1} + f_{n-2} + f_{n-3} + 2f_{n-4} + 4f_{n-5} + \cdots + 2^{n-4} f_1) + 2^{n-2} f_1$. The first group is seen to be equal to S_n when we realize that the last $f_2 = f_1$. The second group is equal to S_{n-1}, so we have $S_{n+1} = S_n + S_{n-1} + 2^{n-1} = 2^{n-2} + 2^{n-2} + 2^{n-1} = 2^n$ by the induction hypothesis. Therefore, by mathematical induction, the proposition is proved.

1.4.13. We proceed by mathematical induction. For the basis step, $\sum_{j=1}^{1} f_j^2 = f_1^2 = f_1 f_2$. To make the inductive step we assume that $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$. Then $\sum_{j=1}^{n+1} f_j^2 = \sum_{j=1}^{n} f_j^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1} f_{n+2}$.

1.4.14. We use mathematical induction. We will use the recursive definition $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$. For $n = 1$ we have $f_2 f_0 - f_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1$. Hence the basis step holds. Now assume that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$. Then $f_{n+2} f_n - f_{n+1}^2 = (f_{n+1} + f_n) f_n - f_{n+1}(f_n + f_{n-1}) = f_n^2 - f_{n+1} f_{n-1} = -(-1)^n = (-1)^{n+1}$. This completes the proof.

1.4.15. From Exercise 13, we have $f_{n+1} f_n - f_{n-1} f_{n-2} = (f_1^2 + \cdots + f_n^2) - (f_1^2 + \cdots f_{n-2}^2) = f_n^2 + f_{n-1}^2$. The identity in Exercise 10 shows that this is equal to f_{2n-1} when n is a positive integer, and in particular when n is greater than 2.

1.4.16. Since $f_1 f_2 = 1 \cdot 1 = 1^2 = f_2^2$, the basis step holds. By the induction hypothesis we have $f_1 f_2 + \cdots + f_{2n-1} f_{2n} + f_{2n} f_{2n+1} + f_{2n+1} f_{2(n+1)} = f_{2n}^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2(n+1)} = f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1} f_{2(n+1)} =$

$f_{2n}f_{2(n+1)} + f_{2n+1}f_{2(n+1)} = (f_{2n} + f_{2n+1})f_{2(n+1)} = f_{2(n+1)}^2$.

1.4.17. For fixed m, we proceed by induction on n. The basis step is $f_{m+1} = f_m f_2 + f_{m-1}f_1 = f_m \cdot 1 + f_{m-1} \cdot 1$ which is true. Assume the identity holds for $1, 2, \ldots, k$. Then $f_{m+k} = f_m f_{k+1} + f_{m-1}f_k$ and $f_{m+k-1} = f_m f_k + f_{m-1}f_{k-1}$. Adding these equations gives us $f_{m+k} + f_{m+k-1} = f_m(f_{k+1} + f_k) + f_{m-1}(f_k + f_{k-1})$. Applying the recursive definition yields $f_{m+k+1} = f_m f_{k+2} + f_{m-1}f_{k+1}$, which is precisely the identity.

1.4.18. We're given that $L_1 = 1$ and $L_2 = 3$. Adding each consecutive pair to generate the next Lucas number yields the sequence $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots$.

1.4.19. A few trial cases lead us to conjecture that $\sum_{i=1}^n L_i = L_{n+2} - 3$. We prove that this formula is correct by induction. The basis step is $L_1 = 1$ and $L_3 - 3 = 4 - 3 = 1$, which checks. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_i = \sum_{i=1}^n L_i + L_{n+1} = L_{n+2} - 3 + L_{n+1}$ by the induction hypothesis. This last equals $(L_{n+2} + L_{n+1}) - 3 = L_{n+3} - 3$, which completes the induction step.

1.4.20. A few trial cases lead us to conjecture that $\sum_{i=1}^n L_{2i-1} = L_{2n} - 2$. We prove that this formula is correct by induction. The basis step is $L_1 = 1 = L_2 - 2$. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_{2i-1} = \sum_{i=1}^n L_{2i-1} + L_{2n+1} = L_{2n} - 2 + L_{2n+1} = L_{2n+2} - 2$, which completes the induction step.

1.4.21. A few trial cases lead us to conjecture that $\sum_{i=1}^n L_{2i} = L_{2n+1} - 1$. We prove that this formula is correct by induction. The basis step is $L_2 = 3 = L_3 - 1$. Assume that the formula holds for n and compute $\sum_{i=1}^{n+1} L_{2i} = \sum_{i=1}^n L_{2i} + L_{2n+2} = L_{2n+1} - 1 + L_{2n+2} = L_{2n+3} - 1$, which completes the induction step.

1.4.22. We proceed by induction. The basis step is when $n = 2$, and we have $L_2^2 - L_3L_1 = 3^2 - 4 \cdot 1 = 5 = 5(-1)^2$. Now assume the identity holds for n. Then for $n + 1$ we have $L_{n+1}^2 - L_{n+2}L_n = (L_n + L_{n-1})L_{n+1} - (L_{n+1} + L_n)L_n = L_nL_{n+1} + L_{n-1}L_{n+1} - L_{n+1}L_n - L_n^2 = -(L_n^2 - L_{n-1}L_{n+1}) = -(5(-1)^n) = 5(-1)^{n+1}$, where we apply the induction hypothesis at the penultimate step.

1.4.23. We proceed by induction. The basis step is $L_1^2 = 1 = L_1L_2 - 2 = 1 \cdot 3 - 2$. Assume the formula holds for n and consider $\sum_{i=1}^{n+1} L_i^2 = \sum_{i=1}^n L_i^2 + L_{n+1}^2 = L_nL_{n+1} - 2 + L_{n+1}^2 = L_{n+1}(L_n + L_{n+1} - 2 = L_{n+1}L_{n+2} - 2$, which completes the induction step.

1.4.24. For $n = 2$, we have $L_2 = 3 = 1 + 2 = f_1 + f_3$. For $n = 3$, we have $L_3 = 4 = 1 + 3 = f_2 + f_4$. This serves as the basis step. Now assume that the statement is true for $k = 2, 3, 4, \ldots, n$. Then $L_{n+1} = L_n + L_{n-1} = (f_{n+1} + f_{n-1}) + (f_n + f_{n-2}) = (f_{n+1} + f_n) + (f_{n-1} + f_{n-2}) = f_{n+2} + f_n$, which completes the induction.

1.4.25. For the basis step, we check that $L_1f_1 = 1 \cdot 1 = 1 = f_2$ and $L_2f_2 = 3 \cdot 1 = 3 = f_4$. Assume the identity is true for all positive integers up to n. Then we have $f_{n+1}L_{n+1} = (f_{n+2} - f_n)(f_{n+2} - f_n)$ from Exercise 16. This equals $f_{n+2}^2 - f_n^2 = (f_{n+1} + f_n)^2 - (f_{n-1} + f_{n-2})^2 = f_{n+1}^2 + 2f_{n+1}f_n + f_n^2 - f_{n-1}^2 - 2f_{n-1}f_{n-2} - f_{n-2}^2 = (f_{n+1}^2 - f_{n-1}^2) + (f_n^2 - f_{n-2}^2) + 2(f_{n+1}f_n - f_{n-1}f_{n-2}) = (f_{n+1} - f_{n-1})(f_{n+1} + f_{n-1}) + (f_n - f_{n-2})(f_n + f_{n-2}) + 2(f_{2n-1})$, where the last parenthetical expression is obtained from Exercise 8. This equals $f_nL_n + f_{n-1}L_{n-1} + 2f_{2n-1}$. Applying the induction hypothesis yields $f_{2n} + f_{2n-2} + 2f_{2n-1} = (f_{2n} + f_{2n-1}) + (f_{2n-1} + f_{2n-2}) = f_{2n+1} + f_{2n} = f_{2n+2}$, which completes the induction.

1.4.26. For the basis step, we check that when $n = 1$, $5f_2 = 5 \cdot 1 = 1 + 4 = L_1 + L_3$ and when $n = 2$, $5f_3 = 10 = 3 + 7 = L_2 + L_4$. Now assume the identity holds for integers less than n, and compute $5f_{n+1} = 5f_n + 5f_{n-1} = (L_{n-1} + L_{n+1}) + (L_{n-2} + L_n) = (L_{n-1} + L_{n-2}) + (L_{n+1} + L_n) = L_n + L_{n+1}$, which completes the induction step.

1.4.27. We prove this by induction on n. Fix m a positive integer. If $n = 2$, then for the basis step we need to show that $L_{m+2} = f_{m+1}L_2 + f_mL_1 = 3f_{m+1} + f_m$, for which we will use induction on m. For $m = 1$ we have $L_3 = 4 = 3 \cdot f_2 + f_1$ and for $m = 2$ we have $L_4 = 7 = 3 \cdot f_3 + f_2$, so the basis step for m holds. Now assume that the basis step for n holds for all values of m less than and equal to m. Then $L_{m+3} = L_{m+2} + L_{m+1} = 3f_{m+1} + f_m + 3f_m + f_{m-1} = 3f_{m+2} + f_{m+1}$, which completes the induction step on m and proves the basis step for n. To prove the induction step on n, we compute $L_{m+n+1} = L_{m+n} +$

$L_{m+n-1} = (f_{m+1}L_n + f_mL_{n-1}) + (f_{m+1}L_{n-1} + f_mL_{n-2}) = f_{m+1}(L_n + L_{n-1}) + f_m(L_{n-1} + L_{n-2}) = f_{m+1}L_{n+1} + f_mL_n$, which completes the induction on n and proves the identity.

1.4.28. First check that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. We proceed by induction. The basis steps are $\alpha + \beta = (1 + \sqrt{5})/2 + (1 - \sqrt{5})/2 = 1 = L_1$ and $\alpha^2 + \beta^2 = (1 + \alpha) + (1 + \beta) = 2 + L_1 = 3 = L_2$. Assume the identity is true for all positive integers up to n. Then $L_{n+1} = L_n + L_{n-1} = \alpha^n + \beta^n + \alpha^{n-1} + \beta^{n-1} = \alpha^{n-1}(\alpha + 1) + \beta^{n-1}(\beta + 1) = \alpha^{n-1}(\alpha^2) + \beta^{n-1}(\beta^2) = \alpha^{n+1} + \beta^{n+1}$, which completes the induction.

1.4.29. We find that $50 = 34 + 13 + 3 = f_9 + f_7 + f_4$, $85 = 55 + 21 + 8 + 1 = f_{10} + f_8 + f_6 + f_2$, $110 = 89 + 21 = f_{11} + f_8$ and $200 = 144 + 55 + 1 = f_{12} + f_{10} + f_2$. In each case, we used the "greedy" algorithm, always subtracting the largest possible Fibonacci number from the remainder.

1.4.30. Suppose there is a positive integer that has no Zeckendorf representation. Then by the well-ordering property, there is a smallest such integer, n. Let f_k be the largest Fibonacci number less than or equal to n. Note that if $n = f_k$, then n has a Zeckendorf representation, contrary to our assumption. Then $n - f_k$ is a positive integer less than n, so it has a Zeckendorf representation $n - f_k = \sum_{i=1}^{m} f_{a_i}$. Since n has no Zeckendorf representation, it must be that one of the f_{a_i}'s is equal to or consecutive to f_k. That is, one of f_{k-1}, f_k, or f_{k+1} appears in the summation for $n - f_k$. Then $n = \sum_{i=1}^{m} f_{a_i} + f_k \geq f_{k-1} + f_k = f_{k+1}$. But this contradicts the choice of f_k as the largest Fibonacci number less than n. This establishes existence. To establish uniqueness of the Zeckendorf representation, suppose that there is a positive integer that has two distinct representations. Then the well-ordering property gives us a smallest such integer, n. Suppose $n = \sum_{i=1}^{m} f_{a_i} = \sum_{j=1}^{l} f_{b_l}$ are two distinct representations for n. Then no $f_{a_i} = f_{b_j}$, else we could cancel this term from each side and have a smaller integer with two distinct representations. Without loss of generality, assume that $f_{a_1} > f_{a_2} > \cdots > f_{a_m}$ and $f_{b_1} > f_{b_2} > \cdots > f_{b_l}$ and that $f_{a_1} > f_{b_1}$. If b_1 is even, we compute $n = \sum_{i=1}^{l} f_{b_i} \leq f_{b_1} + f_{b_1-2} + f_{b_1-4} + \cdots + f_2 = f_{b_1+1} - 1$ by Exercise 4. But this last is less than or equal to $f_{a_1} - 1 < n$, a contradiction. If b_1 is odd, we compute, now using Exercise 3, $n = \sum_{i=1}^{l} f_{b_i} \leq f_{b_1} + f_{b_1-2} + f_{b_1-4} + \cdots + f_3 = f_{b_1+1} - f_1 \leq f_{a_1} - 1 < n$, which is also a contradiction. This proves uniqueness.

1.4.31. We proceed by mathematical induction. The basis steps ($n = 2$ and 3) are easily seen to hold. For the inductive step, we assume that $f_n \leq \alpha^{n-1}$ and $f_{n-1} \leq \alpha_{n-2}$. Now, $f_{n+1} = f_n + f_{n-1} \leq \alpha^{n-1} + \alpha^{n-2} = \alpha^n$, since α satisfies $\alpha^n = \alpha^{n-1} + \alpha^{n-2}$.

1.4.32. We proceed by the second principle of mathematical induction on n. For the basis step, we observe that $\binom{0}{0} = f_{0+1} = 1$. For the inductive step, we assume that $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = f_{n+1}$, and that $\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots = f_n$. Now, $\binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \cdots = \binom{n}{0} + [\binom{n-1}{1} + \binom{n-1}{0}] + [\binom{n-2}{2} + \binom{n-2}{1}] + \cdots = \binom{n}{0} + \binom{n-1}{1}\binom{n-2}{2} + \cdots + \binom{n-1}{0} + \binom{n-2}{1} + \cdots = f_{n+1} + f_n = f_{n+2}$.

1.4.33. Using Theorem 1.3 and the notation therein, we have $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, since they are roots of $x^2 - x - 1 = 0$. Then we have $f_{2n} = (\alpha^{2n} - \beta^{2n})/\sqrt{5} = (1/\sqrt{5})((\alpha + 1)^n - (\beta + 1)^n) = (1/\sqrt{5})\left(\sum_{j=0}^{n} \binom{n}{j}\alpha^j - \sum_{j=0}^{n} \binom{n}{j}\beta^j\right) = (1/\sqrt{5})\sum_{j=0}^{n} \binom{n}{j}(\alpha^j - \beta^j) = \sum_{j=1}^{n} \binom{n}{j} f_j$ since the first term is zero in the penultimate sum.

1.4.34. We prove this using mathematical induction. For $n = 1$ we have

$$\mathbf{F}^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix}$$

where $f_0 = 0$. Now assume that this formula is true for n. Then

$$\mathbf{F}^{n+1} = \mathbf{F}^n\mathbf{F} = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{n+1} + f_n & f_{n+1} \\ f_n + f_{n-1} & f_n \end{pmatrix} = \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix}.$$

1.4.35. On one hand, $\det(\mathbf{F}^n) = \det(\mathbf{F})^n = (-1)^n$. On the other hand,

$$\det\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = f_{n+1}f_{n-1} - f_n^2.$$

1.4.36. We proceed by induction. Clearly the basis step holds. For the inductive step, we assume that $g_n = af_{n-2} + bf_{n-1}$. Then, $g_{n+1} = g_n + g_{n-1} = af_{n-2} + bf_{n-1} + af_{n-3} + bf_{n-2} = af_{n-1} + bf_n$.

1.4.37. We use the relationship $f_n = f_{n+2} - f_{n+1}$ to extend the definition to include negative indices. Thus, $f_0 = 0, f_{-1} = 1, f_{-2} = -1, f_{-3} = 2, f_{-4} = -3, f_{-5} = 5, f_{-6} = -8, f_{-7} = 13, f_{-8} = -21, f_{-9} = 34, f_{-10} = -55$.

1.4.38. We conjecture that $f_{-n} = (-1)^{n+1} f_n$. The basis step is given in Exercise 55. Assume the conjecture is true for n. Then $f_{-(n+1)} = f_{-(n-1)} - f_{-n} = (-1)^n f_{n-1} - (-1)^{n+1} f_n = (-1)^n (f_{n-1} + f_n) = (-1)^{n+2} f_{n+1}$, which completes the induction step.

1.4.39. The square has area 64 square units, while the rectangle has area 65 square units. This corresponds to the identity in Exercise 7, which tells us that $f_7 f_5 - f_6^2 = 1$. Notice that the slope of the hypotenuse of the triangular piece is $3/8$, while the slope of the top of the trapezoidal piece is $2/5$. We have $2/5 - 3/8 = 1/40$. Thus, the "diagonal" of the rectangle is really a very skinny parallelogram of area 1, hidden visually by the fact that the two slopes are nearly equal.

1.4.40. First check that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$ as in the solution to Exercise 18. We compute $a_1 = (1/\sqrt{5})(\alpha - \beta) = (1/\sqrt{5})\left((1 + \sqrt{5}/2 - (1 - \sqrt{5}/2\right) = (1/\sqrt{5})\left(2\sqrt{5}/2\right) = 1$ and $a_2 = (1/\sqrt{5})(\alpha^2 - \beta^2) = (1/\sqrt{5})(\alpha + 1 - \beta - 1) = (1/\sqrt{5})(\alpha - \beta) = 1$. Finally, we check that $a_{n-1} + a_{n-2} = (1/\sqrt{5})(\alpha^{n-1} - \beta^{n-1}) + (1/\sqrt{5})(\alpha^{n-2} - \beta^{n-2}) = (1/\sqrt{5})(\alpha^{n-1} + \alpha^{n-2} - \beta^{n-1} - \beta^{n-2}) = (1/\sqrt{5})(\alpha^{n-2}(\alpha + 1) - \beta^{n-2}(\beta + 1)) = (1/\sqrt{5})(\alpha^{n-2}\alpha^2 - \beta^{n-2}\beta^2) = (1/\sqrt{5})(\alpha^n - \beta^n) = a_n$. Since these a_n satisfy the defining relationships of the Fibonacci numbers, we can conclude that $a_n = f_n$ for $n = 1, 2, \ldots$.

1.4.41. We solve the equation $r^2 - r - 1 = 0$ to discover the roots $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$. Then according to the theory in the paragraph above, $f_n = C_1 r_1^n + C_2 r_2^n$. For $n = 0$ we have $0 = C_1 r_1^0 + C_2 r_2^0 = C_1 + C_2$. For $n = 1$ we have $1 = C_1 r_1 + C_2 r_2 = C_1(1 + \sqrt{5})/2 + C_2(1 - \sqrt{5})/2$. Solving these two equations simultaneously yields $C_1 = 1/\sqrt{5}$ and $C_2 = -1/\sqrt{5}$. So the explicit formula is $f_n = (1/\sqrt{5})r_1^n - (1/\sqrt{5})r_2^n = (r_1^n - r_2^n)/\sqrt{5}$.

1.4.42. First note that $G(x) - xG(x) - x^2G(x) = \sum_{k=0}^{\infty} f_k x^k - \sum_{k=0}^{\infty} f_k x^{k+1} - \sum_{k=0}^{\infty} f_k x^{k+2} = \sum_{k=0}^{\infty} f_k x^k - \sum_{k=1}^{\infty} f_{k-1} x^k - \sum_{k=2}^{\infty} f_{k-2} x^k = f_0 x^0 + f_1 x - f_0 x + \sum_{k=2}^{\infty}(f_k - f_{k-1} - f_{k-2})x^k = 0 + x - 0 + \sum_{k=2}^{\infty} 0x^k = x$. Solving this for $G(x)$ yields $G(x) = x/(1 - x - x^2)$. Let α and β be defined as in Exercise 30. Then the denominator of $G(x)$ factors as $-(x + \beta)(x + \alpha)$. Expand $G(x)$ into partial fractions to get $G(x) = (1/\sqrt{5})\left(\beta/(x + \beta) - \alpha/(x + \alpha)\right)$. Since $1/\alpha = -\beta$ we can write the above as $G(x) = (1/\sqrt{5})\left(1/(1 - x\alpha) - 1/(1 - x\beta)\right)$. But these last two fractions represent the sums of geometric series, so we have $G(x) = (1/\sqrt{5})\left((1 + \alpha x + (\alpha x)^2 + \cdots) - (1 + \beta x + (\beta x)^2 + \cdots)\right) = (1/\sqrt{5})(0 + (\alpha - \beta)x + (\alpha^2 - \beta^2)x^2 + \cdots)$. Thus the coefficient on the nth power of x is given by $(1/\sqrt{5})(\alpha^n - \beta^n) = f_n$, for all $n \geq 0$.

1.4.43. We seek to solve the recurrence relation $L_n = L_{n-1} + L_{n-1}$ subject to the initial conditions $L_1 = 1$ and $L_2 = 3$. We solve the equation $r^2 - r - 1 = 0$ to discover the roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Then according to the theory in the paragraph above Exercise 31, $L_n = C_1\alpha^n + C_2\beta^n$. For $n = 1$ we have $L_1 = 1 = C_1\alpha + C_2\beta$. For $n = 2$ we have $3 = C_1\alpha^2 + C_2\beta^2$. Solving these two equations simultaneously yields $C_1 = 1$ and $C_2 = 1$. So the explicit formula is $L_n = \alpha^n + \beta^n$.

1.4.44. Let $H(x) = \sum_{k=0}^{\infty} L_k x^k$ be the generating function for the Lucas numbers. Note that we define $L_0 = 2$ so that $L_0 + L_1 = 3 = L_2$. Consider $H(x) - xH(x) - x^2H(x) = \sum_{k=0}^{\infty} L_k x^k - \sum_{k=0}^{\infty} L_k x^{k+1} - \sum_{k=0}^{\infty} L_k x^{k+2} = \sum_{k=0}^{\infty} L_k x^k - \sum_{k=1}^{\infty} L_{k-1} x^k - \sum_{k=2}^{\infty} L_{k-2} x^k = L_0 x^0 + L_1 x - L_0 x + \sum_{k=2}^{\infty}(L_k - L_{k-1} - L_{k-2})x^k = 2 + x - 2x + \sum_{k=2}^{\infty} 0x^k = 2 - x$. We solve for $H(x)$ and find its partial fraction expansion $H(x) = (2 - x)/(1 - x - x^2) = (1/(2\sqrt{5}))\left((5 + \sqrt{5})/(x + \alpha) - (5 - \sqrt{5})/(x + \beta)\right)$, where α and β are defined as in Exercise 30. We multiply the top and bottom of the first fraction by β and use the fact that $\alpha\beta = 1$, and similarly treat the second fraction to get the above equal to $1/(1 - \alpha x) + 1/(1 - \beta x)$. But these are the representations for the sums of geometric series, so we have $H(x) = (1 + \alpha x + (\alpha x)^2 + \cdots) + (1 + \beta x + (\beta x)^2 + \cdots) =$

$2 + (\alpha + \beta)x + (\alpha^2 + \beta^2)x^2 + \cdots$. Therefore, $L_n = \alpha^n + \beta^n$ the coefficient on the nth power of x.

1.4.45. First check that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. We proceed by induction. The basis steps are $(1/\sqrt{5})(\alpha - \beta) = (1/\sqrt{5})(\sqrt{5}) = 1 = f_1$ and $(1/\sqrt{5})(\alpha^2 - \beta^2) = (1/\sqrt{5})((1 + \alpha) - (1 + \beta)) = (1/\sqrt{5})(\alpha - \beta) = 1 = f_2$. Assume the identity is true for all positive integers up to n. Then $f_{n+1} = f_n + f_{n-1} = (1/\sqrt{5})(\alpha^n - \beta^n) + (1/\sqrt{5})(\alpha^{n-1} - \beta^{n-1}) = (1/\sqrt{5})(\alpha^{n-1}(\alpha + 1) - \beta^{n-1}(\beta + 1)) = (1/\sqrt{5})(\alpha^{n-1}(\alpha^2) - \beta^{n-1}(\beta^2)) = (1/\sqrt{5})(\alpha^{n+1} - \beta^{n+1})$, which completes the induction.

1.5. Divisibility

1.5.1. We find that $3 \mid 99$ since $99 = 3 \cdot 33$, $5 \mid 145$ since $145 = 5 \cdot 29$, $7 \mid 343$ since $343 = 7 \cdot 49$, and $888 \mid 0$ since $0 = 888 \cdot 0$.

1.5.2. We see that 1001 is divisible by 7, 11, and 13.

1.5.3. a. Yes, $0 = 7 \cdot 0$.

b. Yes, $707 = 7 \cdot 101$.

c. By the division algorithm, we have $1717 = 245 \cdot 7 + 2$. Since the remainder is nonzero, we know that $7 \nmid 1717$.

d. By the division algorithm, we have $123321 = 17617 \cdot 7 + 2$. Since the remainder is nonzero, we know that $7 \nmid 123321$.

e. By the division algorithm, we have $-285714 = -40817 \cdot 7 + 5$. Since the remainder is nonzero, we know that $7 \nmid -285714$.

f. By the division algorithm, we have $-430597 = -61514 \cdot 7 + 1$. Since the remainder is nonzero, we know that $7 \nmid -430597$.

1.5.4. a. Yes, $0 = 22 \cdot 0$.

b. By the division algorithm, we have $444 = 20 \cdot 22 + 4$. Since the remainder is nonzero, we know that $22 \nmid 444$.

c. Yes, $1716 = 22 \cdot 78$.

d. Yes, $192544 = 22 \cdot 8752$.

e. Yes, $-32516 = 22 \cdot -1478$.

f. By the division algorithm, we have $-195518 = -8888 \cdot 22 + 18$. Since the remainder is nonzero, we know that $22 \nmid -195518$.

1.5.5. a. We have $100 = 5 \cdot 17 + 15$, so the quotient is 5 and the remainder is 15.

b. We have $289 = 17 \cdot 17$, so the quotient is 17 and the remainder is 0.

c. We have $-44 = -3 \cdot 17 + 7$, so the quotient is -3 and the remainder is 7.

d. We have $-100 = -6 \cdot 17 + 2$, so the quotient is -6 and the remainder is 2.

1.5.6. Suppose that $a \mid b$ and $b \mid a$. Then there are integers k and l such that $b = ka$ and $a = lb$. This implies that $b = klb$, so that $kl = 1$. Hence either $k = l = 1$ or $k = l = -1$. It follows that either $a = b$ or $a = -b$.

1.5.7. By hypothesis we know $b = ra$ and $d = sc$, for some r and s. Thus $bd = rs(ac)$ and $ac \mid bd$.

1.5.8. We have $6 \mid 2 \cdot 3$, but 6 divides neither 2 nor 3.

1.5.9. If $a \mid b$, then $b = na$ and $bc = n(ca)$, i.e. $ac \mid bc$. Now, suppose $ac \mid bc$. Thus $bc = nac$ and, as $c \neq 0$, $b = na$, i.e., $a \mid b$.

1.5.10. Suppose $a \mid b$. Then $b = na$, and $b - a = na - a = (n-1)a$. Since a and b are positive $(n-1)a$ is positive and $a \leq b$.

1.5.11. By definition, $a \mid b$ if and only if $b = na$ for some integer n. Then raising both sides of this equation to the kth power yields $b^k = n^k a^k$ whence $a^k \mid b^k$.

1.5.12. Suppose that x and y are even. Then $x = 2k$ and $y = 2l$ where k and l are integers. Hence $x + y = 2k + 2l = 2(k + l)$ so that $x + y$ is also even. Suppose that x and y are odd. Then $x = 2k + 1$ and $y = 2l + 1$ where k and l are integers. Hence $x + y = (2k + 1) + (2l + 1) = 2k + 2l + 2 = 2(k + l + 1)$, so that $x + y$ is even. Suppose that x is even and y is odd. Then $x = 2k$ and $y = 2l + 1$ where k and l are integers. Hence $x + y = 2k + (2l + 1) = 2(k + l) + 1$. It follows $x + y$ is odd.

1.5.13. Let a and b be odd, and c even. Then $ab = (2x + 1)(2y + 1) = 4xy + 2x + 2y + 1 = 2(2xy + x + y) + 1$, so ab is odd. On the other hand, for any integer n, we have $cn = (2z)n = 2(zn)$ which is even.

1.5.14. By the division algorithm, there exist integers s, t such that $a = bs + t, 0 < t < b$ since $b \nmid a$. If t is odd, then we are done. If t is even, then $b - t$ is odd, $|t - b| < b$, and $a = b(s + 1) + (t - b)$.

1.5.15. By the division algorithm, $a = bq + r$, with $0 \leq r < b$. Thus $-a = -bq - r = -(q + 1)b + b - r$. If $0 \leq b - r < b$ then we are done. Otherwise $b - r = b$, or $r = 0$ and $-a = -qb + 0$.

1.5.16. We have $a = qb + r = (tc + s)b + r = tcb + bs + r$.

1.5.17. a. The division algorithm covers the case when b is positive. If b is negative, then we may apply the division algorithm to a and $|b|$ to get a quotient q and remainder r such that $a = q|b| + r$ and $0 \leq r < |b|$. But since b is negative, we have $a = q(-b) + r = (-q)b + r$, as desired.

b. We have $17 = -7(-2) + 3$. Here $r = 3$.

1.5.18. This is called the *l*east remainder algorithm. Suppose that a and b are positive integers. By the division algorithm there are integers s and t with $a = bs + t$ and $0 \leq t < b$. If $0 \leq t \leq \frac{b}{2}$ set $r = t, e = 1$, and $q = s$, so that $a = bq + er$ with $0 \leq r \leq \frac{b}{2}$. If $\frac{b}{2} < t < b$ set $r = b - t, e = -1$, and $q = s + 1$ so that $bq + er = b(s + 1) + (t - b) = bs + t = a$ and $0 < r = t - b < \frac{b}{2}$. Hence there are integers q, e and r such that $a = bq + er$ where $e = \pm 1$ and $0 \leq r \leq \frac{b}{2}$.

1.5.19. By the division algorithm, let $m = qn + r$, with $0 \leq r < n - 1$ and $q = [m/n]$. Then $[(m + 1)/n] = [(qn + r + 1)/n] = [q + (r + 1)/n] = q + [(r + 1)/n]$ as in Example 1.31. If $r = 0, 1, 2, \ldots, n - 2$, then $m \neq kn - 1$ for any integer k and $1/n \leq (r + 1)/n < 1$ and so $[(r + 1)/n] = 0$. In this case, we have $[(m + 1)/n] = q + 0 = [m/n]$. On the other hand, if $r = n - 1$, then $m = qn + n - 1 = n(q + 1) - 1 = nk - 1$, and $[(r + 1)/n] = 1$. In this case, we have $[(m + 1)/n] = q + 1 = [m/n] + 1$.

1.5.20. Suppose $n = 2k$. Then $n - 2[n/2] = 2k - 2[2k/2] = 0$. On the other hand, suppose $n - 2[n/2] = 0$. Then $n/2 = [n/2]$ and $n/2$ is an integer. In other words, n is even.

1.5.21. The positive integers divisible by the positive integer d are those integers of the form kd where k is a positive integer. The number of these that are less than x is the number of positive integers k with $kd \leq x$, or equivalently with $k \leq x/d$. There are $[x/d]$ such integers.

1.5.22. There are $[1000/5] = 200$ positive integers not exceeding 1000 that are divisible by 5, $[1000/25] = 40$ such integers that are divisible by 25, $[1000/125] = 8$ such integers that are divisible by 125, and $[1000/625] = 1$ such integer that is divisible by 625.

1.5.23. There are $[1000/7] - [100/7] = 142 - 14 = 128$ integers between 100 and 1000 that are divisible by 7. There are $[1000/49] - [100/49] = 20 - 2 = 18$ integers between 100 and 1000 that are divisible by 49.

1.5.24. The number of integers not exceeding 1000 that are not divisible by either 3 or 5 equals $1000 - ([1000/3] + [1000/5]) + [1000/15] = 533$.

1.5.25. Using the Principle of Inclusion-Exclusion, the answer is $1000 - ([1000/3] + [1000/5] + [1000/7]) + ([1000/15] + [1000/21] + [1000/35]) - ([1000/105) = 1000 - (333 + 200 + 142) + (66 + 47 + 28) - 9 = 462$.

1.5.26. For an integer to be divisible by 3, but not by 4, an integer must be divisible by 3, but not by 12. There are $[1000/3] = 333$ positive integers not exceeding 1000 that are divisible by 3. Of these $[1000/12] = 82$ are divisible by 12 (since anything that is divisible by 12 is automatically divisible by 3). Hence there are $333 - 83 = 250$ possible integers not exceeding 1000 that are divisible by 3, but not by 4.

1.5.27. Let w be the weight of a letter in ounces. Note that the function $-[-x]$ rounds x up to the least integer less than or equal to x. (That is, it's the equivalent of the ceiling function.) The cost of mailing a letter weighing w ounces is, then, 33 cents plus 22 cents for each ounce or part thereof more than 1, so we need to round $w - 1$ up to the next integer. So the cost is $c(w) = 33 - [1 - w]23$ cents. Suppose that $33 - [1 - w]22 = 145$. then $-[1 - w]22 = 145 - 33 = 112$ which is not a multiple of 22, so no letter can cost \$1.45. Suppose that $33 - [1 - w]22 = 231$. then $-[1 - w]22 = 231 - 33 = 198 = 9 \cdot 22$. Then $[1 - w] = -9$, so $-9 \leq 1 - w < -8$, or $9 < w \leq 10$. So a letter weight at least 9 ounces but less than 10 ounces would cost \$2.31.

1.5.28. Note that $a^3 - a = a(a^2 - 1) = (a - 1)a(a + 1)$. By the division algorithm $a = 3k, a = 3k + 1$, or $a = 3k + 2$, where k is an integer. If $a = 3k$, 3 divides a, if $a = 3k + 1$ then $a - 1 = 3k$, so that 3 divides $a - 1$, and if $a = 3k + 2$, then $a + 1 = 3k + 3 = 3(k + 1)$, so that 3 divides $a + 1$. Hence 3 divides $(a - 1)a(a + 1) = a^3 - a$ for every nonnegative integer a. (Note: This can also be proved using mathematical induction.)

1.5.29. Multiplying two integers of this form gives us $(4n + 1)(4m + 1) = 16mn + 4m + 4n + 1 = 4(4mn + m + n) + 1$. Similarly, $(4n + 3)(4m + 3) = 16mn + 12m + 12n + 9 = 4(4mn + 3m + 3n + 2) + 1$.

1.5.30. Suppose that n is odd. Then $n = 2t + 1$ where t is an integer. It follows that $n^2 = (2t + 1)^2 = 4t^2 + 4t + 1 = 4t(t + 1) + 1$. Now if t is even, then $t = 2u$ where u is an integer. Hence $n^2 = 8u(2u + 1) + 1 = 8k + 1$, where $k = u(2u + 1)$ is an integer. If t is odd, then $t = 2u + 1$ where u is an integer. Hence $n^2 = (8u + 4)(2u + 2) + 1 = 8(2u + 1)(u + 1) + 1 = 8k + 1$, where $k = (2u + 1)(u + 1)$.

1.5.31. Every odd integer may be written in the form $4k + 1$ or $4k + 3$. Observe that $(4k + 1)^4 = 16^2k^4 + 4(4k)^3 + 6(4k)^2 + 4(4k) + 1 = 16(16k^4 + 16k^3 + 6k^2 + k) + 1$. Proceeding further, $(4k + 3)^4 = (4k)^4 + 12(4k)^3 + 54(4k)^2 + 108(4k) + 3^4 = 16(16k^4 + 48k^3 + 54k^2 + 27k + 5) + 1$.

1.5.32. The product of the integers $6k + 5$ and $6l + 5$ is $(6k + 5)(6l + 5) = 36kl + 30(k + l) + 25 = 6[6kl + 5(k + l) + 4] + 1 = 6N + 1$ where $N = 6kl + 5(k + l) + 4$. Hence this product is of the form $6N + 1$.

1.5.33. Of any consecutive three integers, one is a multiple of three. Also, at least one is even. Therefore, the product is a multiple of $2 \cdot 3 = 6$.

1.5.34. The basis step is completed by noting that $1^5 - 1 = 0$ is divisible by 5. For the inductive hypothesis, assume that $n^5 - n$ is divisible by 5. This implies that there is an integer k such that $n^5 - n = 5k$. It follows that $(n + 1)^5 - (n + 1) = (n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) - (n + 1) = (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n) = 5k + 5l = 5(k + l)$. Hence $(n + 1)^5 - (n + 1)$ is also divisible by 5.

1.5.35. For the basis step note that $0^3 + 1^3 + 2^3 = 9$ is a multiple of 9. Suppose that $n^3 + (n + 1)^3 + (n + 2)^3 = 9k$ for some integer k. Then $(n + 1)^3 + (n + 2)^3 + (n + 3)^3 = n^3 + (n + 1)^3 + (n + 2)^3 + (n + 3)^3 - n^3 =$

$9k + n^3 + 9n^2 + 27n + 27 - n^3 = 9k + 9n^2 + 27n + 27 = 9(k + n^2 + 3n + 3)$ which is a multiple of 9.

1.5.36. We prove this by mathematical induction. We will prove that f_{3n-2} is odd, f_{3n-1} is odd, and f_{3n} is even whenever n is a positive integer. For $n = 1$ we see that $f_{3\cdot1-2} = f_1 = 1$ is odd, $f_{3\cdot1-1} = f_2 = 1$ is odd, and $f_{3\cdot1} = f_3 = 2$ is even. Now assume that f_{3n-2} is odd, f_{3n-1} is odd, and f_{3n} is even where n is a positive integer. Then $f_{3(n+1)-2} = f_{3n+1} = f_{3n} + f_{3n-1}$ is odd since f_{3n} is even and f_{3n-1} is odd, $f_{3(n+1)-1} = f_{3n+2} = f_{3n+1} + f_{3n}$ is odd since f_{3n+1} is odd and f_{3n} is even, and $f_{3(n+1)} = f_{3n+3} = f_{3n+2} + f_{3n+1}$ is even since f_{3n+2} and f_{3n+1} are odd. This completes the proof.

1.5.37. We proceed by mathematical induction. The basis step is clear. Assume that only f_{4n}'s are divisible by 3 for $f_i, i \leq 4k$. Then, as $f_{4k+1} = f_{4k} + f_{4k-1}, 3 \mid f_{4k}$ and $3 \mid f_{4k+1}$ gives us the contradiction $3 \mid f_{4k-1}$. Thus $3 \nmid f_{4k+1}$. Continuing on, if $3 \mid f_{4k}$ and $3 \mid f_{4k+2}$, then $3 \mid f_{4k+1}$, which contradicts the statement just proved. If $3 \mid f_{4k}$ and $3 \mid f_{4k+3}$, then since $f_{4k+3} = 2f_{4k+1} + f_{4k}$, we again have a contradiction. But, as $f_{4k+4} = 3f_{4k+1} + 2f_{4k}$, and $3 \mid f_{4k}$ and $3 \mid 3 \cdot f_{4k+1}$, we see that $3 \mid f_{4k+4}$.

1.5.38. We proceed by induction. The basis step is clear. Suppose f_n is divisible by 4. By Exercise 34, $f_{n+1}, f_{n+2}, f_{n+4}, f_{n+5}$ are all odd. Suppose f_{n+3} is divisible by 4. Now, $f_{n+3} = 2f_{n+1} + f_n$. Since f_n and f_{n+3} are divisible by 4, so must by $2f_{n+1}$. This is a contradiction. On the other hand, $f_{n+6} = 8f_{n+1} + f_n$. Since both terms are multiples of 4, so is f_{n+6}.

1.5.39. First note that for $n > 5$, $5f_{n-4} + 3f_{n-5} = 2f_{n-4} + 3(f_{n-4} + f_{n-5}) = 2f_{n-4} + 3f_{n-3} = 2(f_{n-4} + f_{n-3}) + f_{n-3} = 2f_{n-2} + f_{n-3} = f_{n-2} + f_{n-2} + f_{n-3} = f_{n-2} + f_{n-1} = f_n$, which proves the first identity. Now note that $f_5 = 5$ is divisible by 5. Suppose that f_{5n} is divisible by 5. From the identity above $f_{5n+5} = 5f_{5n+5-4} + 3f_{5n+5-5} = 5f_{5n+1} + 3f_{5n}$, which is divisible by 5 since $5f_{5n+1}$ is a multiple of 5 and, by the induction hypothesis, so is f_{5n}. This completes the induction.

1.5.40. We use mathematical induction on the integer m. For $m = 1$ we have $f_{n+1} = f_{n-1}f_1 + f_nf_2 = f_{n-1} + f_n$ which is true from the definition of the Fibonacci numbers. For $m = 2$ we have $f_{n+2} = f_{n-1}f_2 + f_nf_3 = f_{n-1} + 2f_n = f_{n-1} + f_n + f_n = f_{n+1} + f_n$ which is true from the definition of the Fibonacci numbers. This finishes the basis step of the proof. Now assume that $f_{n+m} = f_mf_{n+1} + f_{m-1}f_n$ holds for all integers m with $m < k$. We will show that it must also hold for $m = k$. We have $f_{n+k-2} = f_{k-2}f_{n+1} + f_{k-3}f_n$ and $f_{n+k-1} = f_{k-1}f_{n+1} + f_{k-2}f_n$. Adding these two equations gives $f_{n+k-2} - f_{n+k-1} = f_{n+1}(f_{k-2} + f_{k-1}) + f_n(f_{k-3} + f_{k-2})$. Hence $f_{n+k} = f_{n+1}f_k + f_nf_{k-1}$. Hence the identity is also true for $m = k$. We now show that $f_m \mid f_n$ if $m \mid n$. Since $m \mid n$ we have $n = km$. We prove this using mathematical induction on k. For $k = 1$ we have $n = m$ so $f_m \mid f_n$ since $f_m = f_n$. Now assume f_{mk} is divisible by f_m. Note that $f_{m(k+1)} = f_{mk+m} = f_{mk-1}f_m + f_{mk}f_{m+1}$. The first product is divisible by f_m since f_m is a factor in this term and the second product is divisible by f_m by the inductive hypothesis. Hence $f_m \mid f_{m(k+1)}$. This finishes the inductive proof.

1.5.41. Iterating the transformation T starting with 39 we find that $T(39) = 59$; $T(59) = 89$; $T(89) = 134$; $T(134) = 67$; $T(67) = 101$; $T(101) = 152$; $T(152) = 76$; $T(76) = 38$; $T(38) = 19$; $T(19) = 29$; $T(29) = 44$; $T(44) = 22$; $T(22) = 11$; $T(11) = 17$; $T(17) = 26$; $T(26) = 13$; $T(13) = 20$; $T(20) = 10$; $T(10) = 5$; $T(5) = 8$; $T(8) = 4$; $T(4) = 2$; $T(2) = 1$.

1.5.42. If $3n$ is odd, then so is n. So, $T(n) = (3n + 1)/2 = 2^{2k}/2 = 2^{2k-1}$. Because $T(n)$ is a power of 2, the exponent will decrease down to one with repeated applications of T.

1.5.43. We prove this using the second principle of mathematical induction. Since $T(2) = 1$, the Collatz conjecture is true for $n = 2$. Now assume that the conjecture holds for all integers less that n. By assumption there is an integer k such that k iterations of the transformation T, starting at n, produces an integer m less than n. By the inductive hypothesis there is an integer l such that iterating T l times starting at m produces the integer 1. Hence iterating T $k + l$ times starting with n leads to 1. This finishes the proof.

1.5.44. Suppose $n = 2k$ for some k. Then $T(n) = k < 2k = n$. Suppose that $n = 4k + 1$ for some k. Then $T(T(n)) = T(6k + 2) = 3k + 1 < 4k + 1 = n$. Now suppose that $n = 8k + 3$, where k is an even number. $T(T(T(T(n)))) = 9k/2 + 1 < 8k + 3 = n$. This leaves 17 numbers to be considered,

$7, 11, 15, 23, 27, 31, 39, 43, 47, 55, 59, 63, 71, 75, 79, 87, 91, 95$. These can be methodically tested. The worst of them is 27, which requires over 70 applications of T to reach 1.

1.5.45. We first show that $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer. By the binomial theorem it follows that $(2+\sqrt{3})^n + (2-\sqrt{3})^n = \sum_{j=0}^{n} \binom{n}{j} 2^j \sqrt{3}^{n-j} + \sum_{j=0}^{n} \binom{n}{j} 2^j (-1)^{n-j} \sqrt{3}^{n-j} = 2(2^n + \binom{n}{2} 3 \cdot 2^{n-2} + \binom{n}{4} 3^2 \cdot 2^{n-4} + \cdots) = 2l$ where l is an integer. Next, note that $(2-\sqrt{3})^n < 1$. We see that $[(2+\sqrt{3})^2] = (2+\sqrt{3})^n + (2-\sqrt{3})^n - 1$. It follows that $[(2+\sqrt{3})^2]$ is odd.

CHAPTER 2

Integer Representations and Operations

2.1. Representations of Integers

2.1.1. We have $1999 = 7 \cdot 285 + 4, 285 = 7 \cdot 40 + 5$, and $40 = 7 \cdot 5 + 5$, and $5 = 7 \cdot 0 + 5$. The sequence of remainders gives the base 7 digits. Hence $(1999)_{10} = (5554)_7$. We have $(6105)_7 = 6 \cdot 7^3 + 1 \cdot 7^2 + 0 \cdot 7 + 5 = (2112)_{10}$.

2.1.2. We have $89156 = 8 \cdot 11144 + 4$, $11144 = 8 \cdot 1393 + 0$, $1393 = 8 \cdot 174 + 1$, $174 = 8 \cdot 21 + 6$, $21 = 8 \cdot 2 + 5$, and $2 = 8 \cdot 0 + 2$. The sequence of remainders gives us $(89156)_{10} = (256104)_8$. We have $(706113)_8 = 7 \cdot 8^5 + 6 \cdot 8^3 + 8^2 + 8 + 3 = (232523)_{10}$.

2.1.3. We have $(10101111)_2 = (175)_{10}$, and $(999)_{10} = (1111100111)_2$.

2.1.4. We have $(101001000)_2 = 2^3 + 2^6 + 2^8 = (328)_{10}$.

2.1.5. We group together blocks of four binary digits starting from the right. We have $(0101)_2 = (5)_{16}$, $(1111)_2 = (F)_{16}, (1000)_2 = (8)_{16}$. Hence $(100011110101)_2 = (8F5)_{16}$. Likewise, $(1110)_2 = (E)_{16}$, $(0100)_2 = (4)_{16}$, and $(0111)_2 = (7)_{16}$. Therefore, $(11101001110)_2 = (74E)_{16}$.

2.1.6. Each hexadecimal digit corresponds to a block of four binary digits. Translating each hexadecimal digit into the corresponding block of four binary digits gives $(ABCDEF)_{16} = (101010111100110111101111)_2$, $(DEFACED)_{16} = (11011110111111010110011101101)_2$, and $(9A0B)_{16} = (1001101000001011)_2$.

2.1.7. This is because we are using the blocks of three digits as one "digit," which has 1000 possible values.

2.1.8. The proof of Theorem 1.10 goes through exactly, with the inequality $0 \leq a_i \leq b - 1$ replaced by $0 \leq a_i <\mid b \mid$ at each step.

2.1.9. We find that $(101001)_{-2} = 1(-2)^5 + 0(-2)^4 + 1 \cdot (-2)^3 + 0(-2)^2 + 0(-2)^1 + 1(-2)^0 = -39$ and $(12012)_{-3} = 1(-3)^4 + 2(-3)^3 + 0(-3)^2 + 1(-3)^1 + 2(-3)^0 = 26$.

2.1.10. $-7 = (-2) \cdot 4 + 1, 4 = (-2) \cdot (-2) + 0, -2 = (-2) \cdot 1 + 0, 1 = (-2) \cdot 0 + 1$, so $(-7)_{10} = (1001)_{-2}$. $-17 = (-2) \cdot 9 + 1, 9 = (-2) \cdot -4 + 1, -4 = (-2) \cdot 2 + 0, 2 = (-2) \cdot -1 + 0, -1 = (-2) \cdot 1 + 1, 1 = (-2) \cdot 0 + 1$, so $(-17)_{10} = (110011)_{-2}$. $61 = (-2) \cdot -30 + 1, -30 = (-2) \cdot 15 + 0, 15 = (-2) \cdot -7 + 1 - 7 = (-2) \cdot 4 + 1, 4 = (-2) \cdot 2 + 0, 2 = (-2) \cdot 1 + 1, 1 = (-2) \cdot 0 + 1$, so $(61)_{10} = (1001101)_{-2}$.

2.1.11. If m is any integer weight less than 2^k, then by Theorem 1.10, m has a base two expansion $m = a_{k-1}2^{k-1} + a_{k-2}2^{k-2} + \cdots + a_1 2^1 + a_0 2^0$, where each a_i is 0 or 1. The 2^i weight is used if and only if $a_i = 1$.

2.1.12. To show existence, mimic the proof of Theorem 2.1 using Exercise 18 of Section 1.5. To show uniqueness, assume that a given number has two representations and look at the difference of these representations. Observe that a number is equal to 0 if and only if e_j is 0 for all j. The result follows.

2.1.13. Let w be the weight to be measured. By Exercise 10, w has a unique balanced ternary expansion. Place the object in pan 1. If $e_i = 1$ then place a weight of 3^i into pan 2. If $e_i = -1$ then place a weight of 3^i in pan 1. If $e_i = 0$ then do not use the weight of 3^i. Now the pans will be balanced.

2.1.14. Each base 9 digit corresponds to two base 3 digits and vice versa. The correspondence is $(0)_9 = (00)_3, (1)_9 = (01)_3, (2)_9 = (02)_3, (3)_9 = (10)_3, (4)_9 = (11)_3, (5)_9 = (12)_3, (6)_9 = (20)_3, (7)_9 = (21)_3, (8)_9 = (22)_3$. To convert a base 9 expansion to a base 3 expansion we simply replace each base 9 digit with the corresponding two base 3 digits. To convert a base 3 expansion to a base 9 expansion, we start at the right of the expansion and replace blocks of two base 3 digits to the corresponding base 9 digit, putting an initial 0 in the last block from the left if it consists only of 1 digit.

2.1.15. To convert a number from base r to base r^n, take the number in blocks of size n. To go the other way, convert each digit of a base r^n number to base r, and concatenate the results.

2.1.16. If $n = (a_k a_{k-1} \dots a_1 a_0)_b$, then $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 + a_0$. Now it follows directly that $n = (a_k b^{k-j} + a_{k-1} b^{k-j-1} + \cdots + a_j) b^j + a_{j-1} b^{j-1} + \cdots + a_0$.

2.1.17. Multiplying n by b^m gives $b^m n = b^m (a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0) = (a_k b^{k+m} + a_{k-1} b^{k+m-1} + \cdots + a_1 b^{m+1} + a_0 b^m + 0 \cdot b^{m-1} + \cdots + 0) = (a_k a_{k-1} \dots a_1 a_0 00 \dots 00)_b$, where we have placed m zeroes at the end of the base b expansion of n.

2.1.18. a. $22 = (10110)_2$, and since $22 > 0$, the one's complement representation is 22 is 010110.

b. $31 = (11111)_2$, and since $31 > 0$, the one's complement representation of 31 is 011111.

c. $7 = (00111)_2$, and since $-7 < 0$, the one's complement of -7 is a 1 followed by the complement of the binary representation of 7, to wit, 111000.

d. $19 = (10011)_2$, and since $-19 < 0$, the one's complement of -19 is a 1 followed by the complement of the binary representation of 19, to wit, 101100.

2.1.19. a. The lead digit is a one, so the number is negative. Its absolute value has a binary representation of the complement of 1001, i.e. 0110. Thus 11001 is the one's complement representation of -6.

b. 01101 is the one's complement representation of 13.

c. 10001 is the one's complement representation of -14.

d. 11111 is a one's complement representation of 0. Note that 00000 also represents 0.

2.1.20. Take the complement of each and every digit.

2.1.21. If m is positive, then $a_{n-1} = 0$ and $a_{n-2} a_{n-3} \dots a_0$ is the binary expansion of m. Hence, $m = \sum_{i=0}^{n-2} a_i 2^i$ as desired. If m is negative, then the one's complement expansion for m has its leading bit equal to 1. If we view the bit string $a_{n-2} a_{n-3} \dots a_0$ as a a binary number, then it represents $(2^{n-1} - 1) - (-m)$, since finding the one's complement is equivalent to subtracting the binary number from $111 \cdots 1$. That is $(2^{n-1} - 1) - (-m) = \sum_{i=0}^{n-2} a_i 2^i$. Solving for m gives us the desired identity.

2.1.22. a. $22 = (10110)_2$. Since 22 is positive, we append a leading 0 to this expansion to obtain 010110 as the two's complement representation of 22.

b. $31 = (11111)_2$. Since 31 is positive, we append a leading 0 to this expansion to obtain 011111 as the two's complement representation of 31.

c. Since -7 is negative, we consider the binary expansion of $2^5 - 7 = 25 = (11001)_2$, and then append a leading 1 to obtain 111001 as the two's complement representation of -7.

d. Since -19 is negative, we consider the binary expansion of $2^5 - 19 = 13 = (01101)_2$, and then append a leading 1 to obtain 101101 as the two's complement representation of -19.

2.1.23. a. Since the first digit is a 1, we know that the integer is negative and that $(1001)_2 = 9$ is the binary expansion of $2^4 - |x|$. So $|x| = 16 - 9 = 7$, and thus $x = -7$.

b. Since the first digit is a 0, we know that the integer is positive and hence $x = (1101)_2 = 13$.

c. Since the first digit is a 1, we know that the integer is negative and that $(0001)_2 = 1$ is the binary expansion of $2^4 - |x|$. So $|x| = 16 - 1 = 15$, and thus $x = -15$.

d. Since the first digit is a 1, we know that the integer is negative and that $(1111)_2 = 15$ is the binary expansion of $2^4 - |x|$. So $|x| = 16 - 15 = 1$, and thus $x = -1$.

2.1.24. If m is positive, then $a_{n-1} = 0$ and $\sum_{i=0}^{n-2} a_i 2^i$ is the binary expansion of m. Hence $m = -a_{n-1}2^{n-1} + \sum_{i=0}^{n-2} a_i 2^i$. If m is negative, then $a_{n-1} = 1$ and $\sum_{i=0}^{n-2} a_i 2^i$ is the binary expansion of $2^{n-1} + m$. Hence, $m = -a_{n-1}2^{n-1} + \sum_{i=0}^{n-2} a_i 2^i$.

2.1.25. If each of the digits in the two's complement representation for m is complemented and then 1 is added to the resulting binary number, the result is the two's complement representation for $-m$. To see this note that $m + (-m) + (-1) =$ (binary expansion of m) + (2^{n-1}+binary expansion for $2^{n-1} - m$) + $(-1) = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1 = (111\ldots1)_2$. Therefore the two's complement representation of $-m-1$ is the complement of m.

2.1.26. If m is positive, the representations are identical. If m is negative, then we compare the solutions to Exercises 25 and 20 to see that we need only add 1 to the one's complement representation of m to obtain the two's complement.

2.1.27. Since 4 bits are required for every decimal digit, $4n$ bits are required to store the number in this manner.

2.1.28. We see that 3! is the largest factorial less than 14. We have $14 = 2 \cdot 3! + 2$. Next, we find that $2 = 1 \cdot 2! + 0$. It follows that $14 = 2 \cdot 3! + 1 \cdot 2! + 0 \cdot 1! = (210)_!$. We see that 4! is the largest factorial less than 56. We have $56 = 2 \cdot 4! + 8$. Next, we find that $8 = 1 \cdot 3! + 2$, and $2 = 1 \cdot 2! + 0$. It follows that $56 = 2 \cdot 4! + 1 \cdot 3! + 1 \cdot 2! + 0 \cdot 1! = (2110)_!$. We see that 5! is the largest factorial less than 384. We have $384 = 3 \cdot 5! + 24$. Next we see that $384 = 1 \cdot 4!$. Hence $384 = 3 \cdot 5! + 1 \cdot 4! + 0 \cdot 3! + 0 \cdot 2! + 0 \cdot 1! = (31000)_!$.

2.1.29. We first show that every positive integer has a Cantor expansion. To find a Cantor expansion of the positive integer n, let m be the unique positive integer such that $m! \leq n < (n+1)!$. By the division algorithm there is an integer a_m such that $n = m! \cdot a_m + r_m$ where $0 \leq a_m \leq m$ and $0 \leq r_m < m!$. We iterate, finding that $r_m = (m-1)! \cdot a_{m-1} + r_{m-1}$ where $0 \leq a_{m-1} \leq m-1$ and $0 \leq r_{m-1} < (m-1)!$. We iterate $m-2$ more times, where we have $r_i = (i-1)! \cdot a_{i-1} + r_{i-1}$ where $0 \leq a_{i-1} \leq i-1$ and $0 \leq r_{i-1} < (i-1)!$ for $i = m+1, m, m-1, \ldots, 2$ with $r_{m+1} = n$. At the last stage we have $r_2 = 1! \cdot a_1 + 0$ where $r_2 = 0$ or 1 and $r_2 = a_1$.

Now that we have shown that every integer has a Cantor expansion, we must show that this expansion is unique. So suppose that n has two different Cantor expansions $n = a_m m! + a_{m-1}(m-1)! + \cdots + a_2 2! + a_1 1! = b_m m! + b_{m-1}(m-1)! + \cdots + b_2 2! + b_1 1!$, where a_j and b_j are integers, and $0 \leq a_j \leq j$ and $0 \leq b_j \leq j$ for $j = 1, 2, \ldots, m$. Suppose that k is the largest integer such that $a_k \neq b_k$, and without loss of generality, assume $a_k > b_k$, which implies that $a_k \geq b_k + 1$. Then $a_k k! + a_{k-1}(k-1)! + \cdots + a_1 1! = b_k k! + b_{k-1}(k-1)! + \cdots + b_1 1!$. Using the identity $\sum_{j=1}^{k} j \cdot j! = (k+1)! - 1$, proved in Exercise 16 of Section 1.3, we see that $b_k k! + b_{k-1}(k-1)! + \cdots + b_1 1! \leq b_k k! + (k-1) \cdot (k-1)! + \cdots + 1 \cdot 1! \leq b_k k! + k! - 1 = (b_k + 1)k! - 1 < a_k k!$. This is a contradiction, so the expansion is unique.

2.1.30. If the first player takes 2 matches then they must be from the same stack. The second player may then win by taking the other two. If player one only takes one match, then player 2 can take one match from the other stack, which is a winning position as discussed in the description of nim.

2.1.31. Call a position *good* if the number of ones in each column is even, and *bad* otherwise. Since a player can only affect one row, he or she must affect some column sums. Thus any move from a good position

produces a bad position. To find a move from a bad position to a good one, construct a binary number by putting a 1 in the place of each column with odd sum, and a 0 in the place of each column with even sum. Subtracting this number of matches from the largest pile will produce a good position.

2.1.32. Let (w, x, y, z) represent the number $wxyz$, where w, x, y, z are single digits. Let a, b, c, d be the digits of a fixed point n of T (a number such that $T(n) = n$). We first show that all four digits of n are different. Suppose, to the contrary, that $b = c$. Then $(a, b, b, d) - (d, b, b, a) = (a - 1 - d, 9, 9, 10 + d - a)$. Since n is a fixed point, we can now see that it must have two 9s, and as $b = c$, in fact it must have three 9s. So $a = b = c = 9$. From this, since $d \neq 10 + d - a = d + 1$, we know that $d = 8 - d$, and $d = 4$. But $(9, 9, 9, 4) - (4, 9, 9, 9) = (4, 9, 9, 5)$, so there is not a fixed point with $b = c$. Therefore, $b \neq c$. Suppose, now, that $a > b > c = d$. Then $(a, b, c, c) - (c, c, b, a) = (a - c, b - 1 - c, c + 9 - b, 10 + c - a)$. As, $b - c - 1 < a - c$, $a - c > b - c - 1 < b$, and $c + 9 - b \geq 10 + c - a$, we know that a and b are $a - c$ and $c + 9 - b$, perhaps not respectively. If $a = a - c$, then $c = 0$. But then $b = 9 - b$, which is impossible. If $a = c + 9 - b$, then $b = a - c$ and $a = c + 9 - a - c$, from which it follows that 9 is even. So we conclude that $c \neq d$. Suppose that $a = b > c > d$. Then $(a, a, c, d) - (d, c, a, a) = (a - d, a - c - 1, c - a + 9, 10 + d - a)$. From the inequalities $a \geq a - d \geq a - c > a - c - 1$ and $c - a + 9 \geq d + 1 - a + 9 = 10 + d - a$ we may conclude that c and d are $a - c - 1$ and $10 + d - a$, perhaps not respectively. If $c = a - c - 1$, then we see that a must be odd. But in this case $d = 10 + d - a$ also, which tells us that a must be even. If, on the other hand, $c = 10 + d - a$ and $d = a - c - 1$, then $c = 10 + a - c - 1 - a = 9 - c$, which is impossible. We conclude here that $a \neq b$. Suppose that $a = b > c = d$. Then $(a, a, c, c) - (c, c, a, a) = (a - c, a - c - 1, c - a + 9, 10 + c - a)$. Since $a - c > a - c - 1, a - c = a$ and $c = 0$. Now $a - c - 1 = c$, so $a = 1$. But $(1, 1, 0, 0) - (0, 0, 1, 1) = (1, 0, 8, 9)$, so clearly this does not give a fixed point. So we now know that $a > b > c > d$. Now, $(a, b, c, d) - (d, c, b, a) = (a - d, -1 + b - c, 9 - b + c, 10 - a + d)$. Note that $a - d > -1 + b - c$, and $9 + c - b > 10 + d - a > d$. So, d is either $-1 + b - c$ or $10 + d - a$. If $d = 10 + d - a$, then $a = 10$, which is not a single digit. Thus, $d = -1 + b - c$. Now, we see that c is either $a - d$ or $10 + d - a$. If $c = a - d$, then $d = -1 + b - c = -1 + b - a + d$. From this, we arrive at $a + 1 = b$, a contradiction. Thus $c = 10 + d - 1$. If $a = a - d$, then $d = 0$. Proceeding along with thought, $b = c + 1 = 9 + c - b$ now, which tells us that $b = 8, c = 7$ and $a = 4$. This is a contradiction. Thus $a = 9 + c - b$ and $b = a - d$. We now have four equations in four unknowns. Solving this system, we find that $a = 7, b = 6, c = 4$, and $d = 1$. This gives a fixed point, namely 6174.

2.1.33. a. First show that the result of the operation must yield a multiple of 9. Then, it suffices to check only multiples of 9 with decreasing digits. There are only 79 of these. If we perform the operation on each of these 79 numbers and reorder the digits, we will have one of the following 23 numbers: 7551, 9954, 5553, 9990, 9981, 8820, 9810, 9620, 8532, 8550, 9720, 9972, 7731, 6543, 8730, 8640, 8721, 7443, 9963, 7632, 6552, 6642, or 6174. It will suffice to check only 9810, 7551, 9990, 8550, 9720, 8640, and 7632.

b. From the solution in part (a), construct a tree from the last seven numbers. The longest branch is six steps. Every number will reach the tree in two steps. The maximum is given by 8500 (for instance) which takes eight steps.

2.1.34. Let $a_0 = (a, b, c, d)$ be a base 5 fixed point of T_5. Then $T_5(a_0) = (a, b, c, d) - (d, c, b, a) = (a - d, b - 1 - c, c + 4 - b, d + 5 - a)$, for all a_0, with $b \neq c$. Note that the center two digits of $T(a_0)$ sum to $(3)_5$, and the outer two to $(10)_5$. Since the order of the digits is irrelevant, we need only examine four cases: $(1034)_5, (1124)_5, (2033)_5$, and $(2124)_5$. By checking these cases one at a time, we find that they all go to $(3032)_5$, which is a fixed point of T_5. Similarly, if $b \neq c$, then $T_5(a_0)$ is one of $(0444)_5, (1443)_5, (2442)_5, (3441)_5$, and $(4440)_5$. By symmetry, we need only check $(0444)_5, (1443)_5$, and $(2442)_5$. All of these do, in fact, go to $(3032)_5$, the Kaprekar's constant for the base 5.

2.1.35. Consider $a_0 = (3043)_6$. Then $T_6((3043)_6) = (3552)_6$, $T_6((3552)_6) = (3133)_6$, $T_6((3133)_6) = (1554)_6$, $T_6((1554)_6) = (4042)_6$, $T_6((4042)_6) = (4132)_6$, and $T_6((4132)_6) = (3043)_6 = a_0$. So T_6 repeats with period 6. Therefore, it never goes to a Kaprekar's constant for the base 6. Hence, there is no Kaprekar's constant for the base 6.

2.1.36. Let $(abc)_{10}$, be the digits of an integer with $a \leq b \leq c$, and a, b, and c not all the same. Then $(abc)_{10} - (cba)_{10} = ((a - c)(9)(10 + c - a))_{10}$, so the form of the next integer is $9bc$. Then $(9bc)_{10} - (cb9)_{10} = ((9 -$

$c-1)(9)(1+c))_{10}$. After re-ordering, we see that after two iterations we must have one of the numbers $891, 792, 693$, or 594. Then $T(981) = 792$, $T(792) = 693$, $T(693) = 594$, and $T(594) = 495$, up to order of the digits. Therefore 495 is a Kaprekar's constant for three-digit base 10 integers.

2.2. Computer Operations with Integers

2.2.1. To add $(101111011)_2$ and $(1100111011)_2$ we first add 1 and 1, obtaining the rightmost bit 0 and the carry 1. Then we add the bits 1 and 1 and the carry 1, obtaining the second bit from the right in the sum 1 and the carry 1. Then we add the bits 0 and 0, and the carry 1, obtaining the third bit from the right in the sum, 1. Then we add the bits 1 and 1, obtaining the fourth bit from the right in the sum, 0, and the carry 1. Then we add the bits 1 and 1 and the carry 1, obtaining the fifth bit from the right in the sum 1, and the carry 1. Then we add the bits 1 and 1 and the carry obtaining the sixth bit from the right in the sum 1, and the carry 1. Then we add the bits 1 and 0 and the carry 1 obtaining the seventh bit from the right in the sum, 0, and the carry, 1. Then we add the bits 0 and 0 and the carry 1, obtaining the eighth bit from the right in the sum 1. Then we add the bits 1 and 1, obtaining the ninth bit from the right, 0, and the carry 1. Then we add the (leading) bit 0 and the bit 1 and the carry 1, obtaining the tenth bit in the sum, 0, and the carry, 1, which is the leading bit from the left. Hence the sum is $(10010110110)_2$.

2.2.2. We have $(10001000111101)_2 + (11111101011111)_2 = (110000110011100)_2$

2.2.3. We have $(1111000011)_2 - (11010111)_2 = (1011101100)_2$

2.2.4. We have $(1101101100)_2 - (101110101)_2 = (111110111)_2$

2.2.5. To multiply $(11101)_2$ and $(110001)_2$ we need to add $2^0(110001)_2 = (110001)_2$, $2^2(110001)_2 = (11000100)_2$, $2^3(110001)_2 = (110001000)_2$, and $2^4(110001)_2 = (1100010000)_2$. The first bit and carry are computed from $1+0+0+0 = 1$. The second bit and carry are computed from $0+0+0+0 = 0$. The third bit and carry are computed from $0+1+0+0 = 1$. The fourth bit and carry are computed from $0+0+1+0 = 1$. The fifth bit and carry are computed from $1+0+0+1 = 10$. The sixth bit and carry are computed from (with the carry 1) $1+1+0+0+0 = 10$. The seventh bit and carry are computed from (with the carry 1) $1+0+1+0+0 = 10$. The eighth bit and carry are computed from (with the carry 1) 1+0+1+1+0=11. The ninth bit and carry are computed from (with the carry 1) 1+0+0+1+1= 11. The tenth bit and eleventh bit are computed from (with the carry 1) 1+0+0+0+1=10. Hence $(11101)_2 \cdot (110001)_2 = (10110001101)_2$.

2.2.6. We have $(1110111)_2 \cdot (10011011)_2 = (100100000001101)_2$

2.2.7. We have $(110011111)_2 = (11111)_2 \cdot (1101)_2 + (1100)_2$

2.2.8. We see that, because of the length of the words $(11101)_2$ and $(110100111)_2$, that our quotient has four digits. We begin with $(110100111)_2 = 2^3(11101)_2 + (10111111)_2$. We continue with $(10111111)_2 = 2^2(11101)_2 + (1001011)_2$ and $(1001011)_2 = 2(11101)_2 + (10001)_2$. Thus, when $(110100111)_2$ is divided by $(11101)_2$, we get a quotient of $(1110)_2$ and a remainder of $(10001)_2$.

2.2.9. We have $(1234321)_5 + (2030104)_5 = (3314430)_5$

2.2.10. We have $(4434201)_5 - (434421)_5 = (3444230)_5$

2.2.11. We have $(1234)_5 \cdot (3002)_5 = (3023)_5 + (4312000)_5 = (4320023)_5$

2.2.12. We have $(14321)_5 = (22)_5 \cdot (334)_5 + (313)_5$

2.2.13. To add $(ABAB)_{16}$ and $(BABA)_{16}$ we first add the rightmost hexadecimal digits B and A obtaining the rightmost digit of the sum, 5, and carry, 1. Then we add the hexadecimal digits in the second position from the right and the carry, namely A, B and 1, obtaining the second digit from the right in the sum, 6, and the carry, 1. Then we add the hexadecimal digits in the third position from the right, namely B, A, and 1, obtaining the digit in the third position from the right, 6, and the carry, 1. Finally, we add

the hexadecimal digits in the leftmost position and the carry, namely A, B, and 1, obtaining the second hexadecimal digit from the left in the sum, 6, and the leftmost hexadecimal digit in the sum 1. Hence the sum is $(16665)_{16}$.

2.2.14. We have $(FEED)_{16} - (CAFE)_{16} = (33EF)_{16}$

2.2.15. We have $(FACE)_{16} \cdot (BAD)_{16} = (B705736)_{16}$

2.2.16. We have $(BEADED)_{16} = (11C)_{16} \cdot (ABBA)_{16} + (2B95)_{16}$

2.2.17. We represent the integer $(18235187)_{10}$ using three words: $((018)(235)(187))_{1000}$ and the integer $(22135674)_{10}$ using three words: $((022)(135)(674))_{1000}$, where each base 1000 digit is represented by three base 10 digits in parentheses. To find the sum, difference, and product of these integers from their base 1000 representations we carry out the algorithms for such computations for base 1000. The details are omitted.

2.2.18. The algorithms for addition, subtraction, multiplication, and integer division for numbers written in a negative base are identical to those written in a positive base.

2.2.19. To add numbers using the one's complement representation, first decide whether the answer will be negative or positive. To do this is easy if both numbers have the same lead (sign) bit; otherwise conduct a bit-by-bit comparison of a positive summand's digits and the complement of the negative's. Now, add the other digits (all but the initial (sign) bit) as an ordinary binary number. If the sum is greater than 2^n we have an overflow error. If not, consider the three quantities of the two summands and the sum. If exactly zero or two of these are negative, we're done. Otherwise, we need to add $(1)_2$ to this answer. Also, add an appropriate sign bit to the front of the number.

2.2.20. To subtract b from a, obtain $-b$ as in Exercise 20, Section 2.1. Then add a and $-b$ as in Exercise 19.

2.2.21. Let $a = (a_m a_{m-1} \ldots a_2 a_1)_!$ and $b = (b_m b_{m-1} \ldots b_2 b_1)_!$. Then $a+b$ is obtained by adding the digits from right to left with the following rule for producing carries. If $a_j + b_j + c_{j-1}$, where c_{j-1} is the carry from adding a_{j-1} and b_{j-1}, is greater than j, then $c_j = 1$, and the resulting jth digit is $a_j + b_j + c_{j-1} - j - 1$. Otherwise, $c_j = 0$. To subtract b from a, assuming $a > b$, we let $d_i = a_i - b_i + c_{i-1}$ and set $c_i = 0$ if $a_i - b_i + c_{i-1}$ is between 0 and j. Otherwise, $d_i = a_i - b_i + c_{i-1} + j + 1$ and set $c_i = -1$. In this manner, $a - b = (d_m d_{m-1} \ldots d_2 d_1)_!$.

2.2.22. a. We have $(374)_{12}$ eggs removed from $(B03)_{12}$ eggs (where B is the base 12 digit that represents the decimal integer 11). Since $(B30)_{12} - (374)_{12} = (778)_{12}$ there are 7 gross, 7 dozen, and 8 eggs left.

b. We have $(5)_{12}$ times $(237)_{12}$ eggs in the delivery. Since $(5)_{12} \cdot (237)_{12} = (B5B)_{12}$ there were 11 gross, 5 dozen, and 11 eggs in the delivery.

c. We have three groups of eggs each containing $(BA6)_{12}/(3)_{12}$ eggs. Since $(BA6)_{12}/(3)_{12} = (3B6)_{12}$, each group contains 3 gross, 11 dozen, and 6 eggs.

2.2.23. We have $(a_n \ldots a_1 5)_{10}^2 = (10(a_n \ldots a_1)_{10} + 5)^2 = 100(a_n \ldots a_1)_{10}^2 + 100(a_n \ldots a_1)_{10} + 25 = 100(a_n \ldots a_1)_{10}((a_n \ldots a_1)_{10} + 1) + 25$. The decimal digits of this number consist of the decimal digits of $(a_n \ldots a_1)_{10}((a_n \ldots a_1)_{10} + 1)$ followed by 25 since this first product is multiplied by 100 which shifts its decimal expansion two digits.

2.2.24. We have $(a_n \ldots a_1 B)_{2B}^2 = (2B(a_n \ldots a_1)_{10} + B)^2 = (2B)^2(a_n \ldots a_1)_{10}^2 + 4B^2(a_n \ldots a_1)_{2B} + B^2 = (2B)^2(a_n \ldots a_1)_{2B}((a_n \ldots a_1)_{2B} + 1) + 25$. The base $2B$ digits of this number consist of the base $2B$ digits of $(a_n \ldots a_1)_{2B}(a_n \ldots a_1)_{2B} + 1$ followed by B^2 since this first product is multiplied by $(2B)^2$ which shifts its base $2B$ expansion two digits. To finish the proof, note that $B^2 = (B/20)_{2B} = (2B)(B/2) + 0$ is valid when B is even. Furthermore, when B is odd, $B^2 = ((B-1)/2B)_{2B} = (2B)((B-1)/2) + B$.

2.3. Complexity and Integer Operations

2.3.1. a. We have $2n + 7$ is $O(n)$ since $2n + 7 \leq 9n$ for every positive integer n.

b. Note that $n^2/3$ is not $O(n)$ for if C is a real number it follows $n^2/3 > Cn$ whenever $n > 3C$.

c. We have 10 is $O(n)$ since $10 \leq 10n$ whenever n is a positive integer.

d. We have $n^2 + 1 \leq 2n^2$ whenever n is a positive integer. Hence $\log(n^2 + 1) \leq \log(2n^2) = 2\log n + \log 2 \leq 3n$ whenever n is a positive integer. It follows that $\log(n^2 + 1)$ is $O(n)$.

e. Note that $\sqrt{n^2 + 1} \leq \sqrt{2n^2} \leq \sqrt{2} \cdot n$ whenever n is a positive integer. Hence $\sqrt{n^2 + 1}$ is $O(n)$.

f. We have $(n^2 + 1)/(n + 1) < (2n^2/n = 2n$ whenever n is a positive integer. Hence $(n^2 + 1)/(n + 1)$ is $O(n)$.

2.3.2. Note that for $n \geq 1$, $2n^4 + 3n^3 + 17 \leq 2n^4 + 3n^4 + 17n^4 = 22n^4$. So we take $K = 22$, in the definition.

2.3.3. First note that $(n^3 + 4n^2 \log n + 101n^2)$ is $O(n^3)$ and that $(14n \log n + 8n)$ is $O(n \log n)$ as in Example 1.36. Now applying Theorem 1.12 yields the result.

2.3.4. Note that $n! = \prod_{j=1}^{n} j \leq \prod_{j=1}^{n} n = n^n$ whenever n is a positive integer. Hence $n! = O(n^n)$.

2.3.5. Use Exercise 4 and follow Example 1.34, noting that $(\log n)^3 \leq n^3$ whenever n is a positive integer.

2.3.6. Note that $n! = \sum_{j=1}^{n} j^m \leq \sum_{j=1}^{n} n^m = n^{m+1}$. Hence $\sum_{j=1}^{n} j^m = O(n^{m+1})$.

2.3.7. Let k be an integer with $1 \leq k \leq n$. Consider the function $f(k) = (n + 1 - k)k$, whose graph is a concave-down parabola with k-intercepts at $k = 0$ and $k = n + 1$. Since $f(1) = f(n) = n$, it is clear that $f(k) \geq n$ for $k = 1, 2, 3, \ldots, n$. Now consider the product $(n!)^2 = \prod_{k=1}^{n} k(n + 1 - k) \geq \prod_{k=1}^{n} n$, by the inequality above. This last is equal to n^n. Thus we have $n^n \leq (n!)^2$. Taking logarithms of both sides yields $n \log(n) \leq 2 \log(n!)$, which shows that $n \log(n)$ is $O(\log(n!))$.

2.3.8. There exist by hypothesis k_1 and k_2 such that $f_1 \leq k_1 O(g_1)$ and $f_2 \leq k_2 O(g_2)$. Let $k = \max\{c_1 k_1, c_2 k_2\}$. Then $c_1 f_1 + c_2 f_2 \leq c_1 k_1 O(g_1) + c_2 k_2 O(g_2) \leq k(O(g_1) + O(g_2)) = kO(g_1 + g_2)$.

2.3.9. Suppose that f is $O(g)$ where $f(n)$ and $g(n)$ are positive integers for every integer n. Then there is an integer C such that $f(n) < Cg(n)$ for all $x \in S$. Then $f^k(n) < C^k g^k(n)$ for all $x \in S$. Hence f^k is $O(g^k)$.

2.3.10. Suppose $f(n) = O(\log_2 n)$. Then $f(n) \leq k \log_2 n = k \log_2 r \log_r n = k' \log_r n$. Conversely, if $f(n) \leq k \log_r n = k(\log_2 n)/(\log_2 r) = k' \log_2 n$, and so $f(n) = O(\log_2 n)$.

2.3.11. The number of digits in the base b expansion of n is $1 + k$ where k is the largest integer such that $b^k \leq n < b^{k+1}$ since there is a digit for each of the powers of $b^0, b^1, \ldots, b^k$. Note that this inequality is equivalent to $k \leq \log_b n < k + 1$, so that $k = [\log_b n]$. Hence there are $[\log_b n] + 1$ digits in the base b expansion of n.

2.3.12. For addition, three numbers (two operations) must be added for each digit. Thus it takes less than or equal to $2n$ operations to add two numbers. Subtraction follows in a similar manner.

2.3.13. To multiply an n-digit integer by an m-digit integer in the conventional manner, one must multiply every digit of the first number by every digit of the second number. There are nm such pairs.

2.3.14. a. There are $n - 1$ addition signs in $1 + 2 + \cdots n$, so there are $n - 1$ additions total. Each addition takes at most $2[\log_2 n] + 2$ bit operations (see solution to Exercise 12 and Exercise 11). So, the total number of bit operations is at most $2(n - 1)([\log_2 n] + 1)$.

b. Here we have one multiplication, which will require at most $([\log_2 n + 1] + 1)^2$ operations. Shifting is one bit operation, so the total number of bit operations is at most $([\log_2 n + 1] + 1)^2 - 1$.

2.3.15. a. We use the result of Theorem 2.6. Let $m = [\log_2 n + 1]$. If we first multiply consecutive pairs of integers in the the product, we have $O(n/2)$ multiplications of integers with at most m bits. By Theorem 2.6, there is an algorithm for doing this using $O(m \log_2 m \log_2 \log_2 m)$ operations. Now we have $[n/2]$ integers of at most $2m$ bits. If we multiply pairs of these integers together, then by Theorem 2.6 again, this results in $O((n/4)(2m) \log_2 m \log_2 \log_2 m)$, where we use the fact that $\log_2 km \log_2 \log_2 km = O(\log_2 m \log_2 \log_2 m)$ for any constant k. Continuing in this manner we find that computing $n!$ takes $O(\sum_{j=1}^{m} n/(2^j) 2^{j-1} \log_2 m \log_2 \log_2 m) = O((n/2)m^2 \log_2 m \log_2 \log_2 m) = O(n \log_2^2 n \log_2 \log_2 n \log_2 \log_2 \log_2 n)$ operations.

b. We need to find three factorials, which will have the same big-O value as in part (a). We will also need to perform one subtraction (which will not affect the big-O value), one multiplication and one division. The factorials have at most $n \log n$ bits, so by Theorem 2.5, the multiplication will take at most $O((n \log n)^{1+\epsilon})$ bit operations. By Theorem 2.7, the division will take $O((n \log n)^{1+\epsilon})$, so in total the number of bit operations is $O((n \log n)^{1+\epsilon})$.

2.3.16. Let m be an integer. Then m has $n = [\log_2(m)+1]$ bits, from Exercise 11. Using the method of Example 2.1, we need to perform the division algorithm n times. Each division takes $O(n^2) = O([\log_2(m)+1]^2) = O(\log^2 m)$. Therefore, the binary expansion can be found in $O(\log^3 m)$ bit operations.

2.3.17. $(1001)_2 \cdot (1011)_2 = (2^4+2^2)(10)_2(10)_2+2^2(10-01)_2(11-10))_2+(2^2+1)(01)_2(11)_2 = (10100)_2(100)_2+(100)_2(01)_2(01)_2 + (101)_2(01)_2(11)_2 = (1010000)_2 + (100)_2 + (1111)_2 = (1100011)_2$

2.3.18. $(10010011)_2 \cdot (11001001)_2 = (2^8 + 2^4)(1001)_2(1100)_2 + 2^4(1001 - 0011)_2(1001 - 1100))_2 + (2^4 + 1)(0011)_2(1001)_2 = (100010000)_2(1101100)_2 - (10000)_2(0110)_2(0011)_2 + (10001)_2(11011)_2 = (110110000000000)_2 + (11011000000)_2 - (100100000)_2 + (111001011)_2 = (111001101101011)_2$, where we have used identity (1.9) with $n = 2$ to do the smaller multiplications.

2.3.19. a. $ab = (10^{2n} + 10^n)A_1B_1 + 10^n(A_1 - A_0)(B_0 - B_1) + (10^n + 1)A_0B_0$ where A_i and B_i are defined as in identity (1.9).

b. $73 \cdot 87 = (10^2 + 10)7 \cdot 8 + 10(7 - 3)(7 - 8) + (11)3 \cdot 7 = 5600 + 560 - 40 + 210 + 21 = 6351$.

c. $4216 \cdot 2733 = (10100)42 \cdot 27 + (100)(42 - 16)(33 - 27) + (101)16 \cdot 33$. Then, $42 \cdot 27 = (10^2 + 10)4 \cdot 2 + 10(4 - 2)(7 - 2) + (11)2 \cdot 7 = 1134$, and, $26 \cdot 06 = (10^2 + 10)2 \cdot 0 + 10(2 - 6)(6 - 0) + (11)6 \cdot 6 = 156$, and $16 \cdot 33 = (10^2 + 10)1 \cdot 3 + 10(1 - 6)(3 - 3) + (11)6 \cdot 3 = 528$. Then $4216 \cdot 2733 = (10100)1134 + (100)156 + (101)528 = 11522328$.

2.3.20. Note that an element of the kth column of A will be multiplied with each element of the kth row of B. Thus, each of the n^2 entries of A will be multiplied n entries of B. In other words, n^3 multiplications will be performed.

2.3.21. That the given equation is an identity may be seen by direct calculation. The seven multiplications necessary to use this identity are: $a_{11}b_{11}$, $a_{12}b_{21}$, $(a_{11} - a_{21} - a_{22})(b_{11} - b_{12} - b_{22})$, $(a_{21} + a_{22})(b_{12} - b_{11})$, $(a_{11} + a_{12} - a_{21} - a_{22})b_{22}$, $(a_{11} - a_{21})(b_{22} - b_{12})$, $a_{22}(b_{11} - b_{21} - b_{12} + b_{22})$.

2.3.22. We proceed by mathematical induction. Exercise 21 serves to complete the basis step. For the inductive hypothesis, assume that it requires 7^k multiplications to multiply two $2^k \times 2^k$ matrices, and fewer than 7^{k+1} additions. Note that the identity from Exercise 21 holds when the entries of the 2×2 matrices are themselves square matrices, all the same size. Thus we may view a $2^{k+1} \times 2^{k+1}$ matrix as a 2×2 matrix whose entries are $2^k \times 2^k$ matrices. Thus we will need to multiply $2^k \times 2^k$ matrices seven times, requiring $7 \cdot 7^k = 7^{k+1}$ multiplications. Similarly, we will need to add $2^k \times 2^k$ matrices 18 times, requiring exactly $18 \cdot 2^k$ additions. But $18 \cdot 2^k < 7 \cdot 3 \cdot 2 \cdot 2^{k-1} < 7^2 \cdot 2^{k-1} < 7^{k+1}$, as desired.

2.3.23. Let $k = [\log_2 n] + 1$. Then the number of multiplications for $2^k \times 2^k$ matrices is $O(7^k)$. But, $7^k = 2^{(\log_2 7)([\log_2 n]+1)} = O(2^{\log_2 n \log_2 7} 2^{\log_2 7}) = O(n^{\log_2 7})$. The other bit operations are absorbed into this term.

CHAPTER 3

Primes and Greatest Common Divisors

3.1. Prime Numbers

3.1.1. a. We see that 101 is prime since it is not divisible by any positive integers other than 1 or 101. To verify this it is sufficient to check that 101 is not divisible by any prime not exceeding $\sqrt{101}$. The only such primes are 2, 3, 5, and 7 and none of these divide 101.

b. We see that 103 is prime since it is not divisible by any positive integers other than 1 or 103. To verify this it is sufficient to check that 103 is not divisible by any prime not exceeding $\sqrt{103}$. The only such primes are 2, 3, 5, and 7 and none of these divide 103.

c. We see that 107 is prime since it is not divisible by any positive integers other than 1 or 107. To verify this it is sufficient to check that 107 is not divisible by any prime not exceeding $\sqrt{107}$. The only such primes are 2, 3, 5, and 7 and none of these divide 107.

d. We see that 111 is not prime since it is divisible by 3.

e. We see that 113 is prime since it is not divisible by any positive integers other than 1 or 113. To verify this it is sufficient to check that 113 is not divisible by any prime not exceeding $\sqrt{113}$. The only such primes are 2, 3, 5, and 7 and none of these divide 113.

f. We see that 121 is not prime since it is divisible by 11.

3.1.2. a. We have $201 = 3 \cdot 67$, so 201 is not prime.

b. We have $203 = 7 \cdot 29$, so 203 is not prime.

c. We have $207 = 9 \cdot 23$, so 207 is not prime.

d. 211 is prime.

e. We have $213 = 3 \cdot 71$, so 213 is not prime.

f. We have $221 = 13 \cdot 17$, so 221 is not prime.

3.1.3. The primes less than 150 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149

3.1.4. In addition to the primes in Exercise 3, we have 151, 157, 163, 167, 173, 179, 181, 191, 193, 197 and 199.

3.1.5. Suppose that $n = x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$, where $x > y$. The integer n can not be prime since it divisible by $x + y$ which can not be 1 or n.

3.1.6. We note that n must be positive. Otherwise $n^3 + 1$ is less than or equal to 1 and no such integers are prime. Since $n^3 + 1 = (n+1)(n^2 - n + 1)$, $n^3 + 1$ is not prime unless one of the two factors on the right hand side of this equation is 1 and the other is $n^3 + 1$. But $n + 1$ is greater than 1 for every positive integer n, and the only way for $n + 1 = n^3 + 1$ is when $n = 1$ as is easily verified. It is this case we have

$1^3 + 1 = (1 + 1)(1^2 - 1 + 1) = 2$. Hence 2 is the only prime of this form.

3.1.7. Using the identity given in the hint with k such that $1 < k < n$ and $k \mid n$, then $a^k - 1 \mid a^n - 1$. Since $a^n - 1$ is prime by hypothesis, $a^k - 1 = 1$. From this, we see that $a = 2$ and $k = 1$, contradicting the fact that $k > 1$. Thus we must have $a = 2$ and n is prime.

3.1.8. Since Q_n is a positive integer greater than 1, by Lemma 3.1 it has a prime divisor p. If $p \leq n$, then $p|n!$, so then $p|Q_n - n! = 1$, a contradiction. Therefore, we must have $p > n$. So we can construct an infinite sequence of primes as follows. Choose p_1 to be a prime divisor of Q_1. Then choose p_2 to be a prime divisor of Q_{p_1}, and in general choose p_{k+1} to be a prime divisor of Q_{p_k}. Then $p_1 < p_2 < \cdots < p_k < \cdots$, which proves that there are infinitely many primes.

3.1.9. We need to assume $n \geq 3$ to assure that $S_n > 1$. Then by Lemma 3.1, S_n has a prime divisor p. If $p \leq n$ then $p|n!$, and so $p|n! - S_n = 1$, a contradiction. Therefore we must have $p > n$. Since we can find arbitrarily large primes, there must be infinitely many.

3.1.10. a. We proceed by induction. When $n = 1$ we have $p_1 = 2 \leq 2^{2^0} = 2$. Now assume that $p_k \leq 2^{2^k}$ for $k = 1, 2, \ldots, n-1$. Then by Euclid's proof, a prime q other than $p_1, p_2, \ldots, p_n$ divides Q_n. Then $p_n < q \leq Q_n = p_1 p_2 \cdots p_n + 1 \leq 2^{2^0} 2^{2^1} \cdots 2^{2^{n-1}} = 2^{2^0 + 2^1 + \cdots + 2^{n-1}} = 2^{2^{n-1}-1} + 1$. Since the inequality is strict and we are dealing with integers we have $p_n \leq 2^{2^{n-1}-1} \leq 2^{2^{n-1}}$, which completes the induction step.

b. By part a., the $(n+1)$st prime is less than or equal to 2^{2^n}, and since a power of 2 can not be prime itself when $n > 0$, we must have at least $n + 1$ primes strictly less than 2^{2^n}.

3.1.11. $Q_1 = 3, Q_2 = 7, Q_3 = 31, Q_4 = 211, Q_5 = 2311, Q_6 = 30031$. The smallest prime factors are 3, 7, 31, 211, 2311, and 59, respectively.

3.1.12. Let $Q = p_1 p_2 \cdots p_{n-1} + 1$, where p_i is the ith prime. Then by Euclid's proof, some prime q different from $p_1, p_2, \ldots, p_{n-1}$ divides Q. Then $p_n \leq q \leq Q$.

3.1.13. If n is prime, we are done. Otherwise $n/p < (\sqrt[3]{n})^2$. If n/p is prime, then we are done. Otherwise, by Theorem 3.2, n/p has a prime factor less than $\sqrt{n/p} < \sqrt[3]{n}$, a contradiction.

3.1.14. Suppose $p = 3k + 1$ for some positive integer k. If k is odd, then $k = 2n + 1$ for some integer n and so $p = 3(2n + 1) + 1 = 6n + 4 = 2(3n + 2)$ which is clearly not prime, a contradiction. Therefore, k must be even, say $k = 2n$ for some integer n. Then $p = 3(2n) + 1 = 6n + 1$ as desired.

3.1.15. a. The arithmetic progression is $3n + 1$ and the first values are $4, 7, 10, \ldots$. The first prime is 7.

b. We list the first few numbers of the shape $5n + 4$ until we find a prime: $9, 14, 19$, which is prime.

c. We list the first few numbers of the shape $11n + 16$ until we find a prime: $27, 38, 49, 60, 71$, which is prime.

3.1.16. a. We list the first few numbers of the shape $5n + 1$ until we find a prime: $6, 11$, which is prime.

b. We list the first few numbers of the shape $7n + 2$ until we find a prime: $9, 16, 23$, which is prime. (But if we begin with $n = 0$, the first term is 2, which is prime.)

c. We list the first few numbers of the shape $23n + 13$ until we find a prime: $36, 59$, which is prime. (But if we begin with $n = 0$, the first term is 13 which is prime.)

3.1.17. If n is prime the statement is true for n. Otherwise, n is composite, so n is the product of two integers a and b such that $1 < a \leq b < n$. Since $n = ab$ and by the inductive hypothesis both a and b are the

product of primes, we conclude that n is also the product of primes.

3.1.18. The number of integers not exceeding n that are prime are the integers other than 1 that are either primes less than $\sqrt{n}$ or integers greater than $\sqrt{n}$ not exceeding n that are not divisible by any of these primes. We can use the principle of inclusion-exclusion to find the number of positive integers not exceeding $\sqrt{n}$ that are not divisible by any of the primes $p_1, p_2, \ldots, p_r$ not exceeding $\sqrt{n}$. Then we can add $\pi(\sqrt{n})$ to count the number of primes not exceeding $\sqrt{n}$, and subtract 1 since the integer 1 is not divisible by any of these primes, but is not itself prime. Since the number of integers not exceeding n that are divisible by the primes $p_{i_1}, p_{i_2}, \ldots, p_{i_r}$ is $[n/(p_{i_1}p_{i_2}\cdots p_{i_r})]$, the principle of inclusion-exclusion tells us that the number of integers not exceeding n that are not divisible by any of these primes is $n - \sum_{i=1}^{r}[n/p_i] + \sum_{1\le i<j\le r}[n/(p_ip_j)] - \sum_{1\le i<j<k\le r}[n/(p_ip_jp_k)] + \cdots + (-1)^r[n/\prod_{i=1}^{r} p_i]$. We see that $\pi(n)$ is obtained by adding $\pi(\sqrt{n}) - 1$ to this quantity.

3.1.19. Using Exercise 18, we have, $\pi(250) = (\pi(\sqrt{250}) - 1) + 250 - ([250/2] + [250/3] + [250/5] + [250/7] + [250/11] + [250/13]) + ([250/6] + [250/10] + [250/14] + [250/22] + [250/26] + [250/15] + [250/21] + [250/33] + [250/39] + [250/35] + [250/55] + [250/65] + [250/77] + [250/91] + [250/143]) - ([250/30] + [250/42] + [250/66] + [250/70] + [250/78] + [250/105] + [250/110] + [250/130] + [250/132] + [250/154] + [250/165] + [250/195] + [250/231]) + ([250/210]) = 5 + 250 - (125 + 83 + 50 + 35 + 22 + 19) + (41 + 25 + 17 + 11 + 9 + 16 + 11 + 7 + 6 + 7 + 4 + 3 + 3 + 2 + 1) - (8 + 5 + 3 + 3 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1) + 1 = 53.$

3.1.20. Let $f(x) = x^2 - x + 41$. Then $f(0) = 41, f(1) = 41, f(2) = 43, f(3) = 47, f(4) = 53, f(5) = 61, f(6) = 71, f(7) = 83, f(8) = 97, f(9) = 113, f(10) = 131, f(11) = 151, f(12) = 173, f(13) = 197, f(14) = 223, f(15) = 251, f(16) = 281, f(17) = 313, f(18) = 347, f(19) = 383, f(20) = 421, f(21) = 461, f(22) = 503, f(23) = 547, f(24) = 593, f(25) = 641, f(26) = 691, f(27) = 743, f(28) = 797, f(29) = 853, f(30) = 911, f(31) = 971, f(32) = 1033, f(33) = 1097, f(34) = 1163, f(35) = 1231, f(36) = 1301, f(37) = 1373, f(38) = 1447, f(39) = 1523$, and $f(40) = 1601$ are all primes, but $f(41) = 41^2 - 41 + 41 = 41^2$ is composite.

3.1.21. For $n = 1, 2, \ldots 10$, the values of the function are $13, 19, 29, 43, 61, 83, 109, 139, 173, 211$, each of which is prime. But $2 \cdot 11^2 + 11 = 11(2 \cdot 11 + 1) = 11 \cdot 23$, so it is not prime.

3.1.22. For $n = 1, 2, \ldots 28$, the values of the function are $31, 37, 47, 61, 79, 101, 127, 157, 191, 229, 271, 317, 367, 421, 479, 541, 607, 677, 751, 829, 911, 997, 1087, 1181, 1279, 1381, 1487, 1597$, each of which is prime. But $2 \cdot 29^2 + 29 = 29(2 \cdot 29 + 1) = 29 \cdot 59$, so it is not prime.

3.1.23. Assume not. Let x_0 be a positive integer. It follows that $f(x_0) = p$ where p is prime. Let k be an integer. We have $f(x_0 + kp) = a_n(x_0 + kp)^n + \cdots + a_1(x_0 + kp) + a_0$. Note that by the binomial theorem, $(x_0 + kp)^j = \sum_{i=1}^{j} \binom{j}{i} x_0^{j-i}(kp)^i$. It follows that $f(x_0 + kp) = \sum_{j=0}^{n} a_j x_0^j + Np = f(x_0) + Np$, for some integer N. Since $p \mid f(x_0)$ it follows that $p \mid (f(x_0) + Np) = f(x_0 + kp)$. Since $f(x_0 + kp)$ is supposed to be prime, it follows that $f(x_0 + kp) = p$ for all integers k. This contradicts the fact that a polynomial of degree n takes on each value no more than n times. Hence $f(y)$ is composite for at least one integer y.

3.1.24. The sequence of lucky numbers less than 100 is: 1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, 67, 69, 73, 75, 79, 87, 93, 99.

3.1.25. At each stage of the procedure for generating the lucky numbers the smallest number left, say k, is designated to be a lucky number and infinitely many numbers are left after the deletion of every kth integer left. It follows that there are infinitely many steps, and at each step a new lucky number is added to the sequence. Hence there are infinitely many lucky numbers.

3.1.26. a. Suppose $p_j | t_k - Q_k + 1 = t_k - (p_1 \cdots p_k + 1) + 1 = t_k - p_1 \cdots p_k$. Since $p_j | p_1 \cdots p_k + 1$, then $p_j | t_k$ which is impossible by Euclid's proof.

b. For $k = 1$ we have $Q_1 = 2 + 1 = 3$, so $t_1 = 5$ and $t_1 - Q_1 + 1 = 5 - 3 + 1 = 3$ is prime. For $k = 2$, we have $Q_2 = 2 \cdot 3 + 1 = 7$, so $t_2 = 11$ and $t_2 - Q_2 + 1 = 11 - 7 + 1 = 5$ which is prime. For $k = 3$ we have $Q_3 = 2 \cdot 3 \cdot 5 + 1 = 31$, so $t_3 = 37$ and $t_3 - Q_3 + 1 = 37 - 31 + 1 = 7$ which is prime. For $k = 4$ we have

$Q = 2\cdot3\cdot5\cdot7+1 = 211$ so $t_4 = 223$ and $t_4 - Q_4 + 1 = 223-211+1 = 13$ which is prime. For $k = 5$ we have $Q_5 = 2\cdot3\cdot5\cdot7\cdot11+1 = 2311$, so $t_5 = 2333$ and $t_5 - Q_5 + 1 = 2333-2311+1 = 23$ which is prime.

3.2. The Distribution of Primes

3.2.1. The smallest 5 consecutive composite integers can be found by locating the first pair of consecutive composite odd integers, 25 and 27. Hence the smallest 5 consecutive composite integers are 24, 25, 26, 27, and 28. (These are considerably smaller than the integers $(5+1)!+j = 6!+j = 720+j$ for $j = 2, 3, 4, 5, 6$.)

3.2.2. $1000001! + j$, is divisible by j for all $j = 2, 3, \ldots, 1000001$, which gives one million consecutive composite integers.

3.2.3. Suppose that $p, p+2$, and $p+4$ were all prime. We consider three cases. First, suppose that p is of the form $3k$. Then p cannot be prime unless $k = 1$, and the prime triplet is 3, 5, and 7. Next, suppose that p is of the form $3k+1$. Then $p+2 = 3k+3 = 3(k+1)$ is not prime. We obtain no prime triplets in this case. Finally, suppose that p is of the form $3k+2$. Then $p+4 = 3k+6 = 3(k+2)$ is not prime. We obtain no prime triplet in this case either. Since the three cases are exhaustive, we have only one prime triplet of this kind, 3, 5, and 7.

3.2.4. By searching through a table we find these triples: $(5, 7, 11)$, $(11, 13, 17)$, $(17, 19, 23)$, and $(41, 43, 47)$.

3.2.5. By searching through a table we find these triples: $(7, 11, 13)$, $(13, 17, 19)$, $(37, 41, 43)$, and $(67, 71, 73)$.

3.2.6. a. The smallest prime between 3 and 6 is 5.

b. The smallest prime between 5 and 10 is 7.

c. The smallest prime between 19 and 38 is 23.

d. The smallest prime between 31 and 62 is 37.

3.2.7. a. The smallest prime between 4 and 8 is 5.

b. The smallest prime between 6 and 12 is 7.

c. The smallest prime between 23 and 46 is 29.

d. The smallest prime between 47 and 94 is 53.

3.2.8. To see that the primes are indeed in the range, we print them as triples $(n^2, \text{smallest prime}, (n+1)^2)$. For $n = 1, 2, \ldots, 10$ we have $(1, 2, 4)$, $(4, 5, 9)$, $(9, 11, 16)$, $(16, 17, 25)$, $(25, 29, 36)$, $(36, 37, 49)$, $(49, 53, 64)$, $(64, 67, 81)$, $(81, 83, 100)$, and $(100, 101, 121)$.

3.2.9. To see that the primes are indeed in the range, we print them as triples $(n^2, \text{smallest prime}, (n+1)^2)$. For $n = 11, 12, \ldots, 20$ we have $(121, 127, 144)$, $(144, 149, 169)$, $(169, 173, 196)$, $(196, 197, 225)$, $(225, 227, 256)$, $(256, 257, 289)$, $(289, 293, 324)$, $(324, 331, 361)$, $(361, 367, 400)$, and $(400, 401, 441)$.

3.2.10. a. We have $50 = 47 + 3$.

b. We have $98 = 87 + 11$.

c. We have $102 = 97 + 5$.

d. We have $144 = 139 + 5$.

e. We have $200 = 197 + 3$.

f. We have $222 = 211 + 11$

3.2.11. a. We have $7 = 3 + 2 + 2$.

b. We have $17 = 11 + 3 + 3$.

c. We have $27 = 23 + 2 + 2$.

d. We have $97 = 89 + 5 + 3$.

e. We have $101 = 97 + 2 + 2$.

f. We have $199 = 191 + 5 + 3$

3.2.12. Let n be an integer greater than 11. Suppose, first, that n is even. Then $n = 4 + (n-4)$ exhibits n as the sum of two composite integers, since 4 is composite and $n - 4$ is composite since it is even and greater than two. Now suppose that n is odd. Then $n = (n-9) + 9$ exhibits n as the sum of two composite integers since 9 is composite and $n - 9$ is an even integer greater than two.

3.2.13. Suppose Goldbach's conjecture is true and let $n > 5$ be an integer. If n is even, then $n - 2$ is an even integer greater than 2 and so is the sum of two primes, p and q. Then $n = p + q + 2$, the sum of three primes. If n is odd, then $n - 3$ is an even integer greater than 2 and so is the sum of two primes p and q. Then $n = p + q + 3$, the sum of three primes.

Conversely, suppose every integer greater than 5 is the sum of three primes. Let $n > 2$ be an even integer. Then $n + 2$ is also even and is greater than 5. (It is not equal to 5 since it is even.) By hypothesis, $n + 2$ is the sum of 3 primes. If all three primes were odd, then $n + 2$ would be odd, a contradiction, so at least one of the primes is even, that is, one of the primes must be 2, so $n + 2 = p + q + 2$ for some primes p and q. Therefore $n = p + q$, the sum of two primes.

3.2.14. a. For $n = 4$, we have $4 = 2 + 2$ so $G(4) = 1$. For $n = 6$, we have $6 = 3 + 3$, so $G(6) = 1$. Since $8 = 5 + 3$, $G(8) = 1$. Since $10 = 5 + 5 = 7 + 3$, $G(10) = 2$. Since $12 = 7 + 5$, $G(12) = 1$. Since $14 = 7 + 7 = 11 + 3$, $G(14) = 2$. Since $16 = 13 + 3 = 11 + 5$, $G(16) = 2$. Since $18 = 13 + 5 = 11 + 7$, $G(18) = 2$. Since $20 = 17 + 3 = 13 + 7$, $G(20) = 2$. Since $22 = 19 + 3 = 17 + 5 = 11 + 11$, $G(22) = 3$. Since $24 = 19 + 5 = 17 + 7 = 13 + 11$, $G(24) = 3$. Since $26 = 23 + 3 = 19 + 7 = 13 + 13$, $G(26) = 3$. Since $28 = 23 + 5 = 17 + 11$, $G(28) = 2$. Since $30 = 23 + 7 = 19 + 11 = 17 + 13$, $G(30) = 3$.

b. The primes less than 158 are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151$, and 157. If we subtract each of these from 158 we get the following list of numbers: $156, 155, 153, 151, 147, 145, 141, 139, 135, 129, 127, 121, 117, 115, 111, 105, 99, 97, 91, 87, 85, 79, 75, 69, 61, 57, 55, 51, 49, 45, 31, 27, 21, 19, 9, 7, 1$, of which only 151, 139, 127, 97, 79, 61, 31, 19, and 7 are primes, so $G(158) = 9$.

c. The primes less than 188 are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179$, and 181 If we subtract each of these from 188 we get the following list of numbers: 186, 185, 183, 181, 177, 175, 171, 169, 165, 159, 157, 151, 147, 145, 141, 135, 129, 127, 121, 117, 115, 109, 105, 99, 91, 87, 85, 81, 79, 75, 61, 57, 51, 49, 39, 37, 31, 25, 21, 15, 9, 7, of which only 181, 157, 151, 127, 109, 79, 61, 37, 31, and 7 are prime, so $G(188) = 10$.

3.2.15. Let $p < n$ be prime. Using the division algorithm, we divide each of the first $p + 1$ integers in the sequence by p to get $a = q_0 p + r_0, a + k = q_1 p + r_1, \ldots, a + pk = q_p + r_p$, with $0 \leq r_i < p$ for each i. By the pigeonhole principle, at least two of the remainders must be equal, say $r_i = r_j$. We subtract the corresponding equations to get $a + ik - a - jk = q_i p + r_i - q_j p + r_j$, which reduces to $(i - j)k = (q_i - q_j)p$. Therefore $p|(i - j)k$, and since p is prime, it must divide one of the factors. But since $(i - j) < p$ we must

have $p|k$.

3.2.16. Exercise 15 tells us that every prime less than six will have to divide the common difference, so we will try a difference of $30 = 2 \cdot 3 \cdot 5$, which generates the sequence $7, 37, 67, 97, 127, 157$, all of which are prime.

3.2.17. From Exercise 15, we know that every prime less than four must must divide the difference, so 6 must divide the difference. Therefore the smallest possible difference is 6. This minimum is achieved with $5, 11, 17, 23$.

3.2.18. From Exercise 15, we know that every prime less than five must must divide the difference, so 6 must divide the difference. Therefore the smallest possible difference is 6. This minimum is Achieved with the sequence $5, 11, 17, 23, 29$.

3.2.19. From Exercise 15, we know that every prime less than six must must divide the difference, so 30 must divide the difference. Therefore the smallest possible difference is 30. This minimum is achieved by the example given in Exercise 16.

3.2.20. a. Since 509 is less than 2^9, we consider the numbers $509 - 2^k$ for $k = 0, 1, 2, \ldots, 8$. This gives us the sequence 508, 507, 505, 501, 493, 477, 445, 381, 253. It is easy to check that none of these are prime, and so the conjecture is false.

b. One way to search for likely candidates is to make a sieve. We can write out the odd numbers in a range, say from 509 to 651. Then we can list the prime numbers in a small range, say from 450 to 650. From the list of odd numbers, we strike out every value that 2 more than a prime in our list. Then we strike out every value that is 4 more than a prime in our list. Then we strike out every value that is 2^3 more than a prime in our list, and continue in this fashion, until we have reduced the size of our list sufficiently. Then each number crossed off will have a representation as a power of two plus a prime so we shouldn't consider it. Only a short sequence should be left over: 533, 547, 569, 583, $599\ldots$, which can be tested more thoroughly. And 599 turns out to be the next smallest counterexample.

3.2.21. If $p^\alpha - q^\beta = 1$, with p, q primes, then p or q is even, so p or q is 2. If $p = 2$, there are several cases: we have $2^\alpha - q^\beta = 1$. If α is even, say $\alpha = 2k, (2^{2k} - 1) = (2^k - 1)(2^k + 1) = q^\beta$. So $q|(2^k - 1)$ and $q|(2^k + 1)$, hence $q = 1$, a contradiction. If α is odd and β is odd, $2^\alpha = 1 + q^\beta = (1 + q)(q^{\beta-1} - q^{\beta-2} + \cdots + 1)$. So $1 + q = 2^n$ for some n. Then $2^\alpha = (2^n - 1)^\beta + 1 = 2^n$(odd number), since β is odd. So $2^{\alpha-n} =$ odd number and so $\alpha = n$. Therefore $2^\alpha = 1 + (2^\alpha - 1)^\beta$ and so $\beta = 1$ which is not allowed. If $\alpha = 2k + 1$ and $\beta = 2n$ we have $2^{2k+1} = 1 + q^{2n}$. Since q is odd, q^2 is of the form $4m + 1$, and by the binomial theorem, so is q^{2n}. Thus the right hand side of the last equation is of the form $4m + 2$, but this forces $k = 0$, a contradiction. If $q = 2$, we have $p^\alpha - 2^\beta = 1$. Whence $2^\beta = (p - 1)(p^{\alpha-1} + p^{\alpha-2} + \cdots + p + 1)$, where the last factor is the sum of α odd terms but must be a power of 2, therefore, $\alpha = 2k$ for some k. Then $2^\beta = (p^k - 1)(p^k + 1)$. These last two factors are powers of 2 which differ by 2 which forces $k = 1, \alpha = 2, \beta = 3, p = 3$, and $q = 2$ as the only solution: $3^2 - 2^3 = 1$.

3.2.22. The conjecture is true for $n = 1, 2$, and 3. Let n be an odd integer greater than 3, and assume the conjecture is true for all odd integers less than n. Let $k = (n \pm 1)/2$ where the sign is chosen so that k is odd. Then $k \geq 3$ and $n - k$ is even and ≥ 1. If p is a prime such that $k < p \leq n$, then p is odd and $p \mid n!, p \nmid k!$, and $p \nmid (n-k)!$. Therefore, p divides $\binom{n}{k}$. Hence, $\prod_{k<p\leq n} p \mid \binom{n}{k}$ and so $\prod_{k<p\leq n} p \leq \binom{n}{k}$. But $\binom{n}{k} = \binom{n}{n-k}$ and both these binomial coefficients appear in the expansion of $(1 + 1)^n = 2 \cdot 2^{n-1}$. Using the induction hypothesis we have $\prod_{p\leq n} p = \prod_{p\leq k} p \cdot \prod_{k<p\leq n} p < 4^k \cdot 2^{n-1} = 2^{n+2k-1} \leq 2^{2n} = 4^n$. So the conjecture is true for all odd positive integers. If n is even, we have $\prod_{p\leq n} p = \prod_{p\leq(n-1)} p \leq 4^{n-1} < 4^n$.

3.2.23. Since $3p > 2n, p$, and $2p$ are the only multiples of p that appear as factors in $(2n)!$. So p divides $(2n)!$ exactly twice. Since $2p > n, p$ is the only multiple of p that appears as a factor in $n!$. So $p \mid n!$ exactly once. Then since $\binom{2n}{n} = 2n!/(n!n!)$, the two factors of p in the numerator are cancelled by the two in the

denominator.

3.2.24. The theorem holds for $n = 2, 3, \ldots, 127$, as can be seen by (tedious) inspection. Let $n \geq 128$ and suppose there is no prime between n and $2n$. Let $\binom{2n}{n} = \prod_{p \leq 2n} p^r$ be the prime factorization for $\binom{2n}{n}$. But there are no primes between n and $2n$, so $\binom{2n}{n} = \prod_{p \leq n} p^r$. If p is a prime in the range $2n/3 < p \leq n$ then p divides $n!$ exactly once and $(2n)!$ exactly twice. Thus $p \nmid \binom{2n}{n}$. Therefore $\binom{2n}{n} = \prod_{p \leq \sqrt{2n}} p^r \prod_{\sqrt{2n} < p \leq 2n/3} p^r \leq \prod_{p \leq \sqrt{2n}} 2n \prod_{p \leq 2n/3} p$ since if p is in the range $\sqrt{2n} < p \leq 2n/3$, then p divides $\binom{2n}{n}$ exactly once. The number of primes less than $\sqrt{2n}$ is less than the number of odd integers less than $\sqrt{2n}$, i.e. less than $\sqrt{2n}/2 - 1 = \sqrt{n/2} - 1$. Therefore $\prod_{p \leq \sqrt{2n}} 2n \leq (2n)^{\sqrt{n/2}-1}$. Using Exercise 26, we have $\prod_{p \leq 2n/3} p < 4^{2n/3}$. Thus $\binom{2n}{n} < (2n)^{\sqrt{n/2}-1} 4^{2n/3}$. Now $\binom{2n}{n}$ is the largest of $2n+1$ terms in the binomial expansion of $(1+1)^{2n}$, so we have $(2n+1)\binom{2n}{n} > (2n)\binom{2n}{n} > 2^{2n}$, hence $(2n)^{-1}2^{2n} < \binom{2n}{n} < (2n)^{\sqrt{n/2}-1} 4^{2n/3}$, which implies $2^{2n/3} < (2n)^{\sqrt{n/2}}$. Take logarithms and divide by $\sqrt{2n}/6$ and get $\sqrt{8n}\log 2 - 2\log(2n) < 0$. Denote the left side by $f(n)$, and take its derivative to get $f'(n) = (\sqrt{2n}\log 2 - 3)/n$. Note that $f(128) = 8\log 2$, which is positive, and $f'(n)$ is positive for $n \geq 128$, so $f(n)$ is increasing and therefore positive for $n \geq 128$. This contradicts the last inequality.

3.2.25. By Bertrand's conjecture, there must be a prime in each interval of the form $(2^{k-1}, 2^k)$, for $k = 2, 3, 4, \ldots$. Thus, there are at least $k-1$ primes less than 2^k. Since the prime 2 isn't counted here, we have at least k primes less than 2^k.

3.2.26. First check that the statement is true for $n = 7, 8, \ldots, 16$. Let $n \geq 17$. By Bertrand's conjecture, there is a prime p such that $[(n-7)/2] < p \leq 2[(n-7)/2]$, or equivalently, $[(n-5)/2] \leq p \leq 2[(n-7)/2]$. Let $n_1 = n$, and for each n_j, let p_j be a prime between $[(n_j - 5)/2]$ and $2[(n_j - 7)/2]$, inclusive. Then set $n_{j+1} = n_j - p_j$, and if $n_{j+1} \geq 17$, repeat the procedure. The sequence will terminate with some value $n_{k+1} \leq 16$. Since $[(n_j - 5)/2] \leq p_j \leq 2[(n_j - 7)/2]$, we will have $7 \leq n_{k+1} \leq [(n_j + 6)/2]$. Thus the final value n_{k+1} will lie between 7 and 16 inclusive. We will also have $p_{j+1} \leq 2[(n_{j+1} - 7)/2] \leq 2[([n_j + 6)/2] - 7)/2] \leq [(n_j - 8)/2] < [(n_j - 5)/2] \leq p_j$. Hence the sequence p_j will be descending with no duplicates. Note also that $n_j \leq 2p_j + 6$. Thus, since $n_j > 16$ for $j \leq k$, $p_j > 5$, i.e. p_k, the smallest of the p_j, will be at least 7. We also have the following: If $p_j = 7$, then $n_j \leq 20$ and $n_{j+1} \leq 13$; if $p_j = 11$, then $n_j \leq 28$ and $n_{j+1} \leq 17$; if $p_j = 13$, then $n_j \leq 32$ and $n_{j+1} \leq 19$.

Now suppose that $n_{k+1} = 16$. We know from the above that $p_k \geq 11$. Since $16 = 13 + 3 = 11 + 5$, we need only be concerned with the case that $p_k = 11$ and $p_{k-1} = 13$. But then, $n_{k-1} \leq 32$ and $n_{k+1} = n_{k-1} - p_{k-1} - p_k \leq 32 - 13 - 11 = 8$. Thus if $n_{k+1} = 16$, we cannot have both $p_{k-1} = 13$ and $p_k = 11$. Thus by using either $16 = 13 + 3$ or $16 = 11 + 5$, we have a partition of n into distinct primes. Suppose next that $n_{k+1} = 15$. We have again $p_k \geq 11$, and since $15 = 7 + 5 + 3$, we have a partition of n into distinct primes. Suppose next that $n_{k+1} = 14$. We have again $p_k \geq 11$, and since $14 = 7 + 5 + 2$, we have a partition of n into distinct primes. Suppose next that $n_{k+1} = 13$. As in the case $n_{k+1} = 16$, we cannot have both $p_{k-1} = 13$ and $p_k = 11$, since that implies that $n_{k+1} \leq 8$, thus using either $13 = 13$ or $13 = 11 + 2$, we have a partition of n into distinct primes. Suppose next that $n_{k+1} = 12$. If $p_k > 7$, then since $12 = 7 + 5$, we have a partition of n into distinct primes. If $p_k = 7$, then $n_k = 19$. We cannot also have $p_{k-1} = 11$, since then $n_{k-1} \leq 28$ and $n_{k+1} \leq 28 - 11 - 7 = 10$. Thus since $19 = 11 + 5 + 3$, we have a partition of n into distinct primes. Suppose next that $n_{k+1} = 11$. As in the previous case, we cannot have both $p_k = 7$ and $p_{k-1} = 11$, thus if $p_k = 7$ or $p_k > 11$, we have a partition of n into distinct primes. If $p_k = 11$, then $n_k = 22$. As before, we cannot also have $p_{k-1} = 13$, thus with $22 = 13 + 7 + 2$ we have a partition of n into distinct primes.

Suppose next that $n_{k+1} = 10$. Since $p_k \geq 7$ and $10 = 5 + 3 + 2$, we have a partition of n into distinct primes. Finally, suppose that $n_{k+1} \leq 9$. If $p_k = 7$, then $n_k \leq 16$, but by the construction of the sequence we must have $n_k \geq 17$, thus $p_k > 7$. Then with $9 = 7 + 2$, $8 = 5 + 3$ or $7 = 7$, we have a partition of n into distinct primes.

3.2.27. Since $1/1$ is an integer, we may assume $n > 1$. First suppose that $m < n$. Then $1/n + 1/(n+1) + \cdots + 1/(n+m) \leq 1/n + 1/(n+1) + \cdots + 1/(2n-1) < 1/n + 1/n + \cdots + 1/n \leq n(1/n) = 1$, so the sum can not be an integer. Now suppose $m \geq n$. Then by Bertrand's postulate, there is a prime p such that $n < p < n + m$. Let p be the largest such prime. Then $n + m < 2p$, otherwise there would be a prime q with $p <$

$q < 2p \le n+m$ contradicting the choice of p. Suppose that $1/n+1/(n+1)+\cdots+1/p+\cdots+1/(n+m) = a$ where a is an integer. Note that p occurs as a factor in only one denominator, since $2p > n+m$. Let $Q = \prod_{j=n}^{n+m} j$, and let $Q_i = Q/i$, for $i = n, n+1, \ldots, n+m$. If we multiply the equation by Q we get $Q_n + Q_{n+1} + \cdots + Q_p + \cdots + Q_{n+m} = Qa$. Note that every term on both sides of the equation is divisible by p except for Q_p. If we solve the equation for Q_p and factor a p out of the other side we have an equation of the form $Q_p = pN$ where N is some integer. But this implies that p divides Q_p, a contradiction. Therefore a can not be an integer.

3.2.28. a. (Proof by Ed Hook) With the given notation, suppose that p_j divided $p_1p_2\cdots p_{k-1}i-1$ for some $i = 1, 2, \ldots, p_k$ and some $j = 1, 2, \ldots, k-1$. Then, as in Euclid's proof, p_j would also have to divide 1, a contradiction, so none of the first $k-1$ primes can divide any of these p_k numbers. Further, suppose some larger prime p divided two of these numbers. Then it would have to divide the difference and we would have $p|((p_1p_2\cdots p_{k-1}i-1)-(p_1p_2\cdots p_{k-1}j-1)) = (i-j)(p_1p_2\cdots p_{k-1})$. But since $i-j < p_k$, the larger prime can not divide it, so it must divide the product $p_1p_2\cdots p_{k-1}$, but these are all smaller primes, another contradiction. Therefore, a prime larger than p_k can divide at most one of these numbers.

b. Since there are $n-k+1$ primes from p_k up to p_n, and each one can divide at most one of the p_k numbers $p_1p_2\cdots p_{k-1}i-1$, there must be at least one of the numbers which is not divisible by any prime from p_k up to p_n. (There are more numbers than primes.) From part (a), the primes less than p_k also do not divide any of the the numbers, in particular, the one whose existence we have just shown.

c. From part (b), there is a number of the form $p_1p_2\cdots p_{k-1}i-1$ whose least prime divisor is at least p_{n+1}, since none of the primes $p_1, \ldots, p_n$ can divide it. Therefore, $p_{n+1} \le p_1p_2\cdots p_{k-1}p_k$ if $n-k+1 < p_k$. So let k be the smallest positive integer for which this inequality holds. Then $n-(k-1)+1 \ge p_{k-1}$ which reduces to $n-k \ge p_{k-1}-2$. Now since the sequence of primes grows by at least 2 at each step after 3 and $p_{k-1}-2 = 7-2 = 5 = k$ when $k = 5$, we have the left-hand side growing faster than the right. So $p_{k-1}-2 \ge k$ for $k \ge 5$. So if $n \ge 10$, then $n-k+1$ has to be less than p_k and a quick check shows that this forces $k \ge 5$ (since if $n \ge 10$ and $k \le 4$ then $n-k+1 \ge 10-4+1 = 7 = p_4$, which fails the condition.) Therefore, if $n \ge 10$, the condition $n-k+1 < p_k$ is satisfied and so $p_{n+1} \le p_1p_2\cdots p_{k-1}p_k$ for some k such that $n-k \ge p_{k-1}-2 \ge k$. Note that this implies $2k \le n$.

Then assuming $n \ge 10$ we can derive Bonse's inequality as follows. For the k found above we have $p_{n+1}^2 < (p_1p_2\cdots p_k)(p_1p_2\cdots p_k) < (p_1p_2\cdots p_k)(p_{k+1}p_{k+2}\cdots p_{2k}) \le p_1p_2\cdots p_n$, which is the desired inequality.

d. When $n = 4$, we have $p_5^2 = 121 < 210 = 2\cdot3\cdot5\cdot7 = p_1\cdot p_2\cdot p_3\cdot p_4$. When $n = 5$, we have $p_6^2 = 169 < 2310 = 2\cdot3\cdot5\cdot7\cdot11 = p_1\cdots p_5$. When $n = 6$, we have $p_7^2 = 17^2 = 289 < 30030 = p_1\cdots p_6$. When $n = 7$, we have $p_8^2 = 19^2 = 361 < 510510 = p_1\cdots p_7$. When $n = 8$, we have $p_9^2 = 23^2 = 529 < 9699690 = p_1\cdots p_8$. When $n-9$, we have $p_{10}^2 = 29^2 = 841 < 223092870 = p_1\cdots p_9$, which verifies all remaining cases.

3.2.29. Suppose n has the stated property and $n \ge p^2$ for some prime p. Since p^2 is not prime, there must a prime dividing both p^2 and n, and the only possibility for this is p itself, that is, $p|n$. Now if $n \ge 7^2$, then it is greater than $2^2, 3^2$, and 5^2 and hence divisible by $2, 3, 5$, and 7. This is the basis step for induction. Now assume n is divisible by $p_1, p_2, \ldots, p_k$. By Bonse's inequality $p_{k+1}^2 < p_1p_1\cdots p_k < n$, so $p_{k+1}|n$ also. This induction implies that every prime divides n, which is absurd. Therefore if n has the stated property, it must be less than $7^2 = 49$.

Now we note that the integers less that 30 sharing no common prime factor with 30 are 1, 7, 11, 13, 17, 19, 23 and 29, all of which are prime or 1. So 30 has the property. It remains to show that the numbers from 31 to 48 do not have the property. We exhibit a counterexample in each case. For $n = 31$, 33, 35, 37, 39, 41, 43, 45 and 47 we note that $k = 8$ shares no prime factor with n, and yet is not prime. For $n = 32, 34, 38, 40, 44$ and 46, we note that $k = 9$ shares no prime factor with n and yet is not prime. For the remaining cases $n = 36, 42$, and 48, we note that $k = 25$ shares no prime factor with n and yet is not prime.

3.2.30. From part (c) in Exercise 28, we have that when $n \ge 10$ then $p_{n+1} < p_1p_2\cdots p_{k-1}p_k$ for some $k \ge 5$ such that $n-k \ge k$. By Bertrand's postulate, we have $p_{n+1} < p_{n+2} < 2p_{n+1}$,so we have $p_{n+1}p_{n+2} <$

$p_{n+1}2p_{n+1} < (p_1p_2\cdots p_k)(2p_1p_2\cdots p_k)$. Since $2p_1 = 4 < 5 \le p_{k+1}$ and since $p_i < p_{k+i}$ for $i > 1$ we have the last expression less than $p_1p_2\cdots p_kp_{k+1}p_{k+2}\cdots p_{2k} < p_1\cdots p_n$, since $n - k \ge k$ implies $2k < n$. It remains to check the cases for $4 \le n < 10$. When $n = 9$, we have $p_{10}p_{11} = 29\cdot 31 = 899 < 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23 = 223092870$. When $n = 8$, we have $p_9p_{10} = 23\cdot 29 = 677 < 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19 = 9699690$. When $n = 7$, we have $p_8p_9 = 19\cdot 23 = 437 < 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17 = 510510$. When $n = 6$, we have $p_7p_6 = 17\cdot 19 = 232 < 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13 = 30030$. When $n = 5$, we have $p_6p_5 = 13\cdot 17 = 221 < 2310 = 2\cdot 3\cdot 5\cdot 7\cdot 11$. And when $n = 4$ we have $p_5p_6 = 11\cdot 13 = 143 < 2\cdot 3\cdot 5\cdot 7 = 210$, which completes all the cases.

3.2.31. First suppose $n \ge 8$. Note that by Bertrand's postulate we have $p_{n-1} < p_n < 2p_{n-1}$ and $p_{n-2} < p_{n-1} < 2p_{n-2}$. Therefore, $p_n^2 < (2p_{n-1})(2p_{n-1}) < (2p_{n-1})(4p_{n-2}) = 8p_{n-1}p_{n-2} = p_{n-1}p_{n-2}p_5 \le p_{n-1}p_{n-2}p_{n-3}$, since $n \ge 8$. All that remains is to check that the inequality it true for $n = 6$ and 7. When $n = 7$ we have $p_7^2 = 17^2 = 289 < 1001 = 13\cdot 11\cdot 7 = p_6p_5p_4$, and when $n = 6$ we have $p_6^2 = 13^2 = 169 < 385 = 11\cdot 7\cdot 5 = p_5p_4p_3$. This completes the proof. To see that the inequality does not hold for smaller n, we check that for $n = 5$, we have $p_5^2 = 11^2 = 121 > 7\cdot 5\cdot 3 = 105$ and when $n = 4$, we have $p_4^2 = 7^2 = 49 > 5\cdot 3\cdot 2 = 30$.

3.2.32. Let n be a sufficiently large integer and consider the sequence of primes $3 = p_2 < p_3 < \cdots < p_n$. Let S be the set $\{p_{i+1} - p_i | i = 2, \ldots, n-1\}$ of $n-2$ differences between successive primes up to p_n. Note that some differences may be repeated, and since each of the primes is odd, every difference is even. Suppose there are at least $[(n-2)/N]$ elements in S, then there is one difference of at least 2, another difference of at least 4, and so on up to $[(n-2)/N]$. Thus, $p_n - p_2 = (p_n - p_{n-1}) + (p_{n-1} - p_{n-2}) + \cdots + (p_3 - p_2) \ge 2 + 4 + \cdots 2[(n-2)/N] = 2([(n-2)/N]([(n-2)/N] + 1)/2$. Now the right hand side is asymptotic to n^2/M^2, but by the prime number theorem, the left hand side is asymptotic to $n \log n$, which is impossible. Therefore, there are less than $[(n-2)/N]$ elements in S. Since we must assign $n-2$ differences among less than $[(n-2)/N]$ values, at least one value, say K is assigned to more than N differences. Otherwise we would have less than $[(n-2)/N]N < n-2$ differences.

3.3. Greatest Common Divisors

3.3.1. a. The positive divisors of 15 are 1, 3, 5, and 15 and the positive divisors of 25 are 1, 5, and 25. Hence the greatest common divisor of 15 and 25 is 5.

b. Every positive integer is a divisor of 0. Hence the greatest common divisor of 0 and 111 is 111.

c. The positive divisor of -12 are 1, 2, 3, 4, 6, and 12 and the positive divisors of 18 are 1, 2, 3, 6, 9, and 18. Hence the greatest common divisor of -12 and 18 is 6.

d. No positive integer greater than 1 can divide 99 and 100 since any common divisor of 99 and 100 divides $100 - 99 = 1$. Hence the greatest common divisor of 99 and 100 is 1.

e. The positive divisors of 11 are 1 and 11 and the positive divisors of 121 are 1, 11, and 121. Hence the greatest common divisor of 11 and 121 is 11.

f. A common divisor of 100 and 102 is also a divisor of $102 - 100 = 2$. Since 2 is a common divisor of 100 and 102, 2 is the greatest common divisor of these integers.

3.3.2. a. The positive divisors of 5 are 1 and 5 and the positive divisors of 15 are 1, 5, and 15. Therefore, the greatest common divisor of 5 and 15 is 5.

b. Every positive integer is a divisor of 0. Hence the greatest common divisor of 0 and 100 is 100.

c. The positive divisors of -27 are 1, 3, 9, and 27. The positive divisors of -45 are 1, 3, 5, 9, 15, and 45. It follows that the greatest positive divisor of -27 and -45 is 3.

d. The greatest common divisor of -90 and 100 will also divide their sum, 10. As 10 divides -90 and 100, the greatest common divisor of -90 and 100 is 10.

e. The positive divisors of 121 are 1, 11, and 121. Of these, 11 and 121 do not divide 100. Hence the greatest common divisor of 100 and 121 is 1.

f. The positive divisors of 289 are 1, 17, and 289. As neither 17 nor 289 divide 1001, the greatest common divisor of 289 and 1001 is 1.

3.3.3. The greatest common divisor of a and $2a$ is also a divisor of their difference, $2a - a = a$. As a divides both a and $2a$, the greatest common divisor of a and $2a$ is a.

3.3.4. Since a is a common divisor of a and a^2 and a can have no divisor larger than itself, we have $(a, a^2) = a$.

3.3.5. As $(a+1, a)$ is the least positive linear combination of a and $a+1$, it is clear that $(a+1, a) \leq (a+1)-a = 1$. It follows that $(a + 1, a) = 1$.

3.3.6. A common divisor of a and $a + 2$ is also a divisor of $(a + 2) - 2 = 2$. Hence if a is even, the greatest common divisor of a and $a + 2$ is 2, since 2 does divide both of these integers, while if a is odd, then the greatest common divisor of a and $a + 2$ is 1.

3.3.7. By Theorem 3.8, $(ca, cb) = cma + cnb = |c| \cdot |ma + nb|$, where $cma + cnb$ is as small as possible. Therefore, $|ma + nb|$ is as small a positive integer as possible, i.e. equal to (a, b).

3.3.8. Suppose that $d \mid (a + b)$ and $d \mid (a - b)$. Then $d \mid ((a + b) + (a - b)) = 2a$ and $d \mid ((a + b) - (a - b)) = 2b$. Note that by Exercise 5 $(2a, 2b) = 2(a, b) = 2$. Since d is a common divisor of $2a$ and $2b$ it follows that $d \mid 2$. Hence either $d = 1$ or $d = 2$. Moreover, if one of a and b is even and the other odd, then both $a + b$ and $a - b$ are odd, so that $(a + b, a - b) = 1$. If both a and b are odd then both $a + b$ and $a - b$ are even, so that $(a + b, a - b) = 2$.

3.3.9. Let p be a prime dividing $(a^2 + b^2, a + b)$. Then $p \mid (a + b)^2 - (a^2 + b^2) = 2ab$. Now if $p \mid a$, then $p \mid b$ since $p \mid a + b$. But $(a, b) = 1$, so $p \nmid a$. Similarly, $p \nmid b$. Therefore $p \mid 2$ and so $p = 1$ or $p = 2$. If a and b have the same parity, then $2|a + b$ and $2|a^2 + b^2$, and so $(a^2 + b^2, a + b) = 2$. But if a and b have opposite parity, then $a + b$ and $(a^2 + b^2, a + b) = 1$.

3.3.10. Let the least positive linear combination of a and b be $(a, b) = an + bm$. Now, $an + bm = (a/2)(2n) + (b/2)(2m) = 2((a/2)n+(b/2)m) \geq 2(a/2, b/2)$. To see the reverse inequality, expand $(a/2, b/2)$ as a smallest positive linear combination and proceed similarly. As $(a, b) \leq 2(a/2, b/2)$ and $(a, b) \geq 2(a/2, b/2)$, we see that $(a, b) = 2(a/2, b/2)$.

3.3.11. Let $a = 2k$. Since $(a, b) \mid b$, and b is odd, (a, b) is odd. But $(a, b) \mid a = 2k$. Thus $(a, b) \mid k$. So $(a, b) = (k, b) = (\frac{a}{2}, b)$.

3.3.12. As $c \mid (a + b), a + b = cn$ for some n. It can be seen from this that any common divisor of a and c is also a divisor of b, hence of (a, b). Similarly, $(b, c) = 1$.

3.3.13. Let $d = (a, b)$. Then $(a/d, b/d) = 1$, so if $g|a/d$, then $(g, b/d) = 1$. In particular, if we let $e = (a/d, bc/d)$, then $e|a/d$, so $(e, b/d) = 1$, so we must have $e|c$. Since $e|a/d$, then $e|a$, so $e|(a, c)$. Conversely, if $f = (a, c)$, then $(f, b) = 1$, so $(d, f) = 1$, so $f|a/d$ and trivially, $f|bc/d$. Therefore $f|e$, whence $e = f$. Then $(a, b)(a, c) = de = d(a/d, bc/d) = (a, bc)$.

3.3.14. a. By Theorem 3.8 there are integers u,v,r, and s such that $1 = ua + vb = ra + sc$. Multiplication of $ua + vb$ and $ra + sc$ shows that $1 = (uva + usc + vbr)a + (vs)bc$. Hence by Theorem 2.2 it follows that $(a, bc) = 1$.

b. Suppose that $(a_i, b) = 1$ for $i = 1, 2, \ldots, n$. Let $A_i = \prod_{j=1}^{i} a_j$. We wish to prove that $(A_n, b) = 1$. We use mathematical induction. The basis case, $n = 2$ was shown in part (a). For the inductive step, assume that $(A_i, b) = 1$. Then since $(a_{i+1}, b) = 1$, part (a) implies that $(A_{i+1}, b) = 1$ since $A_{i+1} =$

$A_i a_{i+1}$.

3.3.15. Let p, q, r be prime numbers. The set $\{pq, qr, pr\}$ is a set of three integers that are mutually relatively prime, but no two of which are relatively prime.

3.3.16. We can take $30, 42, 70$, and 105. We find these by taking all products of three different primes in the set $\{2, 3, 5, 7\}$. We have $(30, 42, 70, 105) = 1$, but $(30, 42, 70) = 2$, $(30, 70, 105) = 5$, $(30, 42, 105) = 3$, $(42, 70, 105) = 7$.

3.3.17. a. We have $(8, 10, 12) = 2$.

b. We have $(5, 25, 75) = 5$.

c. We have $(99, 9999, 0) = 99$.

d. We have $(6, 15, 21) = 3$.

e. We have $(-7, 28, -35) = 7$.

f. We have $(0, 0, 1001) = 1001$

3.3.18. We have (66,105,42) = 3, (66,105,70) = 1, (66,105,165) = 3, (66,42,70) = 2, (66,42,165) = 3, (66,70,165) = 1, (105,42,70) = 7, (105,70,165) = 5, and (42,70,165) = 1. Hence there are three sets of mutually relatively prime integers in this set, namely $\{66, 105, 70\}$, $\{66, 70, 105\}$,and $\{42, 70, 165\}$.

3.3.19. Let $d \mid a_i, 1 \leq i \leq n$. Then clearly $dc \mid ca_i, 1 \leq i \leq n$. So $dc \mid (ca_1, ca_2, \ldots, ca_n)$. To see the other direction, note that $c \mid ca_i$ for all i, so $c \mid (ca_1, ca_2, \ldots, ca_n) = d$. Express d as $d = cd'$, where d' is as great as possible. But since $cd' \mid ca_i, d' \mid a_i$ and $d'|(a_1, a_2, \ldots, a_n)$, say $d'k = (a_1, a_2, \ldots, a_n)$. If $k > 1$, this contradicts the maximality of d', so we must have $d' = (a_1, a_2, \ldots, a_n)$.

3.3.20. We use induction on n. The basis step is done by Theorem 3.8. For the inductive step, we use Lemma 2.1. Thus $(a_1, \ldots a_n) = (a_1, \ldots, (a_{n-1}, a_n)) = m_1 a_1 + \cdots + m_{n-1}(a_{n-1}, a_n)$, by the inductive hypothesis. Now $m_1 a_1 + \cdots + m_{n-1}(a_{n-1}, a_n) = m_1 a_1 + \cdots + m_{n-1}(m_{n-1}^{'} a_{n-1} + m_n^{'} a_n) = m_1 a_1 + \cdots + m_{n-1} m_{n-1}^{'} a_{n-1} + m_{n-1} m_n^{'} a_n$. This completes the proof.

3.3.21. Suppose that $(6k + a, 6k + b) = d$. Then $d \mid b - a$. We have $a, b \in \{-1, 1, 2, 3, 5\}$, so if $a < b$ it follows that $b - a \in \{1, 2, 3, 4, 6\}$. Hence $d \in \{1, 2, 3, 4, 6\}$. To show that $d = 1$ it is sufficient to show that neither 2 nor 3 divides $(6k + a, 6k + b)$. If $p = 2$ or $p = 3$ and $p \mid (6k + a, 6k + b)$ then $p \mid a$ and $p \mid b$. However, there are no such pairs a, b in the set $\{-1, 1, 2, 3, 5\}$.

3.3.22. We have $5(3k + 2) - 3(5k + 3) = 1$, so that by Theorem 3.8,$3k + 2$ and $5k + 3$ are relatively prime.

3.3.23. We proceed with the Euclidean algorithm. $8a + 3 = 1(5a + 2) + (3a + 1)$. $5a + 2 = 1(3a + 1) + (2a + 1)$. $3a + 1 = 1(2a + 1) + (a)$. $2a + 1 = 2(a) + (1)$. Therefore $(8a + 3, 5a + 2) = 1$.

3.3.24. Let $d = (a + 2b, 2a + b)$. Then $d \mid 2(a + 2b) - (2a + b) = 3b$. Likewise, $d \mid 3a$. Hence, $d \mid (3a, 3b) = 3(a, b) = 3$. Therefore, $d = 1$ or 3.

3.3.25. From Exercise 21, we know that $6k - 1, 6k + 1, 6k + 2, 6k + 3$, and $6k + 5$ are pairwise relatively prime. To represent n as the sum of two relatively prime integers greater than one, let $n = 12k + h, 0 \leq h < 12$. We now examine the twelve cases, one for each possible value of h, in the following chart:

h	n
0	$(6k-1)+(6k+1)$
1	$(6k-1)+(6k+2)$
2	$(6k-1)+(6k+3)$
3	$(6k+1)+(6k+2)$
4	$(6k+1)+(6k+3)$
5	$(6k+2)+(6k+3)$
6	$(6k+1)+(6k+5)$
7	$(6k+2)+(6k+5)$
8	$(6k+3)+(6k+5)$
9	$(12k+7)+2$
10	$(12k+7)+3$
11	$(12k+9)+2$

3.3.26. The Farey series of order 7 is $\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$.

3.3.27. Let S be the set of all fractions $P/Q = (xa+ye)/(xb+yf)$ where x, y are relatively prime positive integers. Then every element of S lies between a/b and e/f and is in lowest terms. The first element of S to appear in a Farey series will have the smallest Q, i.e. $x = y = 1$. This fraction must be c/d by hypothesis.

3.3.28. Let x_0, y_0 be a solution to the Diophantine equation $bx - ay = 1$. Then $x = x_0 + at, y = y_0 + bt$ is a solution for any integer t. Choose t_0 so that $n - b < y_0 + bt_0 \leq n$. Then $x = x_0 + at_0, y = y_0 + bt_0$ is a solution such that $(x, y) = 1$ and $0 \leq n - b < y \leq n$. Since x/y is in lowest terms, and $y \leq n$, it is a fraction of the nth Farey series. Also

$$\frac{x}{y} = \frac{a}{b} + \frac{1}{by} > \frac{a}{b},$$

so that x/y comes later than a/b in the series. If it is not c/d, then it comes later in the series than c/d, and

$$\frac{x}{y} - \frac{c}{d} = \frac{dx - cy}{dy} \geq \frac{1}{dy},$$

and

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd} \geq \frac{1}{bd}.$$

Hence

$$\frac{1}{by} = \frac{bx - ay}{by} = \frac{x}{y} - \frac{a}{b} \geq \frac{1}{dy} + \frac{1}{bd} = \frac{b+y}{bdy} > \frac{n}{bdy} \geq \frac{1}{by},$$

which is a contradiction. Therefore, x/y must be c/d and $bc - ad = 1$.

3.3.29. Since $a/b < (a+c)/(b+d) < c/d$, we must have $b + d > n$, or a/b and c/d would not be consecutive, since otherwise, $(a+c)/(b+d)$ would have appeared in the Farey series of order n.

3.3.30. a. Let $c = a - b$. We may then write $(a^n - b^n)/(a-b)$ as $((b+c)^n - b^n)/c$. The binomial theorem shows that this is $nb^{n-1} + k \cdot c$. Thus $(((b+c)^n - b^n)/c, c)$ divides (nb^{n-1}, c). Rewriting $(a^n - b^n)/(a-b)$ as $(a^n - (a-c)^n)/c$ shows that $(((b+c)^n - b^n)/c, c)$ divides (na^{n-1}, c). Therefore $(((b+c)^n - b^n)/c, c)$ divides $(n(a,b)^{n-1}, c)$. These expansions also make it clear that $(n(a,b)^{n-1}, c)$ is a divisor of $(a^n - b^n)/(a-b)$.

b. If a and b are relatively prime then $(a, b) = 1$. Apply part (a).

3.3.31. Since $(a/b) + (c/d) = (ad+bc)/bd$ is an integer, $bd \mid ad+bc$. Certainly, then, $bd \mid d(ad+bc) = ad^2 + cbd$. Now since $bd \mid cbd$, it must be that $bd \mid ad^2$. From this, $bdn = ad^2$ for some integer n, and it follows that $bn = ad$, or $b \mid ad$. Since $(a, b) = 1$, we must have $b \mid d$. Similarly, we can find that $d \mid b$ hence, $b = d$.

3.3.32. We can conclude that $b = 1$, and $a = c = 1$ or 2. To see this, note that as $(1/a) + (1/b) + (1/c) = (bc + ac + ab)/abc$ is an integer, $abc \mid bc + ac + ab$. Continuing as in Exercise 29, $abc \mid c(bc + ac + ab) = abc + (bc^2 + ac^2)$. Now, we have that $abc \mid bc^2 + ac^2 = c(bc + ac)$, or equivalently (as $c \neq 0$) $ab \mid bc + ac$. But, $b \mid ab$ and $ab \mid bc + ac$, from which it follows that $b \mid ac$. Using Exercise 11, we can now see that $b \mid$

$(a,b)(c,b) = 1 \cdot 1 = 1$, and so $b = 1$. Now, if $(1/a) + (1/b) + (1/c)$ is an integer, then so is $(1/a) + (1/c)$. We now have the situation of Exercise 29, and so $a = c$. And $(1/a) + (1/c) = 2/a$ is an integer only if $a \mid 2$, i.e. when $a = 1$ or 2.

3.3.33. Consider the lattice points inside or on the triangle with vertices $(0,0), (a,0)$, and (a,b). Note that a lattice point lies on the diagonal from $(0,0)$ to (a,b) if and only if $[bx/a]$ is an integer. Let $d = (a,b)$ and $a = cd$, so that $(c,b) = 1$. Then $[bx/a]$ will be an integer exactly when x is a multiple of c, since then $d|b$ and $c|x$ so then $a = cd|bx$. But there are exactly d multiples of c less than or equal to a since $cd = a$, so there are exactly $d+1$ lattice points on the diagonal when we count $(0,0)$ also. So one way to count the lattice points in the triangle is to consider the rectangle which has $(a+1)(b+1)$ points and divide by 2. But we need to add back in half the points on the diagonal, which gives us $(a+1)(b+1)/2+((a,b)+1)/2$ total points in or on the triangle. Another way to count all the points is to count each column above the horizontal axis, starting with $i = 1, 2, \ldots, a-1$. The equation of the diagonal is $y = (b/a)x$, so for a given i, the number of points on or below the diagonal is $[bi/a]$. So the total number of interior points in the triangle plus the points on the diagonal is $\sum_{i=1}^{a-1}[bi/a]$. Then the right-hand boundary has b points (not counting $(a,0)$) and the lower boundary has $a+1$ points, counting $(0,0)$. So in all, we have $\sum_{i=1}^{a-1}[bi/a] + a + b + 1$ points in or on the triangle. If we equate our two expressions and multiply through by 2 we have $(a+1)(b+1) + (a,b) + 1 = 2\sum_{i=1}^{a-1}[bi/a] + 2a + 2b + 2$ which simplifies to our expression.

3.3.34. Let $k = j - i$, then $(n!i+1, n!j+1) = (n!i+1, n!(i+k)+1) = (n!i+1, n!i+1+n!k) = (n!i+1, n!k)$ by Theorem 3.7. But none of the divisors of $n!k$ can divide $n!i+1$, so this last greatest common divisor is equal to 1, as desired.

3.3.35. Assume there are exactly r primes and consider the $r+1$ numbers $(r+1)!+1$. From Lemma 3.1, each of these numbers has a prime divisor, but from Exercise 34, these numbers are pairwise relatively prime, so these prime divisors must be unique, so we must have at least $r+1$ different prime divisors, a contradiction.

3.3.36. First we prove by induction that $(a_i, d) = 1$ for all i. Since $a_0 = c$ and $(c,d) = 1$ by hypothesis, the basis step holds. Now suppose that $(a_i, d) = 1$ for $i = 0, 1, \ldots, k$. Then by Theorem 3.7, we have $(a_{k+1}, d) = (a_0 a_1 \cdots a_k + d, d) = (a_0 a_1 \cdots a_k, d)$, and since d is relatively prime to every factor in $a_0 a_1 \cdots a_k$, we have that $(a_{k+1}, d) = 1$, which completes the induction. Now let $i < j$, and consider $(a_i, a_j) = (a_i, a_0 a_1 \cdots a_i \cdots a_{j-1} + d) = (a_i, d)$ since we can subtract the multiple of a_i from the right side by Theorem 3.7. This last is equal to 1 from our work above, which proves the proposition.

3.4. The Euclidean Algorithm

3.4.1. a. We have $75 = 1 \cdot 45 + 30, 45 = 1 \cdot 30 + 15, 30 = 2 \cdot 15 + 0$, so $(45, 75) = 15$.

b. We have $222 = 2 \cdot 102 + 18, 102 = 5 \cdot 18 + 12, 18 = 1 \cdot 12 + 6, 12 = 2 \cdot 6 + 0$, so $(222, 102) = 6$.

c. We have $1414 = 2 \cdot 666 + 82, 666 = 8 \cdot 82 + 10, 82 = 8 \cdot 10 + 2, 10 = 5 \cdot 2 + 0$, so $(1414, 666) = 2$.

d. We have $44350 = 2 \cdot 20785 + 2780, 20785 = 7 \cdot 2780 + 1325, 2780 = 2 \cdot 1325 + 130, 1325 = 10 \cdot 130 + 25, 130 = 5 \cdot 25 + 5, 25 = 5 \cdot 5 + 0$, so $(44350, 2780) = 5$.

3.4.2. a. We have $87 = 1 \cdot 51 + 36, 51 = 1 \cdot 36 + 15, 36 = 2 \cdot 15 + 6, 15 = 2 \cdot 6 + 3, 6 = 2 \cdot 3$, so $(51, 87) = 3$.

b. We have $300 = 2 \cdot 105 + 90, 105 = 1 \cdot 90 + 15, 90 = 6 \cdot 15$ so $(105, 300) = 15$.

c. We have $1234 = 1 \cdot 981 + 253, 981 = 3 \cdot 253 + 222, 253 = 1 \cdot 222 + 31, 222 = 7 \cdot 31 + 5, 31 = 6 \cdot 5 + 1$, so $(981, 1234) = 1$.

d. We have $100313 = 2 \cdot 34709 + 30895, 34709 = 1 \cdot 30895 + 3814, 30895 = 8 \cdot 3814 + 383, 3814 = 9 \cdot 383 + 367, 383 = 1 \cdot 367 + 16, 367 = 22 \cdot 16 + 15, 16 = 1 \cdot 15 + 1$, so $(34709, 100313) = 1$.

3.4.3. a. We have $q_1 = 1, q_2 = 1, q_3 = 2$, so $s_0 = 1, s_1 = 0, s_2 = s_0 - q_1 s_1 = 1, s_3 = s_1 - q_2 s_2 = -1$ and $t_0 = 0, t_1 = 1, t_2 = t_0 - q_1 t_1 = -1, t_3 = t_1 - q_2 t_2 = 2$. Thus, $(75, 45) = (-1)75 + (2)45$.

b. We have $q_1 = 2, q_2 = 5, q_3 = 1$, so $s_0 = 1, s_1 = 0, s_2 = 1, s_3 = -5, s_4 = 6$ and $t_0 = 0, t_1 = 1, t_2 = -2, t_3 = 11, t_4 = -13$. Thus $(222, 102) = (6)222 + (-13)102$.

c. We have, from Exercise 1(c), that $2 = 82 - 8 \cdot 10 == (1414 - 2 \cdot 666) - 8(666 - 8 \cdot 82) = 1414 - 10 \cdot 666 + 64(1414 - 2 \cdot 666) = -138(666) + (65)1414$.

d. We have, from Exercise 1(d), that $5 = 130 - 5 \cdot 25 = (2780 - 2 \cdot 1325) - 5(1325 - 10 \cdot 130) = (44350 - 2 \cdot 20785) - 7(20785 - 7 \cdot 2780) + 50(2780 - 2 \cdot 1325) = 44350 - 9 \cdot 20785 + 99(44350 - 2 \cdot 20785) - 100(20785 - 7 \cdot 2780) = 100 \cdot 44350 - 307 \cdot 20785 - 7(44350 - 2 \cdot 20785) = -1707(20785) + 800(44350)$.

3.4.4. a. We have, from Exercise 2(a), that $3 = 15 - 2 \cdot 6 = (51 - 36) - 2(36 - 2 \cdot 15) = 51 - 3(87 - 51) + 4(51 - 36) = 8(51) - 3(87) - 4(87 - 51) = 12(51) - 7(87)$.

b. We have, from Exercise 2(b), that $15 = 105 - 90 = 105 - (300 - 2 \cdot 105) = 3(105) - 1(300) = 15$.

c. We have, from Exercise 2(c), that $1 = 31 - 6 \cdot 5 = (253 - 222) - 6(222 - 7 \cdot 31) = (1234 - 981) - 7(981 - 3 \cdot 253) + 42(253 - 222) = 1234 - 8(981) + 63(1234 - 981) - 42(981 - 3 \cdot 235) = 64(1234) - 113(981) + 126(1234 - 981) = -239(981) + 190(1234)$.

d. We have, from Exercise 2(d), that $1 = 16 - 15 = (383 - 367) - (367 - 22 \cdot 16) = (30895 - 8 \cdot 3814) - 2(3814 - 9 \cdot 383) + 22(383 - 367) = (100313 - 2 \cdot 34709) - 10(34709 - 30895) + 40(30895 - 8 \cdot 3814) - 22(3814 - 9 \cdot 383) = 100313 - 12(34709) + 50(100313 - 2 \cdot 34709) - 342(34709 - 30895) + 198(30895 - 8 \cdot 3814) = 51(100313) - 454(34709) + 540(100313 - 2 \cdot 34709) - 1584(34709 - 30895) = 591(100313) - 3118(34709) + 1584(100313 - 2 \cdot 34709) = -6286(34709) + 2175(100313)$.

3.4.5. a. We have $(6, 10, 15)((6, 10), 15) = (2, 15) = 1$.

b. We have $(70, 98, 105) = (70, (98, 105)) = (70, (98, 105 - 98)) = (70, (98, 7)) = (70, 7) = 7$.

c. We have $(280, 330, 405, 490) = (10(28, 33), 5(81, 98)) = (10, 5) = 5$.

3.4.6. a. We have $(15, 35, 90) = ((15, 35), 90) = (5, 90) = 5$.

b. We have $(300, 2160, 5040) = 20(15, 108, 252) = 20((15, 108), 252) = 20(3, 252) = 20 \cdot 3 = 60$.

c. We have $(1240, 6660, 15540, 19980) = 20((62, 333), (777, 999)) = 20(1, 111) = 20$.

3.4.7. a. Since $(6, 10) = 2 = 2 \cdot 6 - 10$, we have $1 = (6, 10, 15) = (2, 15) = 8 \cdot 2 - 15 = 8(2 \cdot 6 - 10) - 15 = 16 \cdot 6 - 8 \cdot 10 - 15$.

b. Since $(70, 98) = 14 = 3 \cdot 70 - 2 \cdot 98$, we have $7 = (70, 98, 105) = (14, 105) = 105 - 7 \cdot (14) = 105 - 7(3 \cdot 70 - 2 \cdot 98) = 105 - 21 \cdot 70 + 14 \cdot 98$.

c. Since $(280, 330) = 10 = 17 \cdot 330 - 20 \cdot 280$, and $(405, 490) = 5 = -75 \cdot 405 + 62 \cdot 490$, we have $(280, 330, 405, 490) = 5 = 0 \cdot 280 + 0 \cdot 330 - 75 \cdot 405 + 62 \cdot 490$.

3.4.8. a. Since $(15, 35) = 5 = -2 \cdot 15 + 35$, we have $(15, 35, 90) = 5 = -2 \cdot 15 + 1 \cdot 35 + 0 \cdot 90$.

b. Since $(300, 2160) = 60 = -7 \cdot 300 + 2160$, we have $(300, 2160, 5040) = 60 = -7 \cdot 300 + 1 \cdot 2160 + 0 \cdot 5040$.

c. We can write $20 = 188 \cdot 1240 - 35 \cdot 6660 + 0 \cdot 15540 + 0 \cdot 19980$, since $(1240, 6660) = 20 = 188 \cdot 1240 - 35 \cdot 6660$.

3.4.9. Applying the reductions in the algorithm we find that $(2106, 8318) = 2(1053, 4159) = 2(3106, 1053) = 2(1553, 1053) = 2(500, 1053) = 2(250, 1053) = 2(125, 1053) = 2(125, 928) = 2(125, 464) = 2(125, 232) = 2(125, 116) = 2(125, 58) = 2(125, 29) = 2(96, 29) = 2(48, 29) = 2(24, 29) = 2(12, 29) = 2(6, 29) = 2(3, 29) = 2(3, 26) = 2(3, 13) = 2(3, 10) = 2(3, 5) = 2(3, 2) = 2(3, 1) = 2(2, 1) = 2(1, 1) = 2$.

3.4.10. Since $(a, b) = (\pm a, \pm b)$, we assume a and b to be always positive. The exercise then follows from Exercises 8 and 9 from Section 3.3, and Theorem 3.7 with $c = -1$. The algorithm terminates since the magnitude of the two arguments' sum is always decreasing and positive.

3.4.11. The algorithm stops after $2n - 2$ steps. To prove this we use mathematical induction. When $n = 2$, $a = 1$ and $b = 2$. The first step leaves $a = 1$ and $b = 1$, and the second step will find the g.c.d.. Thus, the basis step holds. For the inductive hypothesis, we assume that the algorithm uses $2n - 2$ steps to find the g.c.d.. of $(2^n - (-1)^n)/3$ and $(2(2^{n-1} - (-1)^{n-1})/3$. To find the g.c.d. of $(2^{n+1} - (-1)^{n+1})/3$ and $(2(2^n - (-1)^n))/3$, the first step reduces this to the g.c.d. of $(2^{n+1} - (-1)^{n+1})/3$ and $(2^n - (-1)^n)/3$. The next step, as neither of these numbers is even, gives us $(2^n - (-1)^n)/3$ and $(1/3)(2^{n+1} - (-1)^{n+1} - 2^n + (-1)^n) = (1/3)(2^n + 2(-1)^n) = (2/3)(2^{n-1} - (-1)^{n-1})$. By the inductive hypothesis, the algorithm will take $2n - 2$ more steps, for a total of $2n = 2(n - 1) - 2$ steps.

3.4.12. Let $S(a, b)$ be the number of subtractions needed to find (a, b) using this algorithm. Then $S(a, b) = S(b, a)$, and if a is even, $S(a, b) = S(a/2, b)$, so we may assume that both a and b are odd, and $a \geq b$. We proceed by induction on a. Note that $S(b, b) = 1 \leq 1 + [\log_2 \max(b, b)]$. Now suppose that $S(c, b) \leq 1 + [\log_2 \max(c, b)]$ for all $c = b, b + 1, b + 2, \ldots, a - 1$. Since a and b are odd, the first step of the algorithm will be $(a, b) = (a - b, b)$, then since b is odd and $a - b$ is even, the next step will be $(a - b, b) = ((a - b)/2, b)$. So $S(a, b) = 1 + S((a - b)/2, b) \leq 1 + [\log_2 \max((a - b)/2, b)] \leq 1 + [\log_2(a/2 + b/2)] = 1 + [\log_2(a + b)/2] \leq 1 + [\log_2 \max(a, b)]$, which completes the induction step.

3.4.13. Suppose we have the balanced ternary expansions for integers $a \geq b$. If both expansions end in zero, then both are divisible by 3, and we can divide this factor of 3 out by deleting the trailing zeros (a shift) in which case $(a, b) = 3(a/3, b/3)$. If exactly one expansion ends in zero, then we can divide the factor of 3 out by shifting, and we have $(a, b) = (a/3, b)$, say. If both expansions end in 1 or in -1, we can subtract the larger from the smaller to get $(a, b) = (a - b, b)$, say, and then the expansion for $a - b$ ends in zero. Finally, if one expansion ends in 1 and the other in -1, then we can add the two to get $(a + b, b)$, where the expansion of $a + b$ now ends in zero. Since $a + b$ is no larger than $2a$ and since we can now divide $a + b$ by three, the larger term is reduced by a factor of at least $2/3$ after two steps. Therefore this algorithm will terminate in a finite number of steps, when we finally have $a = b = 1$.

3.4.14. We have $384 = 2 \cdot 226 - 68, 226 = 3 \cdot 68 + 14, 68 = 5 \cdot 14 - 2, 14 = 7 \cdot 2$. Hence $(384, 226) = 2$.

3.4.15. Lemma: If c and d are integers and $c = dq \pm r$ where q and r are integers, then $(c, d) = (d, r)$.

Proof of Lemma: If an integer e divides both c and d, then since $r = \pm(c - dq)$, Theorem 1.8 shows that $e \mid r$. If $e \mid d$ and $e \mid r$, then since $c = dq + r$, from Theorem 1.8 we see that $e \mid c$. Since the common divisors of c and d are the same as the common divisors of d and r, we see that $(c, d) = (d, r)$.

Proof of proposition: Let $r_0 = a$ and $r_1 = b$ be positive integers with $a \geq b$. By successively applying the least-remainder division algorithm, we find that

$$\begin{aligned} r_0 &= r_1 q_1 + e_2 r_2, \quad \frac{-r_1}{2} < e_2 r_2 \leq \frac{r_1}{2} \\ &\vdots \\ r_{n-2} &= r_{n-1} q_{n-1} + e_n r_n, \quad \frac{-r_{n-1}}{2} < e_n r_n \leq \frac{r_{n-1}}{2} \\ r_{n-1} &= r_n q_n. \end{aligned}$$

We eventually obtain a remainder of zero since the sequence of remainders $a = r_0 > r_1 > r_2 > \cdots \geq 0$ cannot contain more than a terms. By the Lemma we see that $(a, b) = (r_0, r_1) = (r_1, r_2) = \cdots = (r_{n-2}, r_{n-1}) = (r_{n-1}, r_n) = (r_n, 0) = r_n$. Hence $(a, b) = r_n$, the last nonzero remainder.

3.4.16. Let $E(a,b)$ be the number of steps to find (a,b) with the Euclidean algorithm, and $L(a,b)$ the number of steps to find (a,b) with the least-remainder algorithm. Note that if the first step of the Euclidean algorithm produces r_2 (with $r_0 = a$ and $r_1 = b$), then $E(a,b) = 1 + E(b,r_2)$. For this to work with $r_2 = 0$, we define $E(a,0) = 0$. Similarly, for the least-remainder algorithm, $L(a,b) = 1 + L(b,r_2)$ with $L(a,0) = 0$. Following the hint, we prove that $L(a,b) \le L(a,a-b)$ if a and b are positive integers with $2b < a$. We use mathematical induction on b. Clearly it is true for $b = 1$, since $L(a,1) = 1 \le L(a,a-1)$. So we can assume $L(a',b') \le L(a',a'-b')$ for all positive integers a', b' with $2b' < a'$ and $b' < b$. Consider the first step of the least-remainder algorithm for $(a,a-b)$. We can write $a = (a-b) + b$, giving $r_2 = b$ if $b \le (a-b)/2$, i.e. $a \ge 3b$, or $a = 2(a-b) - (a-2b)$, giving $r_2 = a - 2b$ if $a - 2b < (a-b)/2$, i.e. $a < 3b$. Thus if $a \ge 3b$, we get $L(a,a-b) = 1 + L(a-b,b)$. But $L(a-b,b) = L(a,b)$ (the remainders after division by b are the same), so in this case $L(a,b) < L(a,a-b)$. Now suppose $2b < a < 3b$. We have $L(a,a-b) = 1 + L(a-b,a-2b)$. Consider the first step of the least-remainder algorithm for (a,b). We can write $a = 2b + (a-2b)$, giving $r_2 = a - 2b$ if $a - 2b \le b/2$, i.e. $a \le 5b/2$, or $a = 3b - (3b-a)$, giving $r_2 = 3b - a$ if $3b - a < b/2$, i.e. $a > 5b/2$. If $2b < a \le 5b/2$ we have $L(a,b) = 1 + L(b,a-2b)$. But $L(a-b,a-2b) = L(b,a-2b)$ since $a - b = b + (a-2b)$. So in this case $L(a,b) = L(a,a-b)$. Finally, if $5b/2 < a < 3b$ we have $L(a,b) = 1 + L(b,3b-a)$. We need to show that $L(b,3b-a) \le L(a-b,a-2b) = L(b,a-2b)$. But this is $L(a',b') \le L(a',a'-b')$ with $a' = b$ and $b' = 3b - a$. Note that $a' > 2b'$ (i.e. $a > 5b/2$) and $b' < b$ (i.e. $a > 2b$), so this is true by the induction hypothesis. This completes the proof of the hint. Now, to prove that $L(a,b) \le E(a,b)$, we will again use induction on b. For $b = 1$ we have $L(a,1) = 1 = E(a,1)$. So we can assume $L(a',b') \le E(a',b')$ for all positive integers a', b' with $b' < b$. Consider the first step of the Euclidean algorithm for (a,b): $a = qb + r$ where $0 \le r < b$, and $E(a,b) = 1 + E(b,r)$. Now if $r \le b/2$, this is also the first step of the least-remainder algorithm, i.e. $L(a,b) = 1 + L(b,r) \le 1 + E(b,r) = E(a,b)$ by the induction hypothesis. On the other hand, if $r > b/2$, the first step of the least-remainder algorithm is $a = (q+1)b - (b-r)$ with $b - r < b/2$, and $L(a,b) = 1 + L(b,b-r)$. But since $2(b-r) < b$, the result of the hint says that $L(b,b-r) \le L(b,r)$. So, again using the induction hypothesis, $L(a,b) \le 1 + L(b,r) \le 1 + E(b,r) = E(a,b)$, as desired.

3.4.17. Let $v_2 = v_3 = 2$, and for $i \ge 4, v_i = 2v_{i-1} + v_{i-2}$. Thus the least remainder algorithm will proceed with $e_i = 1$ and $q_i = 2$ for all i. To prove this we use induction. It clearly requires one division in the least-remainder division algorithm to find the g.c.d. of v_2 and v_3. This completes the basis step. For the induction hypothesis, we assume that it takes n steps to find the g.c.d. of v_{n+1} and v_{n+2}. To find the g.c.d. of v_{n+2} and v_{n+3}, the first step will be: $v_{n+3} = 2v_{n+2} + v_{n+1}$ by the definition of our v_i's. From this point, the algorithm will look identical to that for v_{n+1} and v_{n+2}. By our induction hypothesis, this will require n more steps. Hence, the total number of steps is $n + 1$.

3.4.18. In the algorithm, starting with $a = r_0$ and $b = r_1$, we have, if $r_2 \ne 0$, $r_1 = q_2 r_2 + e_3 r_3 \ge 2r_2 + r_3$, (since $r_1 \ge 2r_2$ and $r_2 \ge 2r_3$, so $q_2 = r_1/r_2 - e_3 r_3/r_2 \ge 3/2$, and hence $q_2 \ge 2$, and if $q_2 = 2$ then e_3 must be $+1$ and if $q_2 > 2$ then $r_1 \ge 3r_2 - r_3 \ge 2r_2 + r_3$). Iterating this, induction shows, if $r_j \ne 0$, then $r_1 \ge c_j r_j + c_{j-1} r_{j+1}$ where $c_1 = 1, c_2 = 2$ and $c_{j+2} = 2c_{j+1} + c_j$. In particular, if (a,b) takes at least n steps then $r_n \ge 1$ so $b \ge c_n$. We claim that $c_n \ge 10^{((3n-4)/8)}$. Thus if b has d digits, $b < 10^d$, then (a,b) must take fewer than n steps if $d \le (3n-4)/8$, i.e. if $n \ge (8d+4)/3$. To prove the claim, note first that $c_1 = 1 > 10^{(-1/8)}$ and $c_2 = 2 > 10^{(2/8)}$. If it is true for c_{j-2} and c_{j-1} then $c_j = 2c_{j-1} + c_{j-2} \ge 210^{((3j-7)/8)} + 10^{((3j-10)/8)} \ge 10^{((3j-4)/8)}$ since $210^{(-3/8)} + 10^{(-6/8)} = 1.021221\ldots > 1$, as desired.

3.4.19. Performing the Euclidean algorithm with $r_0 = m$ and $r_1 = n$, we find that $r_0 = r_1 q_1 + r_2, 0 \le r_2 < r_1, r_1 = r_2 q_2 + r_3, 0 \le r_3 < r_2, \ldots, r_{k-3} = r_{k-2} q_{k-2} + r_{k-1}, 0 \le r_{k-1} < r_{k-2}$, and $r_{k-2} = r_{k-1} q_{k-1}$. We have $(m,n) = r_{k-1}$. We will use these steps to find the greatest common divisor $a^m - 1$ and $a^n - 1$. First, we show that if u and v are positive integers, then the least positive residue of $a^u - 1$ modulo $a^v - 1$ is $a^r - 1$ where r is the least positive residue of u modulo v. To see this, note that $u = vq + r$ where r is the least positive residue of u modulo v. It follows that $a^u - 1 = a^{vq+r} - 1 = (a^v - 1)(a^{v(q-1)+r} + \cdots + a^{v+r} + a^r) + (a^r - 1)$. This shows that the remainder is $a^r - 1$ when $a^u - 1$ is divided by $a^v - 1$. Now let $R_0 = a^m - 1$ and $R_1 = a^n - 1$. When we perform the Euclidean algorithm starting with R_0 and R_1 we obtain $R_0 = R_1 Q_1 + R_2$, where $R_2 = a^{r_2} - 1, R_1 = R_2 Q_2 + R_3$ where $R_3 = a^{r_3} - 1, \ldots, R_{k-3} = R_{k-2} Q_{k-2} + R_{k-1}$ where $R_{k-1} = a^{r_{k-1}} - 1$. Hence the last nonzero remainder, $R_{k-1} = a^{r_{k-1}} - 1 = a^{(m,n)} - 1$ is the greatest common divisor of $a^m - 1$ and $a^n - 1$.

3.4.20. Suppose that $m > n$. Performing the Euclidean algorithm with $r_0 = m$ and $r_1 = n$, we find that $r_0 = r_1q_1 + r_2, 0 \leq r_2 < r_1, r_1 = r_2q_2 + r_3$, with $0 \leq r_3 < r_2, \ldots, r_{t-2} = r_{t-1}q_{t-1} + r_t$, with $0 \leq r_t < r_{t-1}$, and $r_{t-1} = r_tq_t$. We have $(m, n) = r_t$. We have $(f_m, f_n) = (f_{r_1q_1+r_2}, f_n)$. Using the result of Exercise 38 of Section 1.5 we have $f_{r_1q_1-1}f_{r_1q_1}f_{r_2+1}$. Since $f_{r_1} \mid f_{r_1q_1}$ it follows that $(f_m, f_n) = (f_{r_1q_1-1}f_{r_2}, f_{r_1})$. Hence $(f_m, f_n) = (f_{r_1q_1-1}, f_{r_1})(f_{r_2}, f_{r_1}) = (f_{r_2}, f_{r_1})$ since $f_{r_1} \mid f_{r_1q_1}$ and $(f_{r_1q_1-1}, f_{r_1q_1}) = 1$. Similarly, we can show that $(f_{r_{i-1}}, f_{r_{i-2}}) = (f_{r_i}, f_{r_{i-1}})$ for all i. It follows that $(f_m, f_n) = (f_{r_t}, f_{r_{t-1}})$. Since r_t is a divisor of r_{t-1} it follows that $f_{r_t} \mid f_{r_{t-1}}$. Hence $(f_{r_t}, f_{r_{t-1}}) = f_{r_t}$. Since $r_t = (m, n)$ it follows that $(f_m, f_n) = f_{(m,n)}$.

3.4.21. Note that $(x, y) = (x - ty, y)$, as any divisor of x and y is also a divisor of $x - ty$. So, every move in the game of Euclid preserves the g.c.d. of the two numbers. Since $(a, 0) = a$, if the game beginning with $\{a, b\}$ terminates, then it must do so at $\{(a, b), 0)\}$. Since the sum of the two numbers is always decreasing and positive, the game must terminate.

3.4.22. First, we show the hint. For convenience, let $g = (1 + \sqrt{5})/2$. If $y < x \leq yg$, then the move $\{x, y\}$ to $x - y, y$ is a legal move. But $x - 2y < x - gy \leq 0$, so there is only one legal move. In this case, we have, since $g^2 = g + 1$, that, $x \leq yg$, so $xg \leq y(g + 1)$ and hence $zg = (x - y)g \leq y$, as desired. Now if $a = b$, then the first player wins immediately. Suppose $a > bg$. Then let k be defined by $kb < a < (k + 1)b$. If $a - kb < b \leq (a - kb)g$, then the first player makes the move $\{a - kb, b\}$, which leaves the second player in the situation of the hint. Therefore, the second player has only one move, which puts the first player back into the situation with $a > bg$ again. If, on the other hand, $(a-kb)g < b$, then the first player makes the move $\{a-(k-1)b, b\}$, in which case, we have $bg > (a-kb)g^2 = (a-kb)(g+1) = (a-kb)g+(a-kb) > b+(a-kb) = a-(k-1)b$. Therefore, the second player is again put into the situation of the hint. Hence, a player in the position $a > bg$ can always force the other player to be in the situation in the hint.

3.4.23. Choose the integer m so that d has no more than m bits and that q has $2m$ bits, appending extra zeros to the front of q if necessary. Then $m = O(\log_2 q) = O(\log_2 d)$. Then from Theorems 2.7 and 2.5 we know that there is an algorithm for dividing q by d in $O(m^2) = O(\log_2 q \log_2 d)$ bit operations. Now let n be the number of steps needed in the Euclidean algorithm to find the greatest common divisor of a and b. Then by Theorem 3.12, $n = O(\log_2 a)$. Let q_i and r_i be as in the proof of Theorem 3.12. Then the total number of bit operations for divisions in the Euclidean algorithm is $\sum_{i=1}^{n} O(\log_2 q_i \log_2 r_i) = \sum_{i=1}^{n} O(\log_2 q_i \log_2 b) = O\left(\log_2 b \sum_{i=1}^{n} \log_2 q_i\right) = O\left(\log_2 b \log_2 \prod_{i=1}^{n} q_i\right)$. By dropping the remainder in each step of the Euclidean algorithm, we have the system of inequalities $r_i \geq r_{i+1}q_{i+1}$, for $i = 0, 1, \ldots, n - 1$. Multiplying these inequalities together yields $\prod_{i=0}^{n-1} r_i \geq \prod_{i=1}^{n} r_iq_i$. Cancelling common factors reduces this to $a = r_0 \geq r_n \prod_{i=1}^{n} q_i$. Therefore, from above we have that the total number of bit operations is $O\left(\log_2 b \log_2 \prod_{i=1}^{n} q_i\right) = O(\log_2 b \log_2 a) = O((\log_2 a)^2)$.

3.4.24. a. From the recursion relation, we have $r_jq_j = r_{j-1} - r_{j+1}$ for $1 \leq j \leq n$, so $\sum_{j=1}^{n} r_jq_j = (r_0 - r_2) + (r_1 - r_3) + \cdots + (r_{n-2} - r_n) + (r_{n-1} - r_{n+1}) = r_0 + r_1 - r_n - r_{n+1} = a + b - (a, b)$, where we notice that the second sum is telescoping.

b. From the recursion relation, we have $r_j^2q_j = r_j(r_{j-1} - r_{j+1}) = r_{j-1}r_j - r_jr_{j+1}$, so $\sum_{j=1}^{n} r_j^2q_j = (r_0r_1 - r_1r_2) + \cdots + (r_{n-1}r_n - r_nr_{n+1}) = r_0r_1 - r_nr_{n+1} = ab$, where we notice that the second sum is telescoping.

3.4.25. We apply the Q_i's one at a time. When we multiply $\begin{pmatrix} q_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_n \\ 0 \end{pmatrix} = \begin{pmatrix} q_nr_n \\ r_n \end{pmatrix} = \begin{pmatrix} r_{n-1} \\ r_n \end{pmatrix}$, the top component is the last equation in the series of equations in the proof of Lemma 3.3. When we multiply this result on the left by the next matrix we get $\begin{pmatrix} q_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{n-1} \\ r_n \end{pmatrix} = \begin{pmatrix} q_{n-1}r_{n-1} + r_n \\ r_{n-1} \end{pmatrix} = \begin{pmatrix} r_{n-2} \\ r_{n-1} \end{pmatrix}$, which is the matrix version of the last two equations the proof of Lemma 3.3. In general, at the ith step we have $\begin{pmatrix} q_{n-i} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{n-i-1} \\ r_{n-i} \end{pmatrix} = \begin{pmatrix} q_{n-i}r_{n-i-1} + r_{n-i} \\ r_{n-i-1} \end{pmatrix} = \begin{pmatrix} r_{n-i-2} \\ r_{n-i-1} \end{pmatrix}$, so that we inductively work our way up the equations in the proof of Lemma 3.3, until finally we have $\begin{pmatrix} r_0 \\ r_1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$.

3.5. The Fundamental Theorem of Arithmetic

3.5.1. a. We have $36 = 6^2 = 2^2 \cdot 3^2$.

b. We have $39 = 3 \cdot 13$.

c. We have $100 = 10^2 = 2^2 \cdot 5^2$.

d. We have $289 = 17^2$.

e. We have $222 = 2 \cdot 111 = 2 \cdot 3 \cdot 37$.

f. We have $256 = 2^8$.

g. We have $515 = 5 \cdot 103$.

h. We have $989 = 23 \cdot 43$.

i. We have $5040 = 10 \cdot 504 = 2 \cdot 5 \cdot 4 \cdot 126 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$.

j. We have $8000 = 8 \cdot 10^3 = 2^6 \cdot 5^3$.

k. We have $9555 = 3 \cdot 5 \cdot 7^2 \cdot 13$.

l. We have $9999 = 9 \cdot 1111 = 3^2 \cdot 11 \cdot 101$.

3.5.2. We have $111111 = 111 \cdot 1001 = 3 \cdot 37 \cdot 7 \cdot 11 \cdot 13$.

3.5.3. We have $4849845 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

3.5.4. a. We have $100000 = 10^5$, so the only prime factors are 2 and 5.

b. We have $10500000 = 105 \cdot 10^5$, so the only prime factors are 2, 3, 5 and 7.

c. If a prime divides $10!$, then it must divide one of the factors from 1 to 10. Thus the only prime factors are those less than or equal to 10, namely 2, 3, 5 and 7.

d. We have $\binom{30}{10} = (21 \cdot 22 \cdot 23 \cdot 24 \cdot 25 \cdot 26 \cdot 27 \cdot 28 \cdot 29 \cdot 30)/(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10) = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 29$.

3.5.5. a. We have $196608 = 2^{16}3$.

b. We have $7290000 = 729 \cdot 10^4 = 2^4 3^6 5^4$.

c. If a prime divides $20!$, then it must divide one of the factors from 1 to 20. Thus the prime factors are exactly those less than or equal to 20.

d. We have $\binom{50}{25} = (26 \cdot 27 \cdot 28 \cdot 29 \cdot 30 \cdot 31 \cdot 32 \cdot 33 \cdot 34 \cdot 35 \cdot 36 \cdot 37 \cdot 38 \cdot 39 \cdot 40 \cdot 41 \cdot 42 \cdot 43 \cdot 44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49 \cdot 50)/(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25) = 2^3 3^2 7^2 13 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47$.

3.5.6. If $n = p_1^{2a_1} p_2^{2a_2} \cdots p_r^{2a_r}$ then $(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r})^2 = n$, so n is a perfect square. Conversely, if $n = d^2$ for some integer d with prime factorization $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then $n = d^2 = p_1^{2a_1} p_2^{2a_2} \cdots p_r^{2a_r}$.

3.5.7. The integers with exactly three positive divisors are those of the form p^2 where p is prime. The integers with exactly four positive divisors are those of the form pq or p^3 where p and q are distinct primes. These results can be proved considering the cases where the integer is a power of a prime, the product of powers of two primes, and the product of powers of more that two primes.

3.5.8. Suppose that the primes in the factorization of n that occur to an even power are $p_1, \ldots, p_k$ and let the power of p_i in the factorization be $2b_i$ and suppose that the primes that occur to an odd power are $q_1, \ldots, q_l$ and let the power of q_j in the factorization be $2c_j + 1$. Then $n = (p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k} q_1^{c_1} q_2^{c_2} \cdots q_l^{c_l})^2 \cdot (q_1 q_2 \cdots q_l)$. This is a factorization of n into a perfect square and a square-free integer.

3.5.9. Let $n = p_1^{2a_1} p_2^{2a_2} \cdots p_k^{2a_k} q_1^{2b_1+3} q_2^{2b_2+3} \cdots q_l^{2b_l+3}$ be the factorization of a powerful number. Then $n = (p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} q_1^{b_1} q_2^{b_2} \cdots q_l^{b_l})^2 (q_1 q_2 \cdots q_l)^3$ is a product of a square and a cube.

3.5.10. Let p be a prime divisor of a, and let p^r be the highest power of p dividing a. Then $p^{3r} \mid a^3$, and hence $p^{3r} \mid b^2$. Let p^s be the highest power of p dividing b. Then $3r \leq 2s$. Therefore, $r \leq (2/3)s < s$, and so $p^r \mid b$. Since this is true for all primes dividing a, we have $a \mid b$.

3.5.11. Suppose that $p^a \mid\mid m$ and $p^b \mid\mid n$. Then $m = p^a Q$ and $n = p^b R$ where both Q and R are products of primes other than p. Hence $mn = (p^a Q)(p^b R) = p^{a+b} QR$. It follows that $p^{a+b} \mid\mid mn$ since p does not divide QR.

3.5.12. If $p^a \mid\mid m$ then $m = p^a n$ where $p \nmid n$. Then $p \nmid n^k$ and we have $m^k = p^{ka} n^k$ and we see that $p^{ka} \mid\mid m^k$.

3.5.13. Suppose that $p^a \mid\mid m$ and $p^b \mid\mid n$ with $a \neq b$. Then $m = p^a Q$ and $n = p^b R$ where both Q and R are products of primes other than p. Suppose, without loss of generality, that $a = \min(a, b)$. Then $m + n = p^a Q + p^b R = p^{\min(a,b)}(Q + p^{b-a} R)$. Then $p \nmid (Q + p^{b-a} R)$ because $p \nmid Q$ but $p \mid p^{b-a} R$. It follows that $p^{\min(a,b)} \mid\mid (m + n)$.

3.5.14. To determine the power of p in the prime factorization of $n!$ we can add the number of positive integers not exceeding n that are divisible by p, the number of positive integers not exceeding n that are divisible by p^2, the number of positive integers not exceeding n that are divisible by p^3, and so on. This will count the total number of factors of p in $n!$ because it will count exactly once each factor of p in each integer not exceeding n. Since there are $[n/p^i]$ positive integers not exceeding n that are divisible by p^i, it follows that the power of p in the prime factorization of n is $[n/p] + [n/p^2] + [n/p^3] + \cdots$.

3.5.15. We know that in the prime power factorization of 20! the number 2 occurs $[20/2] + [20/4] + [20/8] + [20/16] = 10 + 5 + 2 + 1 = 18$ times, 3 occurs $[20/3] + [20/9] = 6 + 2 = 8$ times, 5 occurs $[20/5] = 4$ times, 7 occurs $[20/7] = 2$ times, 11 occurs $[20/11] = 1$ time, 13 occurs $[20/13] = 1$ time, 17 occurs $[20/17] = 1$ time, and 19 occurs $[20/19] = 1$ time. Hence $20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

3.5.16. The number of 0's at the end of 1000! in decimal notation is the minimum of the powers of 2 and 5 in the prime factorization of 1000!. This is the number of 5's in the factorization since there are clearly more 2's than 5's in the prime factorization of 1000!. Since the power of 5 in the prime factorization of 1000! in $\sum_{j=1}^{4} [1000/5^j] = 200 + 40 + 8 + 1 = 249$, there are 249 0's at the end of 1000!.

The number of 0's at the end of 1000! in base eight notation is the highest power of 8 that divides 1000! evenly. This is the quotient obtained when 3 is divided into the power of 2 in the prime factorization of 1000!. Since the power of 2 in the prime factorization of 1000! is $\sum_{j=1}^{9} [1000/2^j] = 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 994$ and since $994 = 331 \cdot 3 + 1$, there are 331 zeros at the end of the base eight expansion of 1000!.

3.5.17. Suppose $n!$ ends with exactly 74 zeroes. Then $5^{74} \cdot 2^{74} = 10^{74} \mid n!$. Since there are more multiples of 2 than 5 in $1, 2, \ldots, n$, we need only concern ourselves with the fact that $5^{74} \mid n!$. Thus, via Exercise 12, we need to find an n such that $74 = [n/5] + [n/25] + \cdots$. By direct calculation, $74 = [300/5] + [300/25] + [300/125]$. It follows that 300!, 301!, 302!, 303!, and 304! end with exactly 74 zeroes.

3.5.18. The number of zeros at the end of $n!$ equals the number of 5's in the prime factorization of $n!$. This is clearly an increasing function of n. There are $\sum_{j=1}^{3} [624/5^j] = 124 + 24 + 4 = 152$ zeros at the end of decimal expansion of 624!. However since 5^4 divides 625, we see that there are 152+4=156 zeros at the end of the decimal expansion of 625!. It follows that there cannot be 153,154, or 155 zeros at the end of

the decimal expansion of n!.

3.5.19. We compute $\alpha\beta = (ac-5bd)+(ad+bc)\sqrt{-5}$. Thus $N(\alpha\beta) = (ac-5bd)^2+5(ad+bc)^2 = a^2c^2-10acbd+25b^2d^2+5a^2d^2+10adbc+5b^2c^2 = a^2(c^2+5d^2)+5b^2(5d^2+c^2) = (a^2+5b^2)(c^2+5d^2) = N(\alpha)N(\beta)$.

3.5.20. Suppose $2 = \alpha\beta$. Then by Exercise 19, $4 = N(2) = N(\alpha)N(\beta)$. Then $N(\alpha) = 1, 2$ or 4. Let $\alpha = a + b\sqrt{-5}$. Then we must have $a^2 + 5b^2 = 1, 2$, or 4. Thus $b = 0$ and $a = \pm 1$ or ± 2 are the only possibilities. Since $\alpha = \pm 1$ is excluded, we must have $\alpha = \pm 2$, which forces $\beta = \pm 1$.

3.5.21. Suppose $3 = \alpha\beta$. Then by Exercise 19, $9 = N(3) = N(\alpha)N(\beta)$. Then $N(\alpha) = 1, 3$ or 9. Let $\alpha = a + b\sqrt{-5}$. Then we must have $a^2 + 5b^2 = 1, 3$, or 9. So either $b = 0$ and $a = \pm 1$ or ± 3, or $b = \pm 1$ and $a = \pm 2$. Since $a = \pm 1, b = 0$ is excluded, and since $a = \pm 3$ forces $\beta = \pm 1$, we must have $b = \pm 1$. That is, $\alpha = \pm 2 \pm \sqrt{-5}$. But then $N(\alpha) = 9$, and hence $N(\beta) = 1$, which forces $\beta = \pm 1$.

3.5.22. Note that $N(1 \pm \sqrt{-5}) = 6$. If $1 \pm \sqrt{-5} = \alpha\beta$ is a nontrivial factorization, then $N(\alpha) = 2$, say. But $N(\alpha) = a^2 + 5b^2 = 2$ has no solution in the integers. Hence, no nontrivial factorization exists.

3.5.23. Note that $21 = 3 \cdot 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5})$. We know 3 is prime from Exercise 21. Similarly if we seek $\alpha = a + b\sqrt{-5}$ such that $N(\alpha) = a^2 + 5b^2 = 7$, we find there are no solutions. For $|b| = 0$ implies $a^2 = 7$, $|b| = 1$ implies $a^2 = 2$ and $|b| > 1$ implies $a^2 < 0$, and in each case there is no such a. Hence if $\alpha\beta = 7$, then $N(\alpha\beta) = N(\alpha)N(\beta) = N(7) = 49$. So one of $N(\alpha)$ and $N(\beta)$ must be equal to 49 and the other equal to 1. Hence 7 is also prime. We have shown that there are no numbers of the form $a + b\sqrt{-5}$ with norm 3 or 7. So in a similar fashion to the argument above, if $\alpha\beta = 1 \pm 2\sqrt{-5}$, then $N(\alpha\beta) = N(\alpha)N(\beta) = N(1 \pm 2\sqrt{-5}) = 21$. And there are no numbers with norm 3 or 7, so one of α and β has norm 21 and the other has norm 1. Hence $1 \pm 2\sqrt{-5}$ is also prime.

3.5.24. Note that, for instance, $25 = 5 \cdot 5 = (1 + 2\sqrt{-6})(1 - 2\sqrt{-6})$. By arguments identical to those in the solutions to Exercises 21 and 22, we see that 5 and $1 \pm 2\sqrt{-6}$ are prime.

3.5.25. The product of $4k + 1$ and $4l + 1$ is $(4k + 1)(4l + 1) = 16kl + 4k + 4l + 1 = 4(4kl + k + l) + 1 = 4m + 1$ where $m = 4kl + k + l$. Hence the product of two integers of the form $4k + 1$ is also of this form.

3.5.26. The twenty smallest Hilbert primes are: 5, 9, 13, 17, 21, 29, 33, 37, 43, 49, 53, 57, 61, 69, 73, 77, 89, 93, 97, 101, 105.

3.5.27. We proceed by mathematical induction on the elements of H. The first Hilbert number greater than 1, 5, is a Hilbert prime because it is an integer prime. This completes the basis step. For the inductive step, we assume that all numbers in H less than or equal to n can be factored into Hilbert primes. The next greatest number in H is $n + 4$. If $n + 4$ is a Hilbert prime, then we are done. Otherwise, $n = hk$, where h and k are less than n and in H. By the inductive hypothesis, h and k can be factored into Hilbert primes. Thus, $n + 4$ can be written as the product of Hilbert primes.

3.5.28. We have $693 = 9 \cdot 77 = 21 \cdot 33$. All of 9, 21, 33, and 77 are Hilbert primes since none of these integers are divisible by any smaller integers of the form $4k + 1$.

3.5.29. Suppose that n is divisible by all primes not exceeding $\sqrt{n}$. Let M be the least common multiple of the integers m with $1 \leq m \leq \sqrt{n}$. Then for every prime p with $p \leq \sqrt{n}, p^k \mid M$ but p^{k+1} does not divide M where p^k is the largest power of p not exceeding $\sqrt{n}$. Then $M = p_1^{k_1} \cdots p_t^{k_t}$ where the powers of the prime p_i is the largest power of this prime not exceeding $\sqrt{n}$. Since $\sqrt{n} < p_i^{k_i+1}$ for $i = 1, 2, \ldots t$ we have $(\sqrt{n})^t < p_1^{k_1+1} \cdots p_t^{k_t+1}$. But note that $p_1^{k_1+1} \cdots p_t^{k_t+1} = (p_1^{k_1} \cdots p_t^{k_t}) \cdot (p_1 \cdots p_k) \leq M \cdot p_1 \cdots p_t \leq M^2$. It follows that $(\sqrt{n})^t < M^2$. Since $M \mid n$ it follows that $M \leq n$, so $(\sqrt{n})^t < n^2$. It follows that $t < 4$. If t is the number of primes less than $\sqrt{n}$ and there are four or fewer primes less than $\sqrt{n}$ and 7 is the fourth prime, it follows that $\sqrt{n} \leq 7$, so $n < 49$. Examining the integers less than 49 shows that the only integers satisfying the conditions are $n = 1, 2, 3, 4, 6, 8, 12$, and 24.

3.5.30. a. We have $[8, 12] = 24$.

b. We have $[14, 15] = 1$.

c. We have $[28, 35] = 140$.

d. We have $[111, 303] = 11211$.

e. We have $[256, 5040] = 80640$.

f. We have $[343, 999] = 342657$.

3.5.31. a. We have $[7, 11] = 77$.

b. We have $[12, 18] = 36$.

c. We have $[25, 30] = 150$.

d. We have $[101, 333] = 33633$.

e. We have $[1331, 5005] = 605605$.

f. We have $[5040, 7700] = 277200$.

3.5.32. a. We have $(23^2 5^3, 2^2 3^3 7^2) = 1$, and $[23^2 5^3, 2^2 3^3 7^2] = 23^2 5^3 2^2 3^3 7^2$.

b. We have $(2 \cdot 3 \cdot 5 \cdot 7, 7 \cdot 11 \cdot 13) = 7$, and $[2 \cdot 3 \cdot 5 \cdot 7, 7 \cdot 11 \cdot 13] = 2 \cdot 3 \cdot 57 \cdot 11 \cdot 13$.

c. We have $(2^8 3^6 5^4 11^{13}, 2 \cdot 3 \cdot 5 \cdot 11 \cdot 13) = 2 \cdot 3 \cdot 5 \cdot 11$, and $[2^8 3^6 5^4 11^{13}, 2 \cdot 3 \cdot 5 \cdot 11 \cdot 13] = 2^8 3^6 5^4 11^{13} 13$.

d. We have $(41^{101} 47^{43} 103^{1001}, 41^{11} 43^{47} 83^{111}) = 41^{11}$, and $[41^{101} 47^{43} 103^{1001}, 41^{11} 43^{47} 83^{111}] = 41^{101} 47^{43} 103^{1001} 43^{47} 83^{111}$.

3.5.33. a. We have $(2^2 3^3 5^5 7^7, 2^7 3^5 5^3 7^2) = 2^2 3^3 5^3 7^2$; $[2^2 3^3 5^5 7^7, 2^7 3^5 5^3 7^2] = 2^7 3^5 5^5 7^7$.

b. We have $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, 17 \cdot 19 \cdot 23 \cdot 29) = 1$; $[2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, 17 \cdot 19 \cdot 23 \cdot 29] = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$.

c. We have $(2^3 5^7 11^{13}, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) = 2 \cdot 5 \cdot 11$; $[2^3 5^7 11^{13}, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13] = 2^3 \cdot 3 \cdot 5^7 \cdot 7 \cdot 11^{13} \cdot 13$.

d. We have $(47^{11} 79^{111} 101^{1001}, 41^{11} 83^{111} 101^{1000}) = 101^{1000}$; $[47^{11} 79^{111} 101^{1001}, 41^{11} 83^{111} 101^{1000}] = 41^{11} 47^{11} 79^{111} 83^{111} 101^{1001}$.

3.5.34. Let $m = [a, b]$ an suppose M is a common multiple of a and b which is not divisible by m. Then by the division algorithm, we have $M = qm + r$, with $0 < r < m$. Since $a \mid m$ and $a \mid M$, then $a \mid M - qm = r$. Similarly, $b \mid r$. Therefore, r is a positive common multiple of a and b which is less than m. This is a contradiction, so no such M exists.

3.5.35. Suppose that both 13-year and 17-year cicadas emerge in a location in 1900. The 13-year cicada will emerge again in years $1900 + 13k$ where k is a positive integer. The 17-year cicadas will emerge again in years $1900 + 17k$ where k is a positive integer. Both 13-year and 17-year cicadas will emerge again in years $1900 + [13, 17]k = 1900 + 221k$ where k is a positive integer. Hence they both will emerge again in the year 2121.

3.5.36. Each of a and b must be a multiple of 18, say $a = 18k$ and $b = 18m$, with $(k, m) = 1$. By Theorem 2.8, $ab = 18k \cdot 18m = 18 \cdot 540$, or $km = 2 \cdot 3 \cdot 5$. The possible values for the pair (k, m) are $(1, 30), (2, 15), (3, 10), (5, 6)$, in either order. So the possible values of a and b are these pairs multiplied

by 18.

3.5.37. Let $a = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ and $b = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, where p_i is a prime and r_i and s_i are nonnegative. $(a,b) = p_1^{\min(r_1,s_1)}\cdots p_k^{\min(r_k,s_k)}$ and $[a,b] = p_1^{\max(r_1,s_1)}\cdots p_k^{\max(r_k,s_k)}$. So $[a,b] = (a,b)p_1^{\max(r_1,s_1)-\min(r_1,s_1)}\cdots p_k^{\max(r_k,s_k)-\min(r_k,s_k)}$. Since $\max(r_i,s_i)-\min(r_i,s_i)$ is clearly nonnegative, we now see that $(a,b) \mid [a,b]$.

3.5.38. Let $e = (a,b)$. Then $(a/e,b) = 1$. Let $c = a/e$ and $d = b$.Then $cb = ab/e = [a,b]$ by Theorem 3.16.

3.5.39. If $[a,b] \mid c$, then since $a \mid [a,b], a \mid c$.Similarly, $b \mid c$. Conversely, suppose that $a = p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n}$ and $b = p_1^{b_1}p_2^{b_2}\cdots p_n^{b_n}$ and $c = p_1^{c_1}p_2^{c_2}\cdots p_n^{c_n}$. If $a|c$ and $b|c$, then $\max(a_i,b_i) \le c_i$ for $i = 1,2,\ldots,n$. Hence, $[a,b] \mid c$.

3.5.40. Suppose that $p \mid a^2$ where p is prime and a is an integer. Then by Lemma 3.5 it follows that $p \mid a$.

3.5.41. Assume that $p \mid a^n = \pm \mid a \mid \cdot \mid a \mid \cdots \mid a \mid$. Then by Lemma 3.5, $p \mid\mid a \mid$ and so $p \mid a$.

3.5.42. Let $p^r \parallel c, p^s \parallel a$, and $p^t \parallel b$. Then $p^r \mid ab$, so $r < s+t$. Then $p^{\max(r,s)} \mid (a,c)$, and $p^{\max(r,t)} \mid (b,c)$. Since $\max(r,s)+\max(r,t) > s+t > r$, we have $p^r \mid (a,c)(b,c)$.

3.5.43. a. Suppose that $(a,b) = 1$ and $p \mid (a^n,b^n)$ where p is a prime. It follows that $p \mid a^n$ and $p \mid b^n$. By Exercise 41, $p \mid a$ and $p \mid b$. But then $p \mid (a,b) = 1$, which is a contradiction.

b. Suppose that a does not divide b, but $a^n \mid b^n$. Then there is some prime power, say p^r that divides a but does not divide b (else $a \mid b$ by the Fundamental Theorem of Arithmetic). Thus, $a = p^rQ$, where Q is an integer. Now, $a^n = (p^rQ)^n = p^{rn}Q^n$, so $p^{rn} \mid a^n \mid b^n$. Then $b^n = mp^{rn}$, from which it follows that each of the n b's must by symmetry contain r p's. But this is a contradiction.

3.5.44. a. Suppose $\sqrt[3]{5} = a/b$, with a and b integers and $(a,b) = 1$. Then $5 = a^3/b^3$, or $5b^3 = a^3$. Then $5 \mid a^3$, so $5 \mid a$ and we have $5^3 \mid a^3$. Then $5^3 \mid 5b^3$, or $5^2 \mid b^3$. But then $5 \mid b$, so $5 \mid (a,b)$, a contradiction. Therefore $\sqrt[3]{5}$ is irrational.

b. By Theorem 2.11, a root of x^3-5 is either an integer or an irrational number. $\sqrt[3]{5}$ is a root, but $1^3 < 5 < 2^3$, so $1 < \sqrt[3]{5} < 2$. Since there are no integers between 1 and 2, $\sqrt[3]{5}$ must be irrational.

3.5.45. Suppose that $x = \sqrt{2}+\sqrt{3}$. Then $x^2 = 2+2\sqrt{2}\sqrt{3}+3 = 5+2\sqrt{6}$. Hence $x^2-5 = 2\sqrt{6}$. It follows that $x^4-10x^2+25 = 24$. Consequently, $x^4-10x^2+1 = 0$. By Theorem 3.17 it follows that $\sqrt{2}+\sqrt{3}$ is irrational, since it is not an integer (we can see this since $3 < \sqrt{2}+\sqrt{3} < 4$).

3.5.46. Suppose that $\log_2 3$ is rational. Then $\log_2 3 = a/b$ where a and b are integers with $b \neq 0$. This implies that $2^{\frac{a}{b}} = 3$. Raising both sides to the bth power gives $2^a = 3^b$. But the fundamental theorem of arithmetic shows that this is impossible since the integer 2^a has a unique factorization into primes, and so cannot equal 3^b.

3.5.47. Suppose that $m/n = \log_p b$. This implies that $p^{\frac{m}{n}} = b$, from which it follows that $p^m = b^n$. Since b is not a power of p, there must be another prime, say q, such that $q \mid b$. But then $q \mid b \mid b^n = p^m = p \cdot p \cdots p$. By Lemma 2.4, $q \mid p$, which is impossible since p is a prime number.

3.5.48. Suppose that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = Q$ where Q is an integer. Let 2^a be the largest power of 2 not exceeding n. Multiply both sides by $2^{a-1}R$ where R is product of the largest powers of odd primes less than n. We obtain $A+\frac{1}{2} = 2^{a-1}RQ$ where A is an integer. This is a contradiction since the left-hand side is not an integer but the right-hand side is an integer.

3.5.49. Let p be a prime that divides a or b. Then p divides $a+b$ and $[a,b]$. Hence p divides both sides of the equation. Define s,t by $p^s \mid\mid a$, $p^t \mid\mid b$, say that $a = xp^s$ and $b = yp^t$. Without loss of generality, suppose $s \le t$. Then $a+b = p^s(x+p^{t-s})$, so $p^s \parallel a+b$. Also, $p^{\max(s,t)} \parallel [a,b]$. But $\max(s,t) = t$, so $p^t \parallel [a,b]$. Therefore $p^{\min(s,t)} \parallel (a+b,[a,b])$. But $\min(s,t) = s$, so the same power of p divides both sides of the

equation. Therefore the two sides must be equal.

3.5.50. By Exercise 41 we know that $(a,b) = (a+b,[a,b]) = (798,10780) = 14$. Let $c = \frac{a}{14}$ and $d = \frac{b}{14}$. Since $ab = (a,b)[a,b]$ it follows that $cd = 10780/14 = 770$ and $c + d = 57$. We can find c and d by solving the equation $(x-c)(x-d) = x^2 - (c+d)x + cd = x^2 - 57x + 770 = 0$. The roots are $c = 35$ and $d = 22$. Hence $a = 490$ and $b = 308$.

3.5.51. Let $a = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, $b = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, and $c = p_1^{t_1}p_2^{t_2}\cdots p_k^{t_k}$, with p_i prime and r_i, s_i, and t_i nonnegative. Observe that $\min(x,\max(y,z)) = \max(\min(x,y),\min(x,z))$. We also know that $[a,b] = p_1^{\max(r_1,s_1)}p_2^{\max(r_2,s_2)}\cdots p_k^{\max(r_k,s_k)}$, and so $([a,b],c) = p_1^{\min(t_1,\max(r_1,s_1))}p_2^{\min(t_2,\max(r_2,s_2))}\cdots p_k^{\min(t_k,\max(r_k,s_k))}$. We also know that $(a,c) = p_1^{\min(r_1,t_1)}p_2^{\min(r_2,t_2)}\cdots p_k^{\min(r_k,t_k)}$ and $(b,c) = p_1^{\min(s_1,t_1)}p_2^{\min(s_2,t_2)}\cdots p_k^{\min(s_k,t_k)}$. Then, $[(a,c),(b,c)] = p_1^{\max(\min(r_1,t_1),\min(s_1,t_1))}p_2^{\max(\min(r_2,t_2),\min(s_2,t_2))}\cdots p_k^{\max(\min(r_k,t_k),\min(s_k,t_k))}$. Therefore, $([a,b],c) = [(a,c),(b,c)]$. In a similar manner, noting that $\min(\max(x,z),\max(y,z)) = \max(\min(x,y),z)$, we find that $[(a,b),c] = ([a,c],[b,c])$.

3.5.52. We have $[6,10,15] = 30$ and $[7,11,13] = 1001$.

3.5.53. Let $c = [a_1,\ldots,a_n]$, $d = [[a_1,\ldots,a_{n-1}],a_n]$, and $e = [a_1,\ldots,a_{n-1}]$. If $c \mid m$, then all a_i's divide m, hence $e \mid m$ and $a_n \mid m$, so $d \mid m$. Conversely, if $d \mid m$, then $e \mid m$ and $a_n \mid m$, so all a_i's divide m, thus $c \mid m$. Since c and d divide all the same numbers, they must be equal.

3.5.54. Let a, b, and n have prime factorizations $a = p_1^{a_1}\cdots p_r^{a_r}$, $b = p_1^{b_1}\cdots p_r^{b_r}$, and $n = p_1^{c_1}\cdots p_r^{c_r}$, where some of the a_i and b_i may be 0. If $n = [a,b]$, we have $\max(a_i,b_i) = c_i$ for each i. So one of each pair a_i, b_i must be equal to c_i. If $a_i = c_i$, there are $c_i + 1$ choices for b_i. If $a_i \neq c_i$, then $b_i = c_i$ and there are c_i choices for a_i, giving $2c_i + 1$ ways in all. Since this occurs for each i, we have $(2c_1+1)\cdots(2c_r+1)$ ways in all.

3.5.55. a. There are six cases, all handled the same way. So without loss of generality, suppose that $a \le b \le c$. Then $\max(a,b,c) = c, \min(a,b) = a, \min(a,c) = a, \min(b,c) = b$, and $\min(a,b,c) = a$. Hence $c = \max(a,b,c) = a + b + c - \min(a,b) - \min(a,c) - \min(b,c) + \min(a,b,c) = a + b + c - a - a - b + a$.

b. The power of a prime p that occurs in the prime factorization of $[a,b,c]$ is $\max(a,b,c)$ where a, b, and c are the powers of this prime in the factorizations of a, b, and c, respectively. Also $a + b + c$ is the power of p in abc, $\min(a,b)$ is the power of p in (a,b), $\min(a,c)$ is the power of p in (a,c), $\min(b,c)$ is the power of p in (b,c), and $\min(a,b,c)$ is the power of p in (a,b,c). It follows that $a + b + c - \min(a,b) - \min(a,c) - \min(b,c)$ is the power of p in $abc(a,b,c)/((a,b)(a,c)(b,c))$. Hence $[a,b,c] = abc(a,b,c)/((a,b)(a,c)(b,c))$.

3.5.56. The formula for $[a_1,a_2,\ldots,a_n]$ is a rational number whose numerator is the product of the greatest common divisors of the a_i's taken $1,3,5,\ldots$ at a time, and whose denominator is the product of the greatest common divisors of the a_i's taken $2,4,6,\ldots$ at a time.

3.5.57. Let $a = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, $b = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, and $c = p_1^{t_1}p_2^{t_2}\cdots p_k^{t_k}$, with p_i prime and r_i, s_i, and t_i nonnegative. Then $p_i^{r_i+s_i+t_i} \parallel abc$, but $p_i^{\min(r_i,s_i,t_i)} \parallel (a,b,c)$ and $p_i^{r_i+s_i+t_i-\min(r_i,s_i,t_i)} \parallel [ab,ac,ab]$, and $p_i^{\min(r_i,s_i,t_i)} \cdot p_i^{r_i+s_i+t_i-\min(r_i,s_i,t_i)} = p_i^{r_i+s_i+t_i}$.

3.5.58. Let a, b, and c have prime factorizations $a = p_1^{a_1}\cdots p_r^{a_r}$, $b = p_1^{b_1}\cdots p_r^{b_r}$, and $c = p_1^{c_1}\cdots p_r^{c_r}$, where some of the a_i and b_i may be 0. Then $[a,b,c](ab,ac,bc) = p_1^{\max(a_1,b_1,c_1)}\cdots p_r^{\max(a_r,b_r,c_r)}p_1^{\min(a_1+b_1,a_1+c_1,b_1+c_1)}\cdots p_r^{\min(a_r+b_r,a_r+c_r,b_r+c_r)} = p_1^{a_1+b_1+c_1}\cdots p_r^{a_r+b_r+c_r} = abc$.

3.5.59. Let $a = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, $b = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, and $c = p_1^{t_1}p_2^{t_2}\cdots p_k^{t_k}$, with p_i prime and r_i, s_i, and t_i nonnegative. Then, using that $(a,b,c) = p_1^{\min(r_1,s_1,t_1)}p_2^{\min(r_2,s_2,t_2)}\cdots p_k^{\min(r_k,s_k,t_k)}$, and $[a,b,c] = p_1^{\max(r_1,s_1,t_1)}p_2^{\max(r_2,s_2,t_2)}\cdots p_k^{\max(r_k,s_k,t_k)}$, we can write the prime factorization of $([a,b],[a,c],[b,c])$ and $[(a,b),(a,c),(b,c)]$. For instance, consider the case where $k = 1$. Then $([a,b],[a,c],[b,c]) = (p_1^{\max(r_1,s_1)},p_1^{\max(r_1,t_1)},p_1^{\max(s_1,t_1)}) = p_1^{\min(\max(r_1,s_1),\max(r_1,t_1),\max(s_1,t_1)}$. Similarly, $[(a,b),(a,c),(b,c)] =$

$p_1^{\max(\min(r_1,s_1),\min(r_1,t_1),\min(s_1,t_1)}$. Clearly, these two are equal (examine the six orderings $r_1 \geq s_1 \geq t_1, \ldots$).

3.5.60. Suppose that there are only finitely many primes $p_1, \ldots, p_t$ of the form $6k+5$. Form $N = 6p_1p_2 \cdots p_t - 1$. Then N is not divisible by any of the primes $p_1, \ldots, p_t$, since each leaves a remainder of -1 when it is divided into N. Now N can only have prime divisors of the form $6k+1$ and $6k+5$ since $(N, 6) = 1$. There also must be at least one prime divisor of the form $6k+5$ since the product of primes of the form $6k+1$ is also of this form. Hence there are infinitely many primes of the form $6k+5$.

3.5.61. First note that there are arbitrarily long sequences of composites in the integers. For example, $(n+2)! + 2, (n+2)! + 3, \ldots, (n+2)! + (n+2)$ is a sequence of n consecutive composites. To find a sequence of n composites in the sequence $a, a+b, a+2b, \ldots$, look at the integers in $a, a+b, a+2b, \ldots$ with absolute values between $(nb+2)! + 2$ and $(nb+2)! + (nb+2)$. There are clearly n or $n+1$ such integers, and all are composite.

3.5.62. a. We have $10^6 - 1 = (10^3+1)(10^3-1)$. Also, we find that $(10^3+1) = (10+1)(10^2-10+1) = 11 \cdot 91 = 11 \cdot 7 \cdot 13$. We also have $(10^3-1) - (10-1)(10^2+10+1) = 9 \cdot 111 = 3^3 \cdot 37$. It follows that $10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$.

b. We have $3^2 11 \cdot 73 \cdot 101 \cdot 137$.

c. We have $7 \cdot 31 \cdot 151$.

d. We have $3^2 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$.

e. We have $3^2 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$.

f. We have $3^3 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 109$.

3.5.63. We have $8137 = 79 \cdot 103$. Since the price of the camera is an integer and less than 99 dollars, it follows that the discounted price of a camera is 79 dollars. Hence they sold 103 cameras at 79 dollars each.

3.5.64. Note that $375961 = 79 \cdot 4759$. So the possible prices of the book are \$1, \$79, \$4759, and \$375,961. The most likely of these is \$79, so we suspect that the number of books sold was 4759.

3.5.65. Since $139499 = 199 \cdot 701$, the price must have been \$199 and so the number of electronic organizers sold was 701.

3.5.66. Suppose that a and b are integers such that $a^2 \mid b^2$. Then there is an integer k such that $b^2 = ka^2$. It follows that $k = (a/b)^2$. Suppose that $\sqrt{k}$ is not an integer. Then by Theorem 2.11 we see that $\sqrt{k}$ is irrational. However $\sqrt{k} = a/b$. It follows that $\sqrt{k} = l$ is an integer. Hence $b = l \cdot a$ where l is an integer. Thus $a \mid b$.

3.5.67. Let $a = \prod_{i=1}^{s} p_i^{\alpha_i}$ and $b = \prod_{i=1}^{t} p_i^{\beta_i}$. The condition $(a, b) = 1$ is equivalent to $\min(\alpha_i, \beta_i) = 0$ for all i and the condition $ab = c^n$ is equivalent to $n \mid (\alpha_i + \beta_i)$ for all i. Hence $n \mid \alpha_i$ and $\beta_i = 0$ or $n \mid \beta_i$ and $\alpha_i = 0$. Let d be the product of $p_i^{\alpha_i/n}$ over all i of the first kind, and let e be the product of $p_i^{\beta_i/n}$ over all i of the second kind. Then $d^n = a$ and $e^n = b$.

3.5.68. We proceed by induction. The basis step is $[a_1, a_2] = a_1a_2/(a_1, a_2) = a_1a_2$, since $(a_1, a_2) = 1$. Suppose the proposition if true for $n-1$. Then by Exercise 45, we have $[a_1, \ldots, a_{n-1}, a_n] = [[a_1, \ldots, a_{n-1}], a_n] = [(a_1 \cdots a_{n-1}), a_n] = a_1 \cdots a_n$.

3.5.69. Suppose the contrary and that $a \leq n$ is in the set. Then $2a$ cannot be in the set. Thus, if there are k elements in the set not exceeding n then, there are k integers between $n+1$ and $2n$ which cannot be in the set. So there are at most $k + (n-k) = n$ elements in the set.

3.5.70. The power of the prime p in the prime factorization of $(m+n)!$ is $\sum_{r=1}^{t}[(m+n)/p^r]$ where p^t is the largest power of p not exceeding $m+n$. The power of this prime in the factorization of $m!$ is $\sum_{r=1}^{t}[m/p^r]$. The power of this prime in the factorization of $n!$ is $\sum_{r=1}^{t}[n/p^r]$. By Exercise 23 of Section 1.4 it follows that $[(m+n)/p^r] \geq [m/p^r] + [n/p^r]$. Hence the prime p occurs to a nonnegative power, namely $[(m+n)/p^r] - [m/p^r] - [n/p^r]$, in the rational number $(m+n)!/(m!n!)$. Since this is true for every prime p, $(m+n)!/(m!n!)$ is an integer.

3.5.71. he fundamental theorem of arithmetic implies that m and n have the same prime divisors. So suppose that m and n have prime-power factorizations $m = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ and $n = p_1^{b_1}p_2^{b_2}\cdots p_k^{b_k}$. From the equation $m^n = n^m$ it follows that $a_i n = b_i m$ for $i = 1, 2, \ldots, k$. We first assume that $n > m$. Then $a_i < b_i$ for $i = 1, 2, \ldots, k$. Hence n is divisible by m, so $n = dm$ for some integer d. This implies that $m^{dm} = (dm)^m$. Taking the mth roots of both sides gives $m^d = dm$, which implies that $m^{d-1} = d$. Since $n > m$ we know that $d > 1$, so $m > 1$. However $2^{2-1} = 2$ and when $d > 2$ it follows that $m^{d-1} > d$. When $d > 2$ and $m \geq 2$ we have $m^{d-1} \geq 2^{d-1} > d$ since $2^{3-1} > 3$ and when $d = 2$ and $m > 2$ we have $m^{d-1} = m > 2 = d$. Hence the only solution with $n > m$ has $m = 2$ and $n = 2d = 2 \cdot 2 = 4$. Consequently all solutions are given by $m = 2$ and $n = 4$, $m = 4$ and $n = 2$, or $m = n$.

3.5.72. Suppose that there are only finitely many primes, say n of them: $p_1, p_2, \ldots, p_n$. Let Q be the product of m primes, and R the product of the remaining $n - m$ primes. Suppose $p_i \mid (Q + R)$. Since p_i is a factor of Q or R, it must also be a factor of the other. This is not possible, therefore no prime divides $Q + R$. But $Q + R$ is larger than p_n, the largest prime. This contradiction tells us that there are infinitely many prime numbers.

3.5.73. By Lemma 3.1, S must have a prime divisor, and by our assumption, it must be one of the p_i, $i = 1, 2, \ldots r$. For $j \neq i$, $p_i | Q_j$, since it is one of the factors. So p_i must divide $S - \sum_{j \neq i} Q_j = Q_i = p_1 \cdots p_{i-1}p_{i+1} \cdots p_r$, but by the Fundamental Theorem of Arithmetic, p_i must be equal to one of these last factors, a contradiction, therefore S must have a prime factor different from the list we have. Since no finite list can contain all the primes, there must be infinitely many primes.

3.5.74. We have $\binom{p}{k} = p!/(k!(p-k)!)$. This is an integer and p divides the numerator and not the denominator. It follows that $(p-1)!/(k!(p-k)!)$ is an integer, so that $\binom{p}{k} = p \cdot (p-1)!/(k!(p-k)!$. It follows that p divides $\binom{p}{k}$.

3.5.75. Let p be the largest prime less than or equal to n. If $2p$ were less than or equal to n then Bertrand's postulate would guarantee another prime q such that $p < q < 2p \leq n$ contradicting the choice of p. Therefore, we know that $n < 2p$. Therefore, in the product $n! = 1 \cdot 2 \cdot 3 \cdots n$, there appears only one multiple of p, namely p itself, and so in the prime factorization of n, p appears with exponent 1.

3.5.76. a. Such an n has prime power factorization $n = p_1^{2a_1+e_1}p_2^{2a_2+e_2}\cdots p_j^{2a_j+e_j}$, where $e_i = 0$ if p_i appears to an even power and $e_i = 1$ if p_i appears to an odd power. Note that some a_i's may be zero in this expression. Then $n = (p_1^{2a_1}\cdots p_j^{2a_j})(p_1^{e_1}\cdots p_j^{e_j}) = (p_1^{a_1}\cdots p_j^{a_j})^2(p_1^{e_1}\cdots p_j^{e_j})$ which is of the desired form.

b. Since s is of the form $p_1^{e_1}\cdots p_j^{e_j}$, and there are two choices for each of the values e_i, $i = 1, 2, \ldots, j$, there are exactly 2^j possible values for s.

c. We have $r^2 \leq r^2 s = n \leq x$, so taking square roots yields $r \leq \sqrt{n}\sqrt{x}$, so there are at most $\sqrt{x}$ possible values for r, and hence for r^2. Then combining this with the result in part (b), there are at most $2^j\sqrt{x}$ possible values for r^2. That is, $N(x) \leq 2^j\sqrt{x}$.

d. Since p_j is assumed to be the largest prime, then no integer can be divisible by any larger prime. So $N(x) = x$ for every x.

e. From part (d) we have $N(x) = x \leq 2^j\sqrt{x}$ by part (c). Squaring both sides gives us $x^2 \leq 2^{2j}x$ and dividing by x yields $x \leq 2^{2j}$. Since j is fixed and x can be as large as we please, this leads to a contradiction.

3.5.77. a. Uniqueness follows from the Fundamental Theorem. If a prime p_i doesn't appear in the prime factorization, then we include it in the product with an exponent of 0. Since $e_i \geq 0$, we have $p_1^{e_1} = p_1^{e_1}p_2^0 \cdots p_r^0 \leq p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r} = m$.

b. Since $p_1^{e_i} < p_i^{e_i} \leq m \leq Q = p_r^n$, we take logs of both sides to get $e_i \log p_1 \leq n \log p_r$. Solving for e_i gives the first inequality. If $1 \leq m \leq Q$, then m has a prime-power factorization of the form given in part (a), so the r-tuples of exponents count the number of integers in the range $1 \leq m \leq Q$.

c. To bound the number of r-tuples, by part (b) there are at most $Cn + 1$ choices for each e_i, therefore there are at most $(Cn + 1)^r$ r-tuples, which by part (b) gives us $p_r^n \leq (Cn + 1)^r = (n(C + 1/n))^r \leq n^r(C + 1)^r$.

d. Taking logs of both sides of the inequality in part (c) and solving for n yields $n \leq (r \log n + \log(C + 1))/\log p_r$, but since n grows much faster than $\log n$, the left side must be larger than the right for large values of n. This contradiction shows there must be infinitely many primes.

3.5.78. From Exercise 80, we know the answer for primes, so $S(2) = 2$, $S(3) = 3$, $S(5) = 5$, $S(7) = 7$, and $S(11) = 11$. Since $4, 8$ and 12 divide $4! = 24$ and no lower factorial, we have $S(4) = S(8) = S(12) = 4$. Since $1|1!$, $S(1) = 1$. Since $6|3!$, $S(6) = 3$. Since $9|6!$ but no lower factorial, $S(9) = 6$, and since $10|5!$, $S(10) = 5$.

3.5.79. Since 40 has lots of small factors in its prime factorization, we expect it to have a small Smarandache value. Since it's divisible by 5, the smallest possible value will be 5, and since 40 does indeed divide $5!$, we have $S(40) = 5$. Since 41 and 43 are primes, after Exercise 80 we have S(41)=41, S(43)=43.

3.5.80. If a prime p divides $n!$, then p must appear as a factor in the product. The smallest value of n for which this happens is $n = p$, and p indeed divides $p!$. Therefore $S(p) = p$.

3.5.81. From Exercise 83, we have $a(2) = 2$, $a(3) = 3$, $a(5) = 5$, $a(7) = 7$, and $a(11) = 11$. The smallest value of m such that $S(m) = 1$ is $m = 1$, so $a(1) = 1$. $S(4) = 4$, but not for any smaller argument, so $a(4) = 4$. To find $a(6)$ we consider the smallest number which would require two factors of 3 in the factorial, and that number would be 9, so $a(6) = 9$. To find $a(8)$, we consider the smallest number which would require 5 factors of 2 in the factorial (one factor from 2, two factors from 4, one factor from 6 and the additional factor from 8.) And that number would be 32, so $a(8) = 32$. Similarly $a(9) = 27$, since 27 is the smallest number requiring the 3 factors of 3 (one from 3, one from 6 and the additional one from 9.) Similarly $a(10) = 25$, since 25 is the smallest number needing both factors of 5. In sum, the sequence is $a(n) = 1, 2, 3, 4, 5, 9, 7, 32, 27, 25, 11$.

3.5.82. If $k|12!$ the last factor of 12 must contribute a new factor of either 2 or 3 which no number smaller than k has. Since $2^8 = 256\|11!$ and $3^4 = 81\|11!$, we see that 3^4 is smaller, and so it must be that the factor of 12 is needed for 3^5 to divide $12!$. So $a(n) = 3^5 = 243$.

3.5.83. From Exercise 80, we have $S(p) = p$ whenever p is prime. If $m < p$ and $m|S(p)! = p!$ then $m|(p-1)!$, so $S(p)$ must be the first time that $S(n)$ takes on the value p. Therefore of all the inverses of p, p is the least.

3.5.84. a. Since $300 = 2^2 \cdot 3 \cdot 5^2$, we have $\text{rad}(300) = 2 \cdot 3 \cdot 5$.

b. Since $444 = 2^2 \cdot 3 \cdot 37$, we have $\text{rad}(444) = 2 \cdot 3 \cdot 37$.

c. Since $44004 = 2^2 \cdot 3 \cdot 19 \cdot 193$, we have $\text{rad}(44004) = 2 \cdot 3 \cdot 19 \cdot 193$.

d. Since $128128 = 2^7 \cdot 7 \cdot 11 \cdot 13$, we have $\text{rad}(128128) = 2 \cdot 7 \cdot 11 \cdot 13$.

3.5.85. Let n be a positive integer and suppose n is square-free. Then no prime can appear to a power greater than one in the prime-power factorization of n. So $n = p_1 p_2 \cdots p_r$ for some distinct primes p_i. Then $rad(n) = p_1 p_2 \cdots p_r = n$. Conversely, if n is not square-free, then some square $d^2|n$ and some prime factor p_1 of d appears to an even power in the prime-power factorization of n. So $n = p_1^{2a} p_2^{b_2} \cdots p_r^{b_r}$. Then $rad(n) = p_1 p_2 \cdots p_r \neq n$.

3.5.86. Since every prime not exceeding n appears in the product, and no prime exceeding n appears in the product, we have that $\text{rad}(n!)$ equals the product of the primes not exceeding n.

3.5.87. Since every prime occurring in the prime-power factorization of mn occurs in either the factorization of m or n, every factor in $\text{rad}(mn)$ occurs at least once in the product $\text{rad}(m)\text{rad}(n)$, which gives us the inequality. If $m = p_1^{a_1} \cdots p_r^{a_r}$ and $n = q_1^{b_1} \cdots q_s^{b_s}$ are relatively prime, then we have $\text{rad}(mn) = p_1 \cdots p_r q_1 \cdots q_s = \text{rad}(m)\text{rad}(n)$.

3.5.88. By Exercise 14, p divides $n!$ exactly $\sum_{i=1}^{\infty}[n/p^i]$ times and $(2n)!$ exactly $\sum_{i=1}^{\infty}[2n/p^i]$ times. Therefore p divides $\binom{2n}{n} = (2n)!/(n!)(n!)$ exactly $\sum_{i=1}^{\infty}[2n/p^i] - 2\sum_{i=1}^{\infty}[n/p^i]$ times, and this is the desired expression.

3.5.89. First note that if $p \mid \binom{2n}{n}$, then $p \leq 2n$. This is true because every factor of the numerator of $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$ is less than or equal to $2n$. Let $\binom{2n}{n} = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the factorization of $\binom{2n}{n}$ into distinct primes. By the definition of π, $k \leq \pi(2n)$. By Exercise 72, $p_i^{r_i} \leq 2n$. It now follows that $\binom{2n}{n} = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \leq (2n)(2n) \cdots (2n) \leq (2n)^{\pi(2n)}$.

3.5.90. If p is a prime between n and $2n$ then $n < p$. Since there are $\pi(2n) - \pi(n)$ such primes, we have $n^{\pi(2n)-\pi(n)} < \prod_{n<p\leq 2n} p$. On the other hand, each prime in this product divides $(2n)!$ but not $n!$, so each prime in the product divides $\binom{2n}{n}$. Since the primes are mutually relatively prime, their product divides $\binom{2n}{n}$, and therefore we have $\prod_{n<p\leq 2n} p < \binom{2n}{n}$.

3.5.91. Note that $\binom{2n}{n} \leq \sum_{a=0}^{2n}\binom{2n}{a} = (1+1)^{2n} = 2^{2n}$. Then from Exercise 74, $n^{\pi(2n)-\pi(n)} < \binom{2n}{n} \leq 2^{2n}$. Taking logarithms gives $(\pi(2n) - \pi(n)) \log n < \log(2^{2n}) = n \log 4$. Now divide by $\log n$.

3.5.92. From Exercise 75, we get the following inequality: $\log(2n)\pi(2n) - \log(n)\pi(n) = \log(2)\pi(2n) + \log(n)(\pi(2n) - \pi(n)) \leq \log(2)\pi(2n) + n\log(4)$. Now for $n > 3$, we have $\pi(2n) \leq n$, (since half of the numbers less than $2n$ are even.) Then we have $\log(2)\pi(2n) + n\log(4) \leq \log(2)n + n\log(4) = 3n\log(2)$. Then $\log(2n)\pi(2n) = (\log(2n)\pi(2n) - \log(n)\pi(n)) + (\log(n)\pi(n) - \log(n/2)\pi(n/2)) + \cdots \leq 4n\log(2) + 2n\log(2) + n\log(2) + \cdots = n\log(2)(3 + 3/2 + 3/4 + \cdots = 6n\log(2)$. Therefore, $\pi(2n) \leq 6n\log(2)/\log(2n) \leq n\log(64)/\log(n)$.

3.5.93. Note that $2^n = \prod_{a=1}^{n} 2 \leq \prod_{a=1}^{n}(n+a)/a = \binom{2n}{n}$. Then by Exercise 73, $2^n \leq (2n)^{\pi(2n)}$. Taking logs gives $\pi(2n) \geq n\log 2/\log 2n$. Hence, for a real number x, we have $\pi(x) \geq [x/2]\log 2/\log[x] > c_1 x/\log x$. For the other half, Exercise 65 gives $\pi(x) - \pi(x/2) < ax/\log x$, where a is a constant. Then $\log x/2^m \pi(x/2^m) - \log x/2^{m+1}\pi(x/2^{m+1}) < ax/2^m$ for any positive integer m. Then, $\log x\pi(x) = \sum_{m=0}^{v}(\log x/2^m\pi(x/2^m) - \log x/2^{m+1}\pi(x/2^{m+1})) < ax\sum_{m=0}^{v} 1/2^m < c_2 x$, where v is the largest integer such that $2^{v+1} \leq x$. Then $\pi(x) < c_2 x/\log x$.

3.6. Factorization Methods and the Fermat Numbers

3.6.1. a. We see that 2 does not divide 33776925. Next we see that 3 does divide 33775925, with $33776925 = 3 \cdot 11258975$. Note that 3 does not divide 11258975. Next note that 5 does divide 11258975 with $11258975 = 5 \cdot 2251795$. We see that 5 also divides 2251795, with $2251795 = 5 \cdot 450359$. Next we see that 5 does not divide 450359. Next we note that 7 does divide 450359 with $450359 = 7 \cdot 64337$. Again dividing by 7 we see that $64337 = 7 \cdot 9191$. Dividing by 7 another time shows that $9191 = 7 \cdot 1313$. Next we note that 7 does not divide 1313. We see that 11 does not divide 1313. Dividing by

13 gives $1313 = 13 \cdot 101$. Since $\sqrt{101} < 13$, we conclude that 101 is prime. Hence the prime factorization is $33776925 = 3 \cdot 5^2 \cdot 7^3 \cdot 13 \cdot 101$.

b. We first note that neither $2, 3, 5$, nor 7 divides 210733237. Next we see that $210733237 = 11 \cdot 19157567$. Dividing by 11 again gives $19157567 = 11 \cdot 1741597$, and dividing by 11 yet again shows that $1741597 = 11 \cdot 158327$. We see that 11 does not divide 158327. Dividing by 13 shows that $158327 = 13 \cdot 12179$. Note that 12179 is not divisible by 13 nor by 17. We see that it is divisible by 19 with $12179 = 19 \cdot 641$. We see that 641 is not divisible by 19 or 23. Since 23 is the largest prime not exceeding $\sqrt{641}$ it follows that 641 is prime. It follows that the prime factorization is $210733237 = 11^3 \cdot 13 \cdot 19 \cdot 641$.

c. We first note that neither $2, 3, 5, 7$, nor 11 divides 1359170111. Next we see that $1359170111 = 13 \cdot 104551547$, and that 13 does not divide 104551547. Dividing by 17 gives $104551547 = 17 \cdot 6150091$, but 17 does not divide 6150091. Next we see that $6150091 = 19 \cdot 323689$, but 19 does not divide 323689. We see that neither $23, 29, 31, 37, 41$, nor 43 divides 323689, but $323689 = 47 \cdot 6887$. We see that 47 does not divide 6887. Neither $53, 59, 61$, nor 67 divides 6887, but $6887 = 71 \cdot 97$. Since 97 is prime, we conclude that $1359170111 = 13 \cdot 17 \cdot 19 \cdot 47 \cdot 71 \cdot 97$.

3.6.2. a. We have $33108075 = 3^3 5^2 7^3 11 \cdot 13$.

b. We have $7300977607 = 7^5 11 \cdot 17 \cdot 23 \cdot 101$.

c. $4165073376607 = 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 43 \cdot 47$.

3.6.3. a. Since $11 < \sqrt{143} < 12$, we begin by noting that $12^2 - 143 = 1$ is a perfect square. So, $143 = 12^2 - 1 = (12 + 1)(12 - 1) = 13 \cdot 11$.

b. Since $47 < \sqrt{2279} < 48$, we begin by noting that $48^2 - 2279 = 25 = 5^2$ is a perfect square. So, $2279 = 48^2 - 5^2 = (48 + 5)(48 - 5) = 53 \cdot 43$.

c. Since $6 < \sqrt{43} < 7$, we begin by looking for a perfect square in the sequence $7^2 - 43 = 6, 8^2 - 43 = 21, 9^2 - 43 = 38, 10^2 - 43 = 57, 11^2 - 43 = 78, \ldots$. The smallest such perfect square is $22^2 - 43 = 21^2$. From this, it follows that $43 = (22 + 21)(22 - 21) = 43 \cdot 1$, which shows that 43 is prime.

d. Since $106 < \sqrt{11413} < 107$, we begin by looking for a perfect square in the sequence $107^2 - 11413 = 36 = 6^2, \ldots$. Thus, $11413 = 107^2 - 6^2 = (107 + 6)(107 - 6) = 113 \cdot 101$.

3.6.4. a. The smallest square greater than 8051 is $90^2 = 8100$. We see that $90^2 - 8051 = 49 = 7^2$, so that $8051 = 90^2 - 7^2 = (90 + 7)(90 - 7) = 97 \cdot 83$.

b. The smallest square greater than 73 is 81. But the smallest square a such that $a^2 - 73$ is a perfect square is 37, for which $37^2 - 73 = 36^2$. If follows that $73 = 37^2 - 36^2 = (37 + 36)(37 - 36) = 73 \cdot 1$. This shows that 73 is prime.

c. The smallest square greater than 10897 is 105^2. But the smallest square a such that $a^2 - 10897$ is a square is $a = 329$. Then $10897 = (329 - 312)(329 + 312) = 17 \cdot 641$.

d. The smallest square greater than 11021 is 105^2, and $105^2 - 11021 = 4 = 2^2$, therefore, $11021 = (105 - 1)(105 + 2) = 103 \cdot 107$.

e. The smallest square greater than 3200399 is 1789^2. But the smallest square a such that $a^2 - 3200399$ is a square is $a = 1800$. Then $3200399 = (1800 - 199)(1800 + 199) = 1601 \cdot 1999$.

f. We have $4968^2 - 24681023 = 1$, so $24681023 = 4967 \cdot 4969$.

3.6.5. Note that $(50+n)^2 = 2500+100n+n^2$ and $(50-n)^2 = 2500-100n+n^2$. The first equation shows that the possible final two digits of squares can be found by examining the squares of the integers $0, 1, \ldots, 49$, and the second equation shows that these final two digits can be found by examining the squares of the integers $0, 1, \ldots, 25$. We find that $0^2 = 0, 1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16, 5^2 = 25, 6^2 = 36, 7^2 = 49, 8^2 = 64, 9^2 = 81, 10^2 = 100, 11^2 = 121, 12^2 = 144, 13^2 = 169, 14^2 = 196, 15^2 = 225, 16^2 = 256, 17^2 = 289, 18^2 = 324, 19^2 = 361, 20^2 = 400, 21^2 = 441, 22^2 = 484, 23^2 = 529, 24^2 = 576$, and $25^2 = 625$. It follows that the last two digits of a square are $00, e1, e4, 25, o6$, and $e9$ where e represents an even digit and o represents an odd digit.

3.6.6. Consider only the last two digits of each number in $x^2 - n = y^2$. Then y^2 and $x^2 = y^2 + n$ must end in one of the given patterns. This will eliminate many possibilities from consideration. For example, in part (a) of Exercise 4, we want to factor 8051. Then $x^2 = 8051 + y^2$, so if y^2 ends in $00, e1, e4, 25, o6$, or $e9$, then x^2 ends in $51, o2, o5, 76, e7$, or $e0$, respectively. But only 76, and $e0$ are patterns for perfect squares, so we only consider squares ending in 76 or $e0$ as candidates for x^2.

3.6.7. Suppose that $x^2 - n$ is a perfect square with $x > (n + p^2)/2p$, say a^2. Now, $a^2 = x^2 - n > ((n + p^2)/2p)^2 - n = ((n-p^2)/2p)^2$. It follows that $a > (n-p^2)/2p$. From these inequalities for x and a, we see that $x + a > n/p$, or $n < p(x + a)$. Also, $a^2 = x^2 - n$ tells us that $(x-a)(x+a) = n$. Now, $(x-a)(x+a) = n < p(x+a)$. Cancelling, we find that $x - a < p$. But since $x - a$ is a divisor of n less than p, the smallest prime divisor of n, $x - a = 1$. In this case, $x = (n = 1)/2$.

3.6.8. Certainly, $m_2 = m_1 - 2q_1 = n_1 - 2q_1$, which is the basis step for induction on k. Suppose $m_{k-1} = n_1 - 2(q_1 + \cdots + q_{k-2})$. Then $m_k = m_{k-1} - 2q_{k-1} = n_1 - 2(q_1 + \cdots + q_{k-2}) - 2q_{k-1} = n_1 - 2(q_1 + \cdots + q_{k-1})$, as desired. For the other formula, note that $n_2 = m_2 + r_1 = (m_1 - 2q_1) + (n_1 - 3q_1) = 2n_1 - 5q_1$, which is the basis step. Assume the formula holds for $k - 1$, then we have $n_k = m_k + r_{k-1} = n_1 - 2(q_1 + \cdots + q_{k-1}) + n_{k-1} - (2k-1)q_{k-1} = n_1 - 2(q_1 + \cdots + q_{k-1}) + (k-1)n_1 - (2k-1)(q_1 + \cdots + q_{k-2}) - (2k-1)q_{k-1} = kn_1 - (2k+1)(q_1 + \cdots + q_{k-1})$, as desired.

3.6.9. From the identity in Exercise 8, it is clear that if $n = n_1$ is a multiple of $2k + 1$, then so is n_k, since it is the sum of two multiples of $2k + 1$. If $(2k+1) \mid n_k$, then $(2k+1) \mid r_k$ and it follows from $r_k < 2k+1$ that $r_k = 0$. Thus, $n_k = (2k + 1)q_k$. Continuing, we see that $n = n + 2n_k - 2(2k + 1)q_k = (2k + 1)n + 2(n_k - kn) - 2(2k + 1)q_k$. It follows from Exercise 8 that $n = (2k + 1)n - 2(2k + 1)\sum_{i=1}^{k-1} q_i - 2(2k + 1)q_k = (2k + 1)n - 2(2k + 1)\sum_{i=1}^{k} q_i$. Using Exercise 8 again, we conclude that $n = (2k + 1)(n - 2\sum_{i=1}^{k} q_i) = (2k + 1)m_{k+1}$.

3.6.10. We compute $n_1 = 5899 = m_1 = 3 \cdot 1966 + 1$, $m2 = 5899 - 2(1966) = 1967, n_2 = 1967 + 1 = 1968 = 5 \cdot 393 + 3$, $m_3 = 1967 - 2 \cdot 393 = 1181, n_3 = 1181 + 3 = 1184 = 7 \cdot 169 + 1$, $m_4 = 1181 - 2(169) = 843, n_4 = 843 + 1 = 844 = 9 \cdot 93 + 7$, $m_5 = 843 - 2(93) = 657, n_5 = 657 + 7 = 664 = 11 \cdot 60 + 4$, $m_6 = 657 - 2(60) = 537, n_6 = 537 + 4 = 541 = 13 \cdot 41 + 8$, $m_7 = 537 - 2(41) = 455, n_7 = 455 + 8 = 463 = 15 \cdot 30 + 13$, $m_8 = 455 - 2(30) = 395, n_8 = 395 + 13 = 408 = 17 \cdot 24$. Therefore $17 \mid 5899$, and we have $5899 = 17 \cdot 347$.

3.6.11. To see that u is even, note that $a - c$ is the difference of odd numbers and that $b - d$ is the difference of even numbers. Thus $a - c$ and $b - d$ are even, and u must be as well. That $(r, s) = 1$ follows trivially from Theorem 2.1 (i). To continue, $a^2 + b^2 = c^2 + d^2$ implies that $(a + c)(a - c) = (d - b)(d + b)$. Dividing both sides of this equation by u, we find that $r(a + c) = s(d + b)$. From this, it is clear that $s \mid r(a + c)$. But since $(r, s) = 1$, $s \mid a + c$.

3.6.12. From Exercise 11, $r(a + c) = s(d + b)$ so $rsv = s(b + d)$ and hence, $rv = b + d$. Now, $(r, s) = 1$, so we have $(a + c, d + b) = (sv, rv) = v(s, r) = v$. Finally, since a and c are odd, $2 \mid (a + c)$, and since b and d are even, $2 \mid (b + d)$, so we have $2 \mid (a + c, b + d) = v$, so v is even.

3.6.13. To factor n, observe that $[(\frac{u}{2})^2 + (\frac{v}{2})^2](r^2 + s^2) = (1/4)(r^2u^2 + r^2v^2 + s^2u^2 + s^2v^2)$. Substituting $a - c, d - b, a + c$, and $d + b$ for ru, su, sv, and rv respectively, will allow everything to be simplified down to n. As u and v are both even, both of the factors are integers.

3.6.14. a. We have $u = (11-5, 10-14) = 2, r = (11-5)/2 = 3, s = (14-10)/2 = 2, v = (11+5, 10+14) = 8$, then $211 = ((2/2)^2 + (8/2)^2)(3^2 + 2^2) = 17 \cdot 13$.

b. We have $u = 8, r = 6, s = 5, v = 10$, then $2501 = ((8/2)^2 + (10/2)^2)(6^2 + 5^2) = 41 \cdot 61$.

c. We have $u = 4, r = 58, s = 7, v = 34$, then $1000009 = ((4/2)^2 + (34/2)^2)(58^2 + 7^2) = 293 \cdot 3413$.

3.6.15. We have $2^{4n+2}+1 = 4(2^n)^4+1 = (2\cdot 2^{2n}+2\cdot 2^n+1)(2\cdot 2^{2n}-2\cdot 2^n+1)$. Using this identity we have the factorization: $2^{18}+1 = 4(2^4)^4+1 = (2\cdot 2^8+2\cdot 2^4+1)(2\cdot 2^8-2\cdot 2^4+1) = (2^9+2^5+1)(2^9-2^5+1) = 545\cdot 481$.

3.6.16. If m has an odd factor, the identity gives a factorization of $a^m + 1$, therefore m must be a power of 2.

3.6.17. We can prove that the last digit in the decimal expansion of F_n is 7 for $n \geq 2$ by proving that the last digit in the decimal expansion of 2^{2^n} is 6 for $n \geq 2$. This can be done using mathematical induction. We have $2^{2^2} = 16$ so the result is true for $n = 2$. Now assume that the last decimal digit of 2^{2^n} is 6, that is $2^{2^n} \equiv 6 \pmod{10}$. It follows that $2^{2^{n+1}} = (2^{2^n})^{2^{n+1}-2^n} \equiv 6^{2^{n+1}-2^n} \equiv 6 \pmod{10}$. This completes the proof.

3.6.18. Note that $\sqrt{2^{2^4}+1} < 257$, so we need only check the primes less than 257 which are of the form $64k+1$. Of $64+1 = 65, 64\cdot 2+1 = 129$, and $64\cdot 3+1 = 193$, only 193 is prime. But $193 \nmid 65537$, so F_4 is prime.

3.6.19. Since every prime factor of $F_5 = 2^{2^5}+1 = 4294967297$ is of the form $2^7k+1 = 128k+1$, attempt to factor F_5 by trial division by primes of this form. We find that $128\cdot 1+1 = 129$ is not prime, $128\cdot 2+1 = 257$ is prime but does not divide 4294967297, $128\cdot 3+1 = 385$ is not prime, $128\cdot 4+1 = 513$ is not prime, and $128\cdot 5+1 = 641$ is prime and does divide 4294967297 with $4294967297 = 641 \cdot 6700417$. Any factor of 6700417 is also a factor of 4294967297. We attempt to factor 6700417 by trial division by primes of the form $128k+1$ beginning with 641. We first note that 641 does not divide 6700417. Among the other integers of the form $128k+1$ less than $\sqrt{6700417}$, namely the integers 769, 897, 1025, 1153, 1281, 1409, 1537, 1665, 1793, 1921, 2049, 2177, 2305, 2433, and 2561, only 769, 1153, and 1409 are prime, and none of them divide 6700417. Hence 6700417 is prime and the prime factorization of F_5 is $641 \cdot 6700417$.

3.6.20. We have $2^{2^0}+5 = 7$. This is the only prime of the form 2^{2^n} since $2^{2^n}+5 \equiv (-1)^{2^n}+5 \equiv 1+5 \equiv 0 \pmod{3}$ when $n > 1$.

3.6.21. The number of decimal digits of F_n is $[\log_{10} F_n]+1 = [\log_2 F_n/\log_2 10]+1$ by the change of base formula for logarithms. But this is approximately $\log_2 2^{2^n}/\log_2 10 + 1 = 2^n/\log_2 10 + 1$.

3.6.22. Suppose that a prime p divides F_n. Then by Theorem 3.20, p is of the form $2^{n+1}k+1$, but this number is larger than n for all $k = 1, 2, \ldots$, so $p \nmid n$. Therefore, $(n, F_n) = 1$.

3.6.23. Suppose $n^a - 2^m = 1$ for some integer n. Then $2^m = (n-1)(n^{a-1}+n^{a-2}+\cdots+n+1)$, where the last factor is the sum of a odd terms but must be a power of 2, therefore, $a = 2k$ for some k. Then $2^m = (n^k-1)(n^k+1)$. These last two factors are powers of 2 which differ by 2 which forces $k = 1, a = 2, m = 3$, and $n = 3$ as the only solution.

3.6.24. For 901, we try $31^1 - 901 = 60, 32^1 - 901 = 123, 33^1 - 901 = 188, 34^1 - 901 = 255, 35^1 - 901 = 324 = 18^2$, so $901 = (35-18)(35+18) = 17 \cdot 53$. On the other hand, for 2703, we try only $52^2 - 2703 = 1$, so $2703 = (52-1)(52+1) = 51 \cdot 53 = 3 \cdot 17 \cdot 53$.

3.7. Linear Diophantine Equations

3.7.1. a. Using the Euclidean algorithm we find that $2\cdot 3 + 5\cdot(-1) = 1$. Multiplying both sides by 11 gives $2\cdot 33+5\cdot(-11) = 11$. Hence $x = 33, y = -11$ is a solution. All solutions are given by $x = 33-5t, y = -11+2t$ where t is an integer.

b. Using the Euclidean algorithm we find that $17 \cdot (-3) + 13 \cdot 4 = 1$. Multiplying both sides by 100 gives $17 \cdot (-300) + 13 \cdot 400 = 100$. Hence $x = -300, y = 400$ is a solution. All solutions are given by $x = -300 + 13t, y = 400 - 17t$, where t is an integer.

c. Using the Euclidean algorithm we see that $21 \cdot 1 + 14 \cdot (-1) = 7$. Multiplying both sides by 21 gives $21 \cdot 21 + 14 \cdot (-21) = 147$. Hence $x = 21, y = -21$ is a solution. All solutions are given by $x = 21 - 2t, y = -21 + 3t$ where t is an integer.

d. Since (60,18)=3 and 97 is not divisible by 3, it follows that there are no solutions in integer of $60x + 18y = 97$.

e. Using the Euclidean algorithm it follows that $1402 \cdot 889 + 1969 \cdot (-633) = 1$. Hence $x = 889, y = -633$ is a solution. All solutions are given by $x = 889 - 1969t, y = -633 + 1402t$ where t is an integer.

3.7.2. a. Using the Euclidean algorithm we find that $3 \cdot 1 + 4 \cdot 1 = 7$. Hence $x = 1, y = 1$ is a solution. All solutions are given by $x = 1 - 4t, y = 1 + 3t$ where t is an integer.

b. Since $(12, 18) = 6$ and $6 \nmid 50$, there are no solutions.

c. Using the Euclidean algorithm we find that $11 \cdot 30 + 47 \cdot (-7) = 1$. Multiplying both sides by -11 gives $-121 \cdot 30 + 47 \cdot (77) = -11$. Hence $x = -121, y = 77$ is a solution. All solutions are given by $x = -121 - 47t, y = 77 + 30t$ where t is an integer.

d. We divide the equation by 5 to get $5x + 19y = 194$. Using the Euclidean algorithm we find that $5 \cdot 4 + 19 \cdot (-1) = 1$. Multiplying both sides by 194 gives $5 \cdot 776 + 19 \cdot 194 = 194$. Hence $x = 776, y = 194$ is a solution. All solutions are given by $x = 776 - 19t, y = 194 + 5t$ where t is an integer.

e. Using the Euclidean algorithm we find that $442 \cdot 102 + 1001 \cdot (-43) = 1$. Hence $x = 442, y = -43$ is a solution. All solutions are given by $x = 442 - 1001t, y = -43 + 102t$ where t is an integer.

3.7.3. Let x be the number of U.S. dollars and y be the number of Canadian dollars the businessman exchanges. Then $122x + 112y = 15286$. Since $(122, 112) \mid 15286$, there exist solutions with integer x and y. Using the Euclidean algorithm we find that $112(12) - 122(11) = 2$. It follows that $122(-84073) + 112(91716) = 15286$. Consequently all solutions of the linear diophantine equation are given by $x = -84073 + 56t, y = 91716 - 61t$. But our situation requires that both x and y be positive. We can see that x is positive when $t > 1501$, and y is positive when $t < 1504$. It follows that the only positive solutions which occur when $t = 1502$ and $t = 1503$, namely $x = 39, y = 94$ and $x = 95, y = 33$, respectively.

3.7.4. Let e be the number of euros and f be the number of francs. Then $111e + 83f = 4626$. Since $(111, 83) = 1$, there exist solutions. Using the Euclidean algorithm we find that $111(3) + 83(-4) = 1$. It follows that $111(13878) + 83(-18504) = 4626$, so all solutions to the diophantine equation are given by $e = 13878 - 83t$, $f = -18504 + 111t$, where t is an integer. For e to be positive we must have $13878 > 83t$ which implies that $t \leq 167$. For f to be positive we must have $111t > 18504$, which implies that $t > 166$, so we must have $t = 167$, which means that $e = 17$ and $f = 33$.

3.7.5. Let e be the number of euros and p be the number of pounds. Then $111e + 169p = 11798$. Since $(111, 169) = 1$, there exist solutions. Using the Euclidean algorithm, we find that $111(-102) + 169(67) = 1$, so that multiplying by 11798 gives us $111(-1203396) + 169(790466) = 11798$, so all solutions are given by $e = -1203396 + 169t$, $p = 790466 - 111t$. Since e is positive we must have $169t > 1203396$, which implies $t > 7120$. Since p is positive we must have $111t < 790466$, which implies that $t \leq 7121$, so we must have $t = 7121$. Therefore $e = 53$ and $p = 35$.

3.7.6. Let x be the number of plantains in each of the 63 equal piles and y be the number of plantains distributed to each traveller. Then $23y = 63x + 7$, which in this context is a diophantine equation for which

we seek positive solutions. The smallest positive solutions are $x = 5$ and $y = 14$, so each pile had 5 plantains. (Mahavira's intent being to find the smallest solution.)

3.7.7. Let x be the number of apples and y the number of oranges. We have $25x + 18y = 839$. Using the Euclidean algorithm we find that $-5 \cdot 25 + 7 \cdot 18 = 1$. It follows that $25(-5 \cdot 839) + 18(7 \cdot 839) = 25(-4195) + 18 \cdot 5873 = 839$. Consequently all solutions of the linear diophantine equation are given by $x = -4195 + 18t, y = 5873 - 25t$ where t is an integer. For x and y to both be nonnegative, we must have $4195/18 \leq t \leq 5873/25$. Since t must be an integer, this requires that $t = 234$. This give the unique nonnegative solution $x = -4195 + 18 \cdot 234 = 17, y = 5873 - 25 \cdot 234 = 23$.

3.7.8. We need to solve the diophantine equation $18x + 33y = 549$. We get the general solution $x = 366 - 11t, y = -183 + 6t$. We seek only positive solutions, and to minimize the number of fruit, we will maximize y, the number of more expensive fruit. This gives us $x = 3, y = 15$ when $t = 33$.

3.7.9. a. Suppose that x 14-cent stamps and y 21-cent stamps are combined to form \$ 3.50. Then $14x + 21y = 350$. Since $(14, 21) = 7$ and $7 \mid 350$ it follows that there are solutions in integers to this diophantine equation. We can find these by first noting that $7 = -1 \cdot 14 + 1 \cdot 21$, so $350 = 50 \cdot 7 = -50 \cdot 14 + 50 \cdot 21$. This implies that all solutions in integers are given by $x = -50 + (21/7)t = -50 + 3t$ and $y = 50 - (14/7)t = 50 - 2t$ where t is an integer. For x to be positive we must have $t \geq 17$ and for y to be positive we must have $t \leq 25$. This gives the solutions, for $17 \leq t \leq 25$, $x = 1, y = 16$; $x = 4, y = 14$; $x = 7, y = 12$; $x = 10, y = 10$; $x = 13, y = 8$; $x = 16, y = 6$; $x = 19, y = 4$; $x = 22, y = 2$; and $x = 25, y = 0$.

b. Let x be the number of 14-cent stamps and y be the number of 21-cent stamps. Then $14x + 21y = 400$. However, $(14, 21) = 7$ but 7 does not divide 400. Hence there are no solutions in integers and it is impossible to use 14-cent and 21-cent stamps to form postage of \$ 4.00.

c. We have 18 solutions: $(0, 37), (3, 35), \ldots, (54, 1)$.

3.7.10. a. We solve the diophantine equation $11x + 8y = 777$ and get the general solution $x = 2331 - 8t, y = -3108 + 11t$. Since we seek only positive solutions, the first equation implies that $2331 - 8t \geq 0$ or $t \leq 291$. The second equation implies that $t \geq 282$. So there are 10 possible configurations for the order.

b. We solve the diophantine equation $11x + 8y = 96$ and get the general solution $x = 288 - 8t, y = -384 + 11t$. Since we seek only positive solutions, the first equation implies that $288 - 8t \geq 0$ or $t \leq 36$. The second equation implies that $t \geq 35$. So either $x = 288 - 8 \cdot 35 = 8, y = -384 + 11 \cdot 35 = 1$ or $x = 288 - 8 \cdot 36 = 0, y = -384 + 11 \cdot 36 = 12$.

c. We solve the diophantine equation $11x + 8y = 69$ and get the general solution $x = 207 - 8t, y = -276 + 11t$. Since we seek only positive solutions, the first equation implies that $207 - 8t \geq 0$ or $t \leq 25$. The second equation implies that $t \geq 26$, so there are no solutions.

3.7.11. a. Since $(2, 3) = 1$, we can take z to be any integer t and solve the diophantine equation $2x + 3y = 5 - 4t$, which leads to the solution $x = -5 + 3s - 2t, y = 5 - 2s, z = t$

b. Since $(7, 21, 35) = 7 \nmid 8$, there are no solutions.

c. Since $(101, 102) = 1$, we can take z to be any integer t and solve the diophantine equation $101x + 102y = 1 - 103z$, which leads to the solution $x = -1 + 102s + t, y = 1 - 101s - 2t, z = t$

3.7.12. a. Since $(2, 3) = 1$, we can choose any values for x_2, and x_3, and solve the remaining equation for x_1 and x_4. We have $2(-1) + 3(1) = 1$, so $2(-1(5 - 5x_2 - 4x_3)) + 3(5 - 5x_2 - 4x_3) = 5 - 5x_2 - 4x_3$. Then a general solution is given by $x_1 = -5 + 5x_2 + 4x_3 + 3t, x_2 = x_2, x_3 = x_3$ and $x_4 = 5 - 5x_2 - 4x_3 - 2t$.

b. The general solution is given by $x_1 = x_1, x + 2 = 3 - x_1 - 2x_4 - 3t, x_3 = -6 + x_1 + 3x_4 + 7t$, and $x_4 = x + 4$.

c. Note that $(6, 35) = 1$, so we can choose x_1, x_3, and x_4 freely and solve for the other variables. This gives us a general solution of $x_1 = x_1, x_2 = 6(1 - 15x_1 - 10x_3 - 21x_4) + 35t, x_3 = x_3, x_4 = x_4$, and $x_5 = -(1 - 15x_1 - 10x_3 - 21x_4) - 6t$.

3.7.13. Let x be the number of pennies, y the number of dimes, and z the number of quarters. Then $x + 10y + 25z = 99$. Since x, y, and z are all nonnegative, it follows that $z = 0, 1, 2$, or 3. First suppose that $z = 0$. Then $x + 10y = 99$. We find the nonnegative solutions to this by letting y range form 0 to 9. We see that $x = 9, y = 9; x = 19, y = 8; x = 29, y = 7; x = 39, y = 6; x = 49, y = 5; x = 59, y = 4; x = 69, y = 3; x = 79, y = 2; x = 89, y = 1$; and $x = 99, y = 0$ are the solutions for $z = 0$. Now let $z = 1$. Then $x + 10y = 74$. The nonnegative solutions to this are determined by letting y range from 0 to 7. We see that $x = 4, y = 7; x = 14, y = 6; x = 24, y = 5; x = 34, y = 4; x = 44, y = 3; x = 54, y = 2; x = 64, y = 1$; and $x = 74, y = 0$ are the solutions with $z = 1$. Now let $z = 2$. Then $x + 10y = 49$. The nonnegative solutions to this are determined by letting y range from 0 to 4. We see that $x = 9, y = 4; x = 19, y = 3; x = 29, y = 2; x = 39, y = 1$; and $x = 49, y = 0$ are the solutions with $z = 2$. Finally, let $z = 3$. Then $x + 10y = 24$. The nonnegative solutions to this are determined by letting y range from 0 to 2. We see that $x = 4, y = 2; x = 14, y = 1$; and $x = 24, y = 0$ are the solutions with $z = 3$. We have exhausted all nonnegative solutions of our equation.

3.7.14. a. We can use either 0, 2, or 4 quarters, and make up the difference with dimes. This gives us 3 ways: (dimes, quarters) $= (10, 0), (5, 2)$, or $(0, 4)$.

b. In part (a) we can replace any dime by two nickels and any quarter by 5 nickels, this gives us the following solutions: (nickels, dimes, quarters) $= (0, 10, 0)$, $(2, 9, 0)$, $(4, 8, 0), \ldots (20, 0, 0)$, $(0, 5, 2), \ldots$ $(10, 0, 2)$, $(5, 5, 1), \ldots, (15, 0, 1)$, $(0, 0, 4)$, $(5, 0, 3)$ for 24 ways in all.

c. Each nickel listed in part (b) can be changed into 5 pennies, giving 175 ways in all.

3.7.15. a. We subtract the first equation from the second to get the diophantine equation $7y + 49y = 56$, which has solutions $y = 8 - 7t, z = t$. Substituting these expressions into the first equation gives us $x = 92 + 6t, y = 8 - 7t, z = t$.

b. We subtract the first equation from the second to get the diophantine equation $5y + 20z = 21$. Since $(5, 20) = 5 \nmid 21$, there is no solution.

c. We subtract the first equation from the other two to get the system $y + 2z + 3w = 200$, and $3y + 8z + 15w = 900$. We subtract 3 times this first equation from the second to get $2z + 6w = 300$, which has solutions $z = 150 - 3t, w = t$. Substituting these expressions into $y + 2z + 3w = 200$ gives us $y = -100 + 3t$, and substituting all three expressions into the first equation gives us $x = 50 - t$.

3.7.16. Suppose that there x nickels, y dimes, and z quarters. Since there are 24 coins in the piggy bank we know that $x + y + z = 24$. since there are two dollars in the bank, we know that $5x + 10y + 25z = 200$. Multiplying the first equation by 5 and subtracting it from the second yields $5y + 20z = 80$. dividing both sides by 5 gives $y + 4z = 16$. The solutions to the linear diophantine equation are $y = 16 - 4t, z = t$ where t is a positive integer. There are 5 nonnegative solutions for $0 \leq t \leq 4$. We have $y = 16$ and $z = 0$, which gives $x = 8, y = 12$ and $z = 1$, which gives $x = 11, y = 8$ and $z = 2$, which gives $x = 14, y = 4$ and $z = 3$, which gives $x = 17, y = 0$ and $z = 4$, which gives $x = 20$. Hence the solutions are 8 nickels, 16 dimes, and 0 quarters; 11 nickels, 12 dimes, and 1 quarter; 14 nickels, 8 dimes, and 2 quarters; 17 nickels, 4 dimes, and 3 quarters; and 20 nickels, 0 dimes, and 4 quarters.

3.7.17. Let x be the number of first-class tickets sold, y be the number of second-class tickets sold, and z be the number of stand-by tickets sold. Then we have the system of diophantine equations $140x + 110y + 78z = 6548$, $x + y + z = 69$. Substituting $z = 69 - x - y$ into the first equation yields $62x + 32y = 1166$, which has solutions $x = 9 + 16t, y = 19 - 31t$. Then $z = 41 + 15t$. The only value of t that leaves all three

quantities positive is $t = 0$, so the only solution is $x = 9, y = 19, z = 41$.

3.7.18. Suppose that there are x pennies, y dimes, and z quarters. Then $x + y + z = 50$ and $x + 10y + 25z = 300$. Subtracting the first equation from the second shows that $9y + 24z = 250$. This linear diophantine equation has no solutions since $(9, 24) = 3$ and 3 does not divide 250. Hence there is no way to have 50 coins, all pennies, dimes, and quarters, that are worth \$ 3.

3.7.19. The quadrilateral with vertices $(b, 0), (0, a), (b - 1, -1)$, and $(-1, a - 1)$, has area $a + b$. Pick's Theorem, from elementary geometry, states that the area of a simple polygon whose vertices are lattice points (points with integer coordinates) is given by $\frac{1}{2}x + y - 1$, where x is the number of lattice points on the boundary and y is the number of lattice points inside the polygon. Since $(a, b) = 1$, $x = 4$, and therefore, by Pick's Theorem, the quadrilateral contains $a + b - 1$ lattice points. Every point corresponds to a different value of n in the range $ab - a - b < n < ab$. Therefore every n in the range must get hit, so the equation is solvable.

3.7.20. If $x = -1$, we can solve the equation $ax + by = ab - a - b$ for b and get $b = a - 1$. Since $(a, ab - a - b) = (b, ab - a - b) = 1$, the general solution is $x = -1 + bt, y = a - 1 - at$. Then for a positive solution, we must have $x = -1 + bt \geq 0$ or $t \geq 1$, but also, $y = a - 1 - at \geq 0$ or $y \leq (a - 1)/a < 1$, a contradiction, so there are no solutions.

3.7.21. See the solution to Exercise 19. The line $ax + by = ab - a - b$ bisects the rectangle with vertices $(-1, a - 1)$, $(-1, -1)$, $(b - 1, a - 1)$, and $(b - 1, -1)$ but contains no lattice points. Hence, half the interior points are below the line and half are above. The half below correspond to $n < ab - a - b$ and there are $(a - 1)(b - 1)/2$ of them.

3.7.22. Let a and b be the values of the stamps, with $a \geq b$. Since there are 33 postages that cannot be formed Exercise 17 tells us that $(a - 1)(b - 1)/2 = 33$. Hence $(a - 1)(b - 1) = 66$. Since a and b are integers, either $a = 67$ and $b = 2$, $a = 34$ and $b = 3$, $a = 23$ and $b = 4$, or $a = 12$ and $b = 7$. However postage of 46 cents cannot be formed, so there are no nonnegative solutions of $ax + by = 46$. Note that $0 \cdot 67 + 23 \cdot 2 = 46, 1 \cdot 34 + 4 \cdot 3 = 46$, and $2 \cdot 23 + 0 \cdot 4 = 46$, but there are no nonnegative solutions of $12x + 7y = 46$, as is easily shown. The values of the two stamps are 7 cents and 12 cents.

3.7.23. Let x, y and z be the number of cocks, hens and chickens respectively. The problem leads to the system of diophantine equations $x + y + z = 100, 5x + 3y + z/3 = 100$. Substituting $z = 100 - x - y$ into the second equation and clearing fractions yields $14x + 8y = 200$, which has solutions $x = 4t, y = 25 - 7t$. It follows that $z = 75 + 3t$. The only values for t which make all three of these numbers nonnegative are $t = 0, 1, 2$, and 3. Thus the solutions to the problem are $(x, y, z) = (0, 25, 75); (4, 18, 78); (8, 11, 81); (12, 4, 84.)$

3.7.24. Suppose that $\frac{1}{x} + \frac{1}{y} = \frac{1}{14}$. Then $14x + 14y = xy$. This implies that $xy - 14x - 14y + 196 = 196$ so that $(x - 14)(y - 14) = 196$. It follows that $x - 14$ and $y - 14$ are divisors of 196. Consequently the values of $x - 14$ and $y - 14$ must be 1 and 196, 2 and 98, 4 and 49, 7 and 28, 14 and 14, 28 and 7, 49 and 4, 98 and 2, 196 and 1, or the negatives of these values. Solving for x and y gives $(x, y) = (15, 210)$, $(16, 112)$, $(18, 63)$, $(21, 42)$, $(28, 28)$, $(42, 21)$, $(63, 18)$, $(112, 16)$, $(210, 15)$, $(13, -182)$, $(12, -84)$, $(10, -35)$, $(7, -14)$, $(-14, 7)$, $(-35, 10)$, $(-84, 12)$, or $(-182, 13)$.

CHAPTER 4

Congruences

4.1. Introduction to Congruences

4.1.1. a. We have $2 \mid (13-1) = 12$, so $13 \equiv 1 \pmod{2}$.

b. We have $5 \mid (22-7) = 15$, so $22 \equiv 7 \pmod{5}$.

c. We have $13 \mid (91-0) = 91$, so $91 \equiv 0 \pmod{13}$.

d. We have $7 \mid (69-62) = 7$, so $69 \equiv 62 \pmod{7}$.

e. We have $3 \mid (-2-1) = -3$, so $-1 \equiv 1 \pmod{3}$.

f. We have $11 \mid (-3-30) = -33$, so $-3 \equiv 30 \pmod{11}$.

g. We have $40 \mid (111-(-9)) = 120$, so $111 \equiv -9 \pmod{40}$.

h. We have $37 \mid (666-0) = 666$, so that $666 \equiv 0 \pmod{37}$.

4.1.2. a. We have $7 \mid (15-1) = 14$, so $15 \equiv 1 \pmod{7}$.

b. We have $7 \mid (42-0) = 42$, so $42 \equiv 0 \pmod{7}$.

c. We have $7 \nmid (99-2) = 97$, so $99 \not\equiv 2 \pmod{7}$.

d. We have $7 \nmid (8-(-1)) = 9$, so $8 \not\equiv -1 \pmod{7}$.

e. We have $7 \mid (-9-5) = -14$, so $-9 \equiv 5 \pmod{7}$.

f. We have $7 \mid (699-(-1)) = 700$, so $699 \equiv -1 \pmod{7}$.

4.1.3. a. Since the positive divisors of $27-5 = 22$ are 1, 2, 11, and 22 it follows that $27 \equiv 5 \pmod{m}$ if and only if $m = 1, m = 2, m = 11$, or $m = 22$.

b. Since the positive divisors of $1000-1 = 999$ are $1, 3, 9, 27, 37, 111, 333$, and 999, it follows that $1000 \equiv 1 \pmod{m}$ if and only if m is one of these eight integers.

c. Since the only positive divisors of $1331-0 = 1331$ are $1, 11, 121$, and 1331 it follows that $1331 \equiv 0 \pmod{m}$ if and only if m is one of these four integers.

4.1.4. Suppose that a is an even integer. Then $a = 2k$ for some integer k. The $a^2 = 4k^2$. Consequently $4 \mid a^2$ so that $a^2 \equiv 0 \pmod{4}$. Suppose that a is an odd integer. Then $a = 2k+1$ for some integer k. Then $a^2 = 4k^2+4k+1 = 4(k^2+k)+1$, so that $a^2-1 = 4(k^2+k)$. It follows that $a^2 \equiv 1 \pmod{4}$.

4.1.5. Suppose that a is odd. Then $a = 2k+1$ for some integer k. Then $a^2 = (2k+1)^2 = 4k^2+4k+1 = 4k(k+1)+1$. If k is even, then $k = 2l$ where l is an integer. Then $a^2 = 8l(2l+1)+1$. Hence $a^2 \equiv 1 \pmod{8}$. If k is odd, then $k = 2l+1$ when l is an integer. Then $a^2 = 4(2l+1)(2l+2)+1 = 8(2l+1)(l+$

$1) + 1$. Hence $a^2 \equiv 1 \pmod 8$. It follows that $a^2 \equiv 1 \pmod 8$ whenever a is odd.

4.1.6. a. $22 \pmod{13} = 9$, since $22 = (1)(13) + 9$.

b. $100 \pmod{13} = 9$, since $100 = (7)(13) + 9$.

c. $1001 \pmod{13} = 0$, since $1001 = (77)(13)$.

d. $-1 \pmod{13} = 12$, since $-1 = (-1)(13) + 12$.

e. $-100 \pmod{13} = 4$, since $-100 = (-8)(13) + 4$.

f. $-1000 \pmod{13} = 1$, since $-1000 = (-77)(13) + 1$.

4.1.7. a. Since $n! \equiv 0 \pmod 2$ if $n \geq 2$, we have $1! + 2! + 3! + \cdots + 100! \equiv 1 \pmod 2$.

b. We have $n! \equiv 0 \pmod 7$ whenever $n \geq 7$. Since $1! \equiv 1 \pmod 7, 2! \equiv 2 \pmod 7, 3! \equiv 6 \pmod 7, 4! = 24 \equiv 3 \pmod 7, 5! = 120 \equiv 1 \pmod 7$ and $6! = 720 \equiv 6 \pmod 7$, we have $1! + 2! + 3! + \cdots + 100! \equiv 1! + 2! + 3! + 4! + 5! + 6! \equiv 1 + 2 + 6 + 3 + 1 + 6 \equiv 5 \pmod 7$.

c. Since $n! \equiv 0 \pmod{12}$ whenever $n \geq 4$, it follows that $1!+2!+3!+\cdots+100! \equiv 1+2+6 \equiv 9 \pmod{12}$.

d. Since $n! \equiv 0 \pmod{25}$ whenever $n \geq 10$, it follows that $1! + 2! + 3! + \cdots + 100! \equiv 1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! \equiv 1 + 2 + 6 + 24 + 20 + 20 + 15 + 20 + 5 \equiv 13 \pmod{25}$.

4.1.8. Since $a \equiv b \pmod m$, there exists an integer k_1 such that $a = b + k_1 m$. Since $n \mid m$, there exists an integer k_2 such that $m = k_2 n$. Thus $a = b + (k_1 k_2)n$, so $a \equiv b \pmod m$.

4.1.9. Since $a \equiv b \pmod m$, there exists an integer k such that $a = b+km$. Thus, $ac = (b+km)c = bc+k(mc)$. By Theorem 4.1, $ac \equiv bc \pmod{mc}$.

4.1.10. Since $a \equiv b \pmod c$, there exists an integer k such that $a = b + kc$. Let $d_1 = (a, c)$, so that $a = d_1 n$ and $c = d_1 m$. Then $d_1 n = a = b + km = b + k d_1 n_2$, so $b = d_1(n - km)$. Thus $d_1 \leq d_2$. A symmetrical argument establishes that $d_2 \leq d_1$, so $d_1 = d_2$.

4.1.11. a. We proceed by induction on n. It is clearly true for $n = 1$. For the inductive step we assume that $\sum_{j=1}^{n} a_j \equiv \sum_{j=1}^{n} b_j \pmod m$ and that $a_{n+1} \equiv b_{n+1} \pmod m$. Now $\sum_{j=1}^{n+1} a_j = (\sum_{j=1}^{n} a_j) + a_{n+1} \equiv (\sum_{j=1}^{n} b_j) + b_{n+1} = \sum_{j=1}^{n+1} b_j \pmod m$ by Theorem 4.5(i). This completes the proof.

b. We use induction on n. For $n = 1$, the identity clearly holds. This completes the basis step. For the inductive step we assume that $\prod_{j=1}^{n} a_j \equiv \prod_{j=1}^{n} b_j \pmod m$ and $a_{n+1} \equiv b_{n+1} \pmod m$. Then $\prod_{j=1}^{n+1} a_j = a_{n+1}(\prod_{j=1}^{n} a_j) \equiv b_{n+1}(\prod_{j=1}^{n} b_j) = \prod_{j=1}^{n+1} b_j \pmod m$ by Theorem 4.5(iii). This completes the proof.

4.1.12.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

4.1.13.

−	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3
3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

4.1.14.

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

4.1.15. a. Since $11 + 29 = 40 \equiv 4 \pmod{12}$, the (12-hour) clock reads 4 o'clock 29 hours after reading 11 o'clock.

b. Since $2 + 100 = 102 \equiv 6 \pmod{12}$, the (12-hour) clock reads 6 o'clock 100 hours after it reads 2 o'clock.

c. Since $6 - 50 = -44 \equiv 4 \pmod{12}$, the (12-hour) clock reads 4 o'clock 50 hours before it reads 6 o'clock.

4.1.16. We find that $1^4 \equiv 3^4 \equiv 7^4 \equiv 9^4 \equiv 1 \pmod{10}, 2^4 \equiv 4^4 \equiv 6^4 \equiv 8^4 = 6 \pmod{10}$, $5^4 \equiv 5 \pmod{10}$, and $0^4 \equiv 0 \pmod{10}$. It follows that the final decimal digit of a fourth power is either 0,1,5, or 6.

4.1.17. If $a^2 \equiv b^2 \pmod{p}$ then $p \mid (a^2 - b^2) = (a + b)(a - b)$. Since p is prime, either $p \mid (a + b)$ or $p \mid (a - b)$. Hence either $a \equiv b \pmod{p}$ or $a \equiv -b \pmod{p}$.

4.1.18. Suppose that $a^k \equiv b^k \pmod{m}$ and $a^{k+1} \equiv b^{k+1} \pmod{m}$. Then multiplying both sides of the congruence $a^k \equiv b^k \pmod{m}$ by b gives $b \cdot a^k \equiv b^{k+1} \pmod{m}$. Since $a^{k+1} \equiv b^{k+1} \pmod{m}$, we see that $b \cdot a^k \equiv a^{k+1} \pmod{m}$. Hence $b \cdot a^k - a^{k+1} = (b - a)a^k \equiv 0 \pmod{m}$. Since $(a, m) = 1$ we see that $(a^k, m) = 1$ and $m \mid (b - a)$. It follows that $a \equiv b \pmod{m}$.

This result is not necessarily true when $(a, m) \neq 1$. Take $m = a = 4$ and $b = 2$. Then $a^2 = b^2 \pmod{m}$ and $a^3 \equiv b^3 \pmod{m}$ but $a \not\equiv ab \pmod{m}$.

4.1.19. Note that $1 + 2 + 3 + \cdots + (n + 1) = (n - 1)n/2$. If n is odd, then $(n - 1)$ is even, so $(n - 1)n/2$ is an integer. Hence $n \mid (1 + 2 + 3 + \cdots + (n - 1))$ if n is odd, and $1 + 2 + 3 + \cdots + (n - 1) \equiv 0 \pmod{n}$. If n is even, then $n = 2k$ where k is an integer. Then $(n - 1)n/2 = (n - 1)k$. We can easily see that n does not divide $(n - 1)k$ since $(n, n - 1) = 1$ and $k < n$. It follows that $1 + 2 + \cdots + (n - 1)$ is not congruent to 0 modulo n if n is even.

4.1.20. Note that $1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = ((n-1)n)^2/4$. If n is odd then $(n-1)$ is even, so $((n-1)n)^2/4$ is an integer. Thus the sum is a multiple of n, and is congruent to 0 modulo n. If n is a multiple of 4, then $n/4$ is an integer, and the sum is again a multiple of n.

If n is even but not a multiple of 4, then $n = 2k$ where k is odd, and $(n - 1)$ is also odd. Thus $((n - 1)n)^2/4 = ((n - 1)2k)^2/4 = ((n - 1)k)^2$, which is odd and thus not congruent to 0 modulo n (which is even).

4.1.21. $1^2 + 2^2 + \cdots + (n - 1)^2 \equiv 0 \pmod{n}$ if and only if n is relatively prime to 6. If $(n, 6) = 1$, then $1^2 + 2^2 + \cdots + (n - 1)^2 = n(n - 1)(2n - 1)/6 \equiv 0 \cdot (n - 1)(2n - 1)/6 \equiv 0 \pmod{n}$, using Exercise 7 from Section 1.2 and Theorem 4.3(iii). This works because $1^2 + 2^2 + \cdots + (n - 1)^2$ is an integer and $(n, 6) = 1$ implies that $(n - 1)(2n - 1)/6$ is an integer, and so we are dealing with integer-only arithmetic. If however, $2 \mid n$ so that $2 \mid (n, 6)$ and if $1^2 + 2^2 + \cdots + (n - 1)^2 = n(n - 1)(2n - 1)/6 \equiv 0 \pmod{n}$, then $nk = n(n - 1)(2n - 1)/6$ for some integer k by Theorem 4.1. It follows that $6k = (n - 1)(2n - 1)$. But $6k$ is even, and $(n - 1)(2n - 1)$ is odd since both $n - 1$ and $2n - 1$ are odd. If $3 \mid n$, and $n(n - 1)(2n - 1)/6 \equiv 0 \pmod{n}$, then $nk = n(n-1)(2n-1)/6$ by Theorem 4.1. Hence, $6k = (n-1)(2n-1)$. But if we look at this equality modulo 3, we see that $0 \equiv 6k = (n-1)(2n-1) \equiv (-1)(-1) = 1 \pmod{3}$. Again, a contradiction.

4.1.22. When $n = 1$ we have $4^1 = 4 = 1+3 \cdot 1$ so the basis step holds. Now suppose that $4^n \equiv 1+3n \pmod{9}$. Then $4^{n+1} = 4 \cdot 4^n \equiv 4(1 + 3n) \equiv 4 + 12n \equiv 4 + 3n \equiv 1 + 3(n + 1) \pmod{9}$. This completes the proof by mathematical induction.

4.1.23. If $n = 1$, then $5 = 5^1 = 1 + 4(1) \pmod{16}$, so the basis step holds. For the inductive step, we assume that $5^n = 1 + 4n \pmod{16}$. Now $5^{n+1} \equiv 5^n 5 \equiv (1 + 4n)5 \pmod{16}$ by Theorem 4.3(iii). Further,

$(1+4n)5 \equiv 5+20n \equiv 5+4n \pmod{16}$. Finally $5+4n = 1+4(n+1)$. So, $5^{n+1} \equiv 1+4(n+1) \pmod{16}$. This completes the proof.

4.1.24. We can take $1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23$, and 25 to form a complete system of residues modulo 13.

4.1.25. Note that if $x \equiv 0 \pmod 4$ then $x^2 \equiv 0 \pmod 4$, if $x \equiv 1 \pmod 4$ then $x^2 \equiv 1 \pmod 4$, if $x \equiv 2 \pmod 4$ then $x^2 \equiv 4 \equiv 0 \pmod 4$, and if $x \equiv 3 \pmod 4$ then $x^2 \equiv 9 \equiv 1 \pmod 4$. Hence $x^2 \equiv 0$ or $1 \pmod 4$ whenever x is an integer. It follows that $x^2 + y^2 \equiv 0, 1$ or $2 \pmod 4$ whenever x and y are integers. We see that n is not the sum of two squares when $n \equiv 3 \pmod 4$.

4.1.26. If x solves $x^2 = x \pmod p$, we know that $p \mid x^2 - x = x(x-1)$. Thus, since p is prime, either $p \mid x$, in which case $x \equiv 0 \pmod p$, or $p \mid (x-1)$, in which case $x \equiv 1 \pmod p$.

4.1.27. By Theorem 4.1, for some integer a, $ap^k = x^2 - x = x(x-1)$. By the Fundamental Theorem of Arithmetic, p^k is a factor of $x(x-1)$. Since p cannot divide both x and $x-1$, we know that $p^k \mid x$ or $p^k \mid x-1$. Thus, $x \equiv 0$ or $x \equiv 1 \pmod{p^k}$.

4.1.28. a. Since $2 \equiv 2 \pmod{47}$, $2^2 \equiv 4 \pmod{47}$, $2^4 \equiv 16 \pmod{47}$, $2^8 = 256 \equiv 21 \pmod{47}$, and $2^{16} \equiv 21^2 \equiv 441 \equiv 18 \pmod{47}$, $2^{32} \equiv 18^2 \equiv 324 \equiv 42 \pmod{47}$.

b. We have $2^{47} = (2^{32})(2^8)(2^4)(2^2)(2^1)$. Using the results of part (a), we have $2^{47} \equiv (42)(21)(16)(4)(2) \equiv (882)(16)(8) \equiv (36)(128) \equiv (36)(34) \equiv 1224 \equiv 2 \pmod{47}$.

c. Continuing our powers of 2, $2^{64} \equiv 42^2 \equiv 1764 \equiv 25 \pmod{47}$, and $2^{128} \equiv 25^2 \equiv 625 \equiv 14 \pmod{47}$. Thus, since $2^{200} = (2^{128})(2^{64})(2^8)$, we have $2^{200} \equiv (14)(25)(21) \equiv (350)(21) \equiv (21)(21) \equiv 18 \pmod{47}$.

4.1.29. First note that there are m_1 possibilities for a_1, m_2 possibilities for a_2, and in general m_i possibilities for a_i. Thus there are $m_1 m_2 \cdots m_k$ expressions of the form $M_1a_1 + M_2a_2 + \cdots M_ka_k$ where $a_1, a_2, \ldots, a_k$ run through complete systems of residues modulo $m_1, m_2, \ldots, m_k$, respectively. Since this is exactly the size of a complete system of residues modulo M, the result will follow if we can show distinctness of each of these expressions modulo M. Suppose, by way of contradiction, that $M_1a_1 + M_2a_2 + \cdots + M_ka_k \equiv M_1a_1' + M_2a_2' + \cdots + M_ka_k' \pmod M$. Then $M_1a_1 \equiv M_1a_1' \pmod{m_1}$, since m_1 divides each of $M_2, M_3, \ldots, M_k$, and further $a_1 \equiv a_1' \pmod{m_1}$ since $(M_1, m_1) = 1$. Similarly $a_i \equiv a_i' \pmod{m_i}$. Thus a_i' is in the same congruence class modulo m_i as a_i for all i. The result now follows.

4.1.30. Let r be the least positive residue of $u+v$, so $u+v \equiv r \pmod m$, or equivalently, there exists an integer k such that $u+v = km+r$. Since u and v are both positive and less than m, we also know that $u+v \leq 2m$ so k is either 0 or 1. Following the hint, assume without loss of generality that $u \leq v$. *Case 1*: $r < u$. Then $u+v > r$, so $u+v = m+r$. *Case 2*: $r > v$. Then $u+v = r$. Note that r cannot be between u and v since that would require one of u or v to be larger than m.

4.1.31. a. Let $\sqrt{n} = a + r$, where a is an integer, and $0 \leq r < 1$. We now consider two cases, when $0 \leq r < \frac{1}{2}$ and when $\frac{1}{2} \leq r < 1$. For the first case, $T = [\sqrt{n} + \frac{1}{2}] = a$, and so $t = T^2 - n = -(2a + r^2)$. Thus $|t| = 2ar + r^2 < 2a(\frac{1}{2}) + (\frac{1}{2})^2 = a + \frac{1}{4}$. Since both T and n are integers, t is also an integer. It follows that $|t| \leq [a + \frac{1}{4}] = a = T$. For the second case, when $\frac{1}{2} \leq r < 1$, we find that $T = [\sqrt{n} + \frac{1}{2}] = a + 1$ and $t = 2a(1-r) + (1-r^2)$. Since $\frac{1}{2} \leq r < 1$, $0 < (1-r) \leq \frac{1}{2}$ and $0 < 1 - r^2 < 1$. It follows that $t \leq 2a(\frac{1}{2}) + (1-r^2)$. Because t is an integer, we can say that $t \leq [a + (1-r^2)] = a < T$.

b. By the division algorithm, we see that if we divide x by T we get $x = aT + b$, where $0 \leq b < T$. If a were negative, then $x = aT + b \leq (-1)T + b < 0$; but we assumed x to be nonnegative. This shows that $0 \leq a$. Suppose now that $a > T$. Then $x = aT + b \geq (T+1)T = T^2 + T \geq (\sqrt{n} - \frac{1}{2})^2 + (\sqrt{n} - \frac{1}{2}) = n - \frac{1}{4}$ and, as x and n are integers, $x \geq n$. This is a contradiction, which shows that $a \leq T$. Similarly, $0 \leq c \leq T$ and $0 \leq d < T$.

c. $xy = (aT + b)(cT + d) = acT^2 + (ad + bc)T + bd \equiv ac(T^2 - n) + zT + bd \equiv act + zT + bd \pmod{n}$.

d. Use part (c), substituting $eT + f$ for ac.

e. The first half is identical to part (b); the second half follows by substituting $gT + h$ for $z + et$ and noting that $T^2 \equiv t \pmod{n}$.

f. Certainly, ft and gt can be computed since all three numbers are less than T, which is less than $\sqrt{n} + 1$. So $(f + g)t$ is less than $2n < w$. Similarly, we can compute $j + bd$ without exceeding the word size. And, finally, using the same arguments, we can compute $hT + k$ without exceeding the word size.

4.1.32. To compute b^N modulo m, first express N in ternary (base 3) notation as $N = (a_k, a_k - 1, \ldots, a_0)$, and then find the least positive residues of b^{3^j}, $j = 1, 2, \ldots, k$ by successively cubing and reducing modulo m. Finally, multiply together the least positive residues, repeating each term b^{3^j} a_j times, reducing modulo m after each multiplication.

4.1.33. a. We have $3^{10} \equiv (3^2)^5 \equiv 9^5 \equiv (-2)^5 \equiv -32 \equiv 1 \pmod{11}$.

b. We have $2^{12} \equiv (2^4)^3 \equiv 16^3 \equiv 3^3 \equiv 27 \equiv 1 \pmod{13}$.

c. We have $5^{16} \equiv (5^2)^8 \equiv 25^8 \equiv 8^8 \equiv (8^2)^4 \equiv 64^4 \equiv (-1)^4 \equiv 1 \pmod{17}$.

d. We have $3^{22} \equiv (3^3)^7 \cdot 3 \equiv 27^7 \cdot 3 \equiv 4^7 \cdot 3 \equiv (4^3)^2 \cdot 4 \cdot 3 \equiv 64^2 \cdot 12 \equiv (-5)^2 \cdot 12 \equiv 2 \cdot 12 \equiv 24 \equiv 1 \pmod{23}$.

e. The theorem is that $a^{p-1} \equiv 1 \pmod{p}$ whenever p is prime and p does not divide a. This is Fermat's little theorem which will be proved in Chapter 5.

4.1.34. a. Since $2! \equiv 2 \pmod{7}$, $3! \equiv 6 \pmod{7}$, $4! = 24 \equiv 3 \pmod{7}$, and $5! \equiv 5 \cdot 3 \pmod{7} \equiv 1 \pmod{7}$, we have $6! \equiv 6 \pmod{7}$.

b. Since $2! \equiv 2 \pmod{11}$, $3! \equiv 6 \pmod{11}$, $4! \equiv 2 \pmod{11}$, $5! \equiv 5 \cdot 2 \pmod{11} \equiv 10 \pmod{11}$, $6! \equiv 6 \cdot 10 \pmod{11} \equiv 5 \pmod{11}$, $7! \equiv 7 \cdot 5 \pmod{11} \equiv 2 \pmod{11}$, $8! \equiv 8 \cdot 2 \pmod{11} \equiv 5 \pmod{11}$, and $9! \equiv 9 \cdot 5 \pmod{11} \equiv 1 \pmod{11}$, we have $10! \equiv 10 \pmod{11}$.

c. Since $2! \equiv 2 \pmod{13}$, $3! \equiv 6 \pmod{13}$, $4! = 24 \equiv 11 \pmod{13}$, $5! \equiv 5 \cdot 11 \pmod{13} \equiv 3 \pmod{13}$, $6! \equiv 6 \cdot 3 \pmod{13} \equiv 5 \pmod{13}$, $7! \equiv 7 \cdot 5 \pmod{13} \equiv 9 \pmod{13}$, $8! \equiv 8 \cdot 9 \pmod{13} \equiv 7 \pmod{13}$, $9! \equiv 9 \cdot 7 \pmod{13} \equiv 11 \pmod{13}$, $10! \equiv 10 \cdot 11 \pmod{13} \equiv 6 \pmod{13}$, and $11! \equiv 11 \cdot 6 \pmod{13} \equiv 1 \pmod{13}$, we have $12! \equiv 12 \pmod{13}$.

d. Since $2! \equiv 2 \pmod{17}$, $3! \equiv 6 \pmod{17}$, $4! = 24 \equiv 7 \pmod{17}$, $5! \equiv 5 \cdot 7 \pmod{17} \equiv 1 \pmod{17}$, $6! \equiv 6 \pmod{17}$, $7! \equiv 7 \cdot 6 \pmod{17} \equiv 8 \pmod{17}$, $8! \equiv 8 \cdot 8 \pmod{17} \equiv 13 \pmod{17}$, $9! \equiv 9 \cdot 13 \pmod{17} \equiv 15 \pmod{17}$, $10! \equiv 10 \cdot 15 \pmod{17} \equiv 14 \pmod{17}$, $11! \equiv 11 \cdot 14 \pmod{17} \equiv 1 \pmod{17}$, $12! \equiv 12 \pmod{17}$, $13! \equiv 13 \cdot 12 \pmod{17} \equiv 3 \pmod{17}$, $14! \equiv 14 \cdot 3 \pmod{17} \equiv 8 \pmod{17}$, and $15! \equiv 15 \cdot 8 \pmod{17} \equiv 1 \pmod{17}$, we have $16! \equiv 16 \pmod{17}$.

e. The theorem is that whenever p is prime, $(p - 1)! \equiv -1 \pmod{p}$. This is Wilson's Theorem which will be proven in Chapter 5.

4.1.35. Since $f_{n-2} + f_{n-1} \equiv f_n \pmod{m}$, if two consecutive numbers recur in the same order, then the sequence must be repeating both as n increases and as it decreases. But there are only m residues, and so m^2 ordered sequence of two residues. As the sequence is infinite, some two elements of the sequence must recur by the pigeonhole principle. Thus the sequence of least positive residues of the Fibonacci numbers repeats. It follows that if m divides some Fibonacci number, that is, if $f_n \equiv 0 \pmod{m}$, then m divides infinitely many Fibonacci numbers. To see that m does divide some Fibonacci number, note

that the sequence must contain a 0, namely $f_0 \equiv 0 \pmod{m}$.

4.1.36. We proceed by induction on the exponent k. We are given that $m \mid a^k - b^k$ is true when $k = 1$. We assume it is true for $k = n \geq 1$ and show it must be true for $n+1$. So $a^{n+1} - b^{n+1} = a^n(a) - b^n(b-a+a) = a^n a - b^n a - b^n(b-a) = a(a^n - b^n) + b^n(a-b)$. Since $m \mid (a^n - b^n)$ by the induction hypothesis, and we are given that $m \mid (a-b)$, we know that $m \mid (a^{n+1} - b^{n+1})$, so $a^{n+1} \equiv b^{n+1} \pmod{m}$.

4.1.37. Let a and b be positive integers less than m. Then they have $O(\log m)$ digits (bits). Therefore by Theorem 2.4, we can multiply them using $O(\log^2 m)$ operations. Division by m takes $O(\log^2 m)$ operations by Theorem 2.7. Then, in all we have $O(\log^2 m)$ operations.

4.1.38. Let N be the number of coconuts. From the division of the coconuts by the first man, giving one to the monkey, we see that $N \equiv 1 \pmod{5}$, so that $N = 5k_0 + 1$ for some positive integer k_0.

From the division of the coconuts by the second man, giving one to the monkey, we see that $N_1 = (\frac{4}{5})(N-1) = 4k_0 \equiv 1 \pmod{5}$, so that $k_0 \equiv 4 \pmod{5}$, or equivalently, that $N = 5(5k_1 + 4) + 1 = 25k_1 + 21$, and $N_1 = 20k_1 + 16$, for some positive integer k_1.

The division of the coconuts by the third man, giving one to the monkey, shows that $N_2 = (\frac{4}{5})(N_1 - 1) = (\frac{4}{5})(20k_1 + 15) = 16k_1 + 12 \equiv 1 \pmod{5}$, so that $k_1 \equiv 4 \pmod{5}$, or equivalently $N = 25(5k_2 + 4) + 21 = 125k_2 + 121$, and $N_2 = (\frac{4}{5})(100k_2 + 95) = 80k_2 + 76$.

The division of the coconuts by the fourth man, giving one to the monkey, shows that $N_3 = (\frac{4}{5})(N_2 - 1) = (\frac{4}{5})(80k_2 + 75) = 64k_2 + 60 \equiv 1 \pmod{5}$, so that $k_2 \equiv 4 \pmod{5}$, or equivalently $N = 125(5k_3 + 4) + 121 = 625k_3 + 621$, and $N_3 = 64(5k_3 + 4) + 60 = 320k_3 + 316$.

The division of the coconuts by the fifth man, giving one to the monkey, shows that $N_4 = (\frac{4}{5})(N_3 - 1) = (\frac{4}{5})(320k_3 + 315) = 256k_3 + 252 \equiv 1 \pmod{5}$, so that $k_3 \equiv 4 \pmod{5}$, or equivalently $N = 625(5k_4 + 4) + 621 = 3125k_4 + 3121$, and $N_4 = 256(5k_4 + 4) + 252 = 1280k_4 + 1276$.

The last division of the coconuts into five equal piles, giving one to the monkey, shows that $N_5 = (\frac{4}{5})(N_4 - 1) = (\frac{4}{5})(1280k_4 + 1275) = 1024k_4 + 1020 \equiv 1 \pmod{5}$, so that $k_4 \equiv 4 \pmod{5}$, or equivalently, that $N = 3125(5k_5 + 4) + 3121 = 15625k_5 + 15621$, for some integer k.

The least number of coconuts is given by the smallest positive integer of the form $15625k_5 + 15621$, which is 15621 with $k_5 = 0$.

4.1.39. Let N_i be the number of coconuts the ith man leaves for the next man and $N_0 = N$. At each stage, the ith man finds N_{i-1} coconuts, gives k coconuts to the monkeys, takes $(1/n)(N_{i-1} - k)$ coconuts for himself and leaves the rest for the next man. This yields the recursive formula $N_i = (N_{i-1} - k)(n-1)/n$. For convenience, let $w = (n-1)/n$. If we iterate this formula a few times we get $N_1 = (N_0 - k)w$, $N_2 = (N_1 - k)w = ((N_0 - k)w - k)w = N_0 w^2 - kw^2 - kw$, $N_3 = N_0 w^3 - kw^3 - kw^2 - kw, \ldots$. The general pattern $N_i = N_0 w^i - kw^i - kw^{i-1} - \cdots - kw = N_0 w^i - kw(w^i - 1)/(w-1)$ may be proved by induction. When the men rise in the morning they find $N_n = N_0 w^n - kw(w^n - 1)/(w-1)$ coconuts, and we must have $N_n \equiv k \pmod{n}$, that is, $N_n = N_0 w^n - kw(w^n - 1)/(w-1) = k + tn$ for some integer t. Substituting $w = (n-1)/n$ back in for w, solving for N_0, and simplifying yields $N = N_0 = n^{n+1}(t+k)/(n-1)^n - kn + k$. For N to be an integer, since $(n, n-1) = 1$, we must have $(t+k)/(n-1)^n$ an integer. Since we seek the smallest positive value for N, we take $t + k = (n-1)^n$, so $t = (n-1)^n - k$. Substituting this value back into the formula for N yields $N = n^{n+1} - kn + k$.

4.1.40. a. Let $f(x) = \sum_{i=0}^{m} c_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$, where the leading coefficients may be zero to keep the limits of summation equal. Since $f(x) \equiv g(x) \pmod{n}$, we have that $c_i \equiv b_1 \pmod{n}$ for $i = 0, 1, \ldots, m$. If a is any integer, by Theorem 4.3 part (iii), $c_i a^i \equiv b_i a^i \pmod{n}$ for $i = 0, 1, \ldots, m$ and so by Theorem 4.4, part (i), $f(x) = \sum_{i=0}^{m} c_i a^i \equiv \sum_{i=0}^{m} b_i a^i \equiv g(x) \pmod{n}$.

b. One counterexample is $x^3 \equiv x \pmod{3}$, which is true for $x = 0, 1$ and 2, but not true as a polynomial congruences, since the coefficient on x^3 on the left side is 1 but on the right side, it is 0. This example was constructed by taking a complete set of residues modulo 3, that is, $\{0, 1, -1\}$ and forming the product $(x-0)(x-1)(x-(-1)) = x^3 - x$. By construction, the value of this polynomial must be congruent to 0 when ever we substitute any residue in for x.

4.1.41. a. Let $f_1(x) = \sum_{i=0}^{m} a_i x^i$, $f_2(x) = \sum_{i=1}^{m} b_i x^i$, $g_1(x) = \sum_{i=1}^{m} c_i x^i$, and $g_2(x) = \sum_{i=1}^{m} d_i x^i$ where the leading coefficients may be zero to keep the limits of summation the same for all polynomials. Then $a_i \equiv c_i \pmod{n}$ and $b_i \equiv d_i \pmod{n}$, for $i = 0, 1, \ldots, m$. Therefore by Theorem 4.5 part (i), $a_i + b_i \equiv c_i + d_i \pmod{n}$ for $i = 0, 1, \ldots, m$. Since $(f_1 + f_2)(x) = \sum_{i=1}^{m}(a_i + b_i)x^i$ and $(g_1 + g_2)(x) = \sum_{i=1}^{m}(c_i + d_i)x^i$, this shows the sums of the polynomials are congruent modulo n.

b. With the same set up as in part (a), the coefficient on x^k in $(f_1 f_2)(x)$ is given by $a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$, and the corresponding coefficient in $(g_1 g_2)(x)$ is given by $c_0 d_k + c_1 d_{k-1} + \cdots + c_k d_0$. Since each $a_i \equiv c_i \pmod{n}$ and $b_i \equiv d_i \pmod{n}$, by Theorem 4.5, the two expressions are congruent modulo n, and so, therefore, are the polynomials.

4.1.42. Note that for i a positive integer, we have $(x^i - a^i)/(x - a) = x^{i-1} + x^{i-2}a + \cdots + xa^{i-2} + a^{i-1}$. Let $f(x) = \sum_{i=0}^{m} b_i x^i$. Then $(f(x) - f(a))/(x - a) = \sum_{i=1}^{m} b_i(x^i - a^i)/(x - a) = \sum_{i=1}^{m} b_i \sum_{j=0}^{i-1} a^j x^{i-j-1} = g(x)$ where $g(x)$ is clearly a polynomial with integer coefficients, and $f(x) - f(a) = (x - a)g(x)$, so the coefficients on both sides must be equal. Since we have $f(a) \equiv 0 \pmod{n}$ as polynomials and $f(x) - f(a) \equiv (x - a)g(x) \pmod{n}$, by Exercise 41, we may add these congruences to get $f(x) \equiv (x - a)g(x) \pmod{n}$.

4.1.43. The basis step for induction on k is Exercise 42. Assume that $f(x) \equiv h(x) \pmod{p}$ and $f(x) = (x - a_1) \cdots (x - a_{k-1})h(x)$, where $h(x)$ is a polynomial with integer coefficients. Substituting a_k for x in this congruence gives us $0 \equiv (a_k - a_1) \cdots (a_k - a_1)h(a_k) \pmod{p}$. None of the factors $a_k - a_i$ can be congruent to zero modulo p, so we must have $h(a_k) \equiv 0 \pmod{p}$. Applying Exercise 42 to $h(x)$ and a_k gives us $h(x) \equiv (x - a_k)g(x) \pmod{p}$ and substituting this in the congruence for $f(x)$ yields $f(x) \equiv (x - a_1) \cdots (x - a_k)g(x) \pmod{p}$, which completes the induction step.

4.1.44. We use induction on n. If $n = 1$, then $f(x) = b_1 x + b_0 \equiv 0 \pmod{p}$. If $f(x)$ has no roots, we're done. If a is a root, from Exercise 43, there exists a polynomial $g(x)$ with integer coefficients such that $f(x) = b_1 x + b_0 \equiv (x - a)g(x) \pmod{p}$. Then every coefficient of $g(x)$ other than the constant term, must be divisible by p. If the constant term of $g(x)$ is also divisible by p, then we would have $g(x) \equiv 0 \pmod{p}$ and so $f(x) \equiv 0 \pmod{p}$ as polynomials, which implies that every coefficient of $f(x)$ is also divisible by p, a contradiction. Therefore, the constant term $g(0)$ of $g(x)$ is not divisible by p. Then since $g(0) \equiv g(x) \pmod{p}$ as polynomials, we have $f(x) \equiv (x - a)g(0) \pmod{p}$. Since the right side has only one root, the left side can have only one root.

Now assume the proposition is true for polynomials of degree $n - 1$ and smaller, and suppose x^n is the largest power of x with coefficient not divisible by p. If $f(x)$ has no roots, we are done. If a is a root, then $(f(x) - f(a))/(x - a) = \sum_{i=1}^{n} b_i(x^i - a^i)/(x - a) = \sum_{i=1}^{m} b_i \sum_{j=0}^{i-1} a^j x^{i-j-1} = g(x)$, where $g(x)$ is a polynomial of degree at most $n - 1$ with with integer coefficients which, by the induction hypothesis, can have at most $d - 1$ roots. Then $f(x) \equiv (x - a)g(x) \pmod{p}$ and $f(x)$ can have at most n roots, namely the roots of g plus a.

4.2. Linear Congruences

4.2.1. a. Since $(2, 7) = 1 \mid 5$, Theorem 4.10 tells us that there is one class of solutions. We solve the diophantine equation $2x + 7y = 5$, to get $x \equiv 6 \pmod{7}$.

b. Since $(3, 9) = 3 \mid 6$, Theorem 4.10 tells us that there are three classes of solutions. We solve the diophantine equation $3x + 9y = 6$, to get $x = 2 + 3t$. All solutions are thus congruent to 2, 5, or 8 modulo 9.

c. Since $(19, 40) = 1 \mid 30$, Theorem 4.10 tells us that there is one class of solutions. We solve the diophantine equation $19x + 40y = 30$, to get $x \equiv 10 \pmod{40}$.

d. All solutions are given by $x \equiv 20 \pmod{25}$.

e. All solutions are given by $x \equiv 111 \pmod{999}$.

f. Since $(980, 1600) = 20 \mid 1500$, Theorem 4.10 tells us that there are twenty classes of solutions. All solutions are given by $x \equiv 75 + 80k \pmod{1600}$ where k is an integer such that $0 \leq k \leq 19$.

4.2.2. a. Suppose that $3x \equiv 2 \pmod 7$. Since $(3, 2) = 1$, by Theorem 4.10 there is a unique solution modulo 7 to this congruence. To solve $3x \equiv 2 \pmod 7$ first translate this to the equation $3x - 7y = 2$ where y is an integer. Using the Euclidean algorithm we find that $-2 \cdot 3 + 1 \cdot 7 = 1$. Multiplying both sides by 2 gives $-4 \cdot 3 + 2 \cdot 3 = 2$. This implies that $x \equiv -4 \equiv 3 \pmod 7$ is the unique solution modulo 7.

b. Suppose that $6x \equiv 3 \pmod 9$. Since (6,3)=3, by Theorem 4.10 there are exactly 3 incongruent solutions modulo 9. To find these solutions, we first translate this congruence into the linear diophantine equation $6x - 9y = 3$. Using the Euclidean algorithm we find that $-1 \cdot 6 + 1 \cdot 9 = 3$. Hence all solutions of $6x - 9y = 3$ are given by $x = -1 + (\frac{9}{3})t = -1 + 3t, y = -1 - (\frac{6}{3})t = -1 - 2t$. We obtain three incongruent solutions modulo 9 by taking the values of x for $t = 0, 1$, and 2. We obtain $x = -1 \equiv 8 \pmod 9, x = -4 \equiv 5 \pmod 9$, and $x = -7 \equiv 2 \pmod 9$.

c. Suppose that $17x \equiv 14 \pmod{21}$. since $(17, 14) = 1$, by Theorem 4.10 there is exactly one solution modulo 21. We find this by translating the congruence into the linear diophantine equation $17x - 21y = 14$. Using the Euclidean algorithm we find that $5 \cdot 17 - 4 \cdot 21 = 1$. Multiplying both sides by 14 gives $70 \cdot 17 - 56 \cdot 21 = 14$. Hence $x = 70, y = 56$ is a solution. This implies that the unique solution modulo 21 is $x = 70 \equiv 7 \pmod{21}$.

d. Suppose that $15x \equiv 9 \pmod{25}$. Then since (15,9)=3 but 3 does not divide 25, it follows by Theorem 4.10 that there are no solutions to this congruence.

e. We check that $(128, 1001) = 1 \mid 833$, so that there is exactly one solution. Solving the diophantine equation $128x + 1001y = 833$ gives us $x \equiv 812 \pmod{1001}$.

f. We check that $(987, 1597) = 1 \mid 610$. Solving the diophantine equation $987x + 1597y = 610$ gives us $x \equiv 1596 \pmod{1597}$.

4.2.3. Since $(28927591, 6789783) = 9163 \mid 2474010$, Theorem 4.10 tells us that there are 9163 classes of solutions. Reducing the congruence by dividing each side of the equation and the modulus by 9163, we look at the congruence $741 \equiv 270 \pmod{3157}$. The single class of solutions of this congruence is congruent to 1074. Thus, the 9163 solutions to the original congruence are given by $x \equiv 1074 + 3157k \pmod{28927591}$ where k is an integer such that $0 \leq k \leq 9162$.

4.2.4. a. Since a_1 is the least positive residue of m modulo a, we have $a_1 = m - [m/a]\, a$. Then $a_1 x \equiv (m - [m/a]\, a)x \equiv -[m/a]\, ax \equiv -[m/a]\, b \pmod m$ as desired.

b. We have a sequence of decreasing positive integers, which, by the well ordering property, must have a least element, a_n. Then we can reduce m modulo a_n and get an a_{n+1} which is smaller than a_n. But a_n is the least positive element of the sequence, so $a_{n+1} = 0$, which is to say $a_n \mid m$. However, since $a_1 = m - [m/a]\, a)$, we have that a common divisor of m and a_1 also divides a. Since $(a, m) = 1$, then we have $(a_1, m) = 1$. By induction, $(a_n, m) = 1$, but we proved $a_n \mid m$, therefore, $a_n = 1$.

c. We have $a_1 = 23 - [23/6]\, 6 = 23 - 3 \cdot 6 = 5$. Then the new congruence is $5x \equiv -7 \cdot 3 \equiv 2 \pmod{23}$. Then $a_2 = 23 - [23/5]\, 5 = 23 - 4 \cdot 5 = 3$, and the next congruence is $3x \equiv -2 \cdot 4 \equiv 15 \pmod{23}$. Then $a_3 = 23 - [23/3]\, 3 = 23 - 7 \cdot 3 = 2$, and the next congruence is $2x \equiv -15 \cdot 7 \equiv 10 \pmod{23}$. Then $a_4 = 23 - [23/2]\, 2 = 23 - 11 \cdot 2 = 1$, and the final congruence is $x \equiv -10 \cdot 11 \equiv 5 \pmod{23}$.

4.2.5. This is equivalent to saying that $11x \equiv 17 \pmod{24}$. This has one solution modulo 24, by Theorem 4.10, $x \equiv 19 \pmod{24}$. So the satellite orbits the Earth every 19 hours.

4.2.6. By Theorem 4.10 there is a solution of $12x \equiv c \pmod{30}$ if and only if $(12, 30) = 6$ divides c. This holds for $c \equiv 0, 6, 12, 18$, and $24 \pmod{30}$. In each of these cases there are $(12, 30) = 6$ incongruent solutions modulo 30.

4.2.7. We know by Theorem 4.10 that $154x \equiv c \pmod{1001}$ has solutions if and only if $(1001, 154) = 77 \mid c$. Also, by Theorem 4.10, we know that when there are solutions, there are exactly 77 of them.

4.2.8. a. We need to solve $2x \equiv 1 \pmod{13}$. Which in turns requires us to solve the Diophantine equation $2x + 13y = 1$, which we do by the Euclidean algorithm. We have $13 = 6 \cdot 2 + 1$, so $1 = 13 - 6 \cdot 2$. Therefore, $x \equiv -6 \equiv 7 \pmod{13}$ and thus $\overline{2} = 7$.

b. We need to solve $3x \equiv 1 \pmod{13}$, or $3x + 13y = 1$ We have $13 = 4 \cdot 3 + 1$, so $1 = 13 - 4 \cdot 3$. Therefore, $x \equiv -4 \equiv 9 \pmod{13}$ and thus $\overline{3} = 9$.

c. We need to solve $5x \equiv 1 \pmod{13}$, or $5x + 13y = 1$ We have $13 = 2 \cdot 5 + 3, 5 = 3 + 2$, and $3 = 2 + 1$, so $1 = 3 - 2 = (13 - 2 \cdot 5) - (5 - 3) = 13 - 3 \cdot 5 + (13 - 2 \cdot 5) = 4 \cdot 13 - 5 \cdot 5$. Therefore, $x \equiv -5 \equiv 8 \pmod{13}$ and thus $\overline{5} = 8$.

d. We need to solve $11x \equiv 1 \pmod{13}$, or $11x + 13y = 1$ We have $13 = 11 + 2$ and $11 = 5 \cdot 2 + 1$, so $1 = 11 - 5 \cdot 2 = 11 - 5(13 - 11) = -5 \cdot 13 + 6 \cdot 11$. Therefore, $x \equiv 6 \pmod{13}$ and thus $\overline{11} = 6$.

4.2.9. a. To find an inverse of 4 modulo 17 we must solve the congruence $4x \equiv 1 \pmod{17}$. Form the Euclidean algorithm we find that $1 \cdot 17 - 4 \cdot 4 = 1$. Hence $x = -4 \equiv 13 \pmod{17}$ is a solution, so that 13 is an inverse of 4 modulo 17.

b. To find an inverse of 5 modulo 17 we must solve the congruence $5x \equiv 1 \pmod{17}$. From the Euclidean algorithm we find that $-2 \cdot 17 + 7 \cdot 5 = 1$. Hence $x = 7 \pmod{17}$ is a solution, so that 7 is an inverse of 5 modulo 17.

c. To find an inverse of 7 modulo 17 we must solve the congruence $7x \equiv 1 \pmod{17}$. From the Euclidean algorithm we find that $-2 \cdot 17 + 5 \cdot 7 = 1$. Hence $x = 5 \pmod{17}$ is a solution, so that 5 is an inverse of 7 modulo 17.

d. To find an inverse of 16 modulo 17 we must solve the congruence $16x \equiv 1 \pmod{17}$. Since $16 \equiv -1 \pmod{17}$, this implies that $-x \equiv 1 \pmod{17}$, or that $x \equiv -1 \equiv 16 \pmod{17}$. Hence $x = 16$ is an inverse of 16 modulo 17.

4.2.10. a. The integers a with inverses modulo 14 are exactly those that are relatively prime to 14. Therefore, only 1, 3, 5, 9, 11, and 13 have inverses modulo 14.

b. For each of the integers a relatively prime to 14, we solve the congruence $ax \equiv 1 \pmod{14}$. We have that 1 and $13 \equiv -1 \pmod{14}$ are their own inverses. The solution to $3x \equiv 1 \pmod{14}$ is $x = 5$, so $3^{-1} = 5$. Note then that $5^{-1} = 3$. Likewise, $-3 \equiv 11$ and $-5 \equiv 9$ are inverses of each other modulo 14.

4.2.11. a. The integers a with inverses modulo 30 are exactly those that are relatively prime to 30. Therefore, only 1, 7, 11, 13, 17, 19, 23, and 29 have inverses modulo 30.

b. Note that 1 and 29 are their own inverses. Solving the congruence $7x \equiv 1 \pmod{30}$ yields $x = 13$, so 7 and 13 are inverses of each other. And so are $-7 \equiv 23$ and $-13 \equiv 17$. Solving $11x \equiv 1 \pmod{30}$ yields $x = 11$, so 11 is its own inverse, and so is $-11 \equiv 19$.

4.2.12. Suppose that $\overline{a}$ and $\overline{b}$ are inverses of a and b modulo m, respectively. Then $a \cdot \overline{a} \equiv b \cdot \overline{b} \equiv 1 \pmod{m}$. We see that $(ab) \cdot (\overline{a}\overline{b}) = (a\overline{a})(b\overline{b}) \equiv 1 \cdot 1 \equiv 1 \pmod{m}$. It follows that $\overline{a}\overline{b}$ is an inverse of ab modulo m.

4.2.13. If $ax + by \equiv c \pmod{m}$, then there exists an integer k such that $ax + by - mk = c$. Since $d \mid ax + by - mk, d \mid c$. Thus there are no solutions when $d \nmid c$. Now, assume that $d \mid c$ and let $a = da'$, $b = db', c = dc'$, and $m = dm'$, so that $(a', b', m') = 1$. Then we can divide the original congruence by d to get (*) $a'x + b'y \equiv c' \pmod{m'}$, or $a'x \equiv c' - b'y \pmod{m'}$, which has solutions if and only if $g = (a', m') \mid c - b'y$, which is equivalent to $b'y \equiv c' \pmod{g}$ having solutions. Since $(a', b', m') = 1$, and $(a', m') = g$, we must have $(b', g) = 1$ and so the last congruence has only one incongruent solution y_0 modulo g. But the m'/g solutions, $y_0, y_0 + g, y_0 + 2g, \ldots, y_0 + (m'/g + 1)g$ are incongruent modulo m'. Each of these yields g incongruent values of x in the congruence (*). Therefore, there are $g(m'/g) = m'$ incongruent solutions to (*).

Now let (x_1, y_1) be one solution of the original congruence. Then the d values $x_1, x_1 + m', x_1 + 2m', \ldots, x_1 + (d-1)m'$ are congruent modulo m' but incongruent modulo m. Likewise, the d values $y_1, y_1 + m', y_1 + 2m', \ldots, y_1 + (d-1)m'$ are congruent modulo m' but incongruent modulo m. So for each solution of (*), we can generate d^2 solutions of the original congruence. Since there are m' solutions to (*), we have $d^2m' = dm$ solutions to the original congruence.

4.2.14. a. Using Exercise 13, we see that $(2, 3, 7) = 1$ and $1 \mid 1$, so there are $1 \cdot 7$ solutions. We get them by letting x take on the values $0, 1, 2, 3, 4, 5$, and 6, and solving the congruence for y. We get, respectively, $y = 5, 2, 6, 3, 0, 4$, and 1, modulo 7.

b. We have $(2, 4, 8) = 2$ and $2 \mid 6$ so there are $2 \cdot 8 = 16$ incongruent solutions modulo 8. If y is even, the congruence reduces to $2x \equiv 6 \pmod{8}$ which has solutions $x \equiv 3$ or $7 \pmod{8}$. This gives us the 8 solutions: $(3, 0), (3, 2), (3, 4), (3, 6), (7, 0), (7, 2), (7, 4)$, and $(7, 6)$. If y is odd, the congruence reduces to $2x \equiv 2 \pmod{8}$ which has solutions $x \equiv 1$ or $5 \pmod{8}$. This gives us the other 8 solutions: $(1, 1), (1, 3), (1, 5), (1, 7), (5, 1), (5, 3), (5, 5)$, and $(5, 7)$.

c. We have $(6, 3, 9) = 3$ and $3 \mid 0$ so there are $3 \cdot 9 = 27$ solutions. We can divide the congruence by 3 and get $2x + y \equiv 0 \pmod{3}$, which has solutions $(0, 0), (1, 1)$, and $(2, 2)$. Then to get all solutions to the original congruence, we add 0, 3, or 6 to each component of the 3 pairs. This gives all 27 solutions.

d. Since $(10, 5, 15) = 5$ and $5 \nmid 9$, there are no solutions.

4.2.15. Suppose that $x^2 \equiv 1 \pmod{p^k}$ where p is an odd prime and k is a positive integer. Then $x^2 - 1 \equiv (x+1)(x-1) \equiv 0 \pmod{p^k}$. Hence $p^k \mid (x+1)(x-1)$. Since $(x+1) - (x-1) = 2$ and p is an odd prime, we know that p divides at most one of $(x-1)$ and $(x+1)$. It follows that either $p^k \mid (x+1)$ or $p^k \mid (x-1)$, so that $p \equiv \pm 1 \pmod{p^k}$.

4.2.16. Suppose that $x^2 \equiv 1 \pmod{2^k}$ where $k > 2$. It follows that $x^2 - 1 \equiv (x+1)(x-1) \equiv 0 \pmod{2^k}$. Hence $2^k \mid (x+1)(x-1)$. Note that $(x+1) - (x-1) = 2$, so that $2^{k-1} \mid x+1$ and $2 \mid x-1$ or $2^k - 1 \mid x-1$ and $2 \mid x+1$. It follows that $x = t2^{k-1} + 1$ or $x = t2^{k-1} - 1$ where t is an integer. We see that there are four incongruent solutions modulo 2^k, taking $t = 0$ or $t = 1$, namely $x \equiv 1, 2^{k-1} + 1, -1$, or $2^{k-1} - 1 \pmod{2^k}$. This can also be stated as $x \equiv \pm$ or $\pm(1 + 2^{k-1}) \pmod{2^k}$, since $x = 2^{k-1} - 1 \equiv (2^{k-1} - 2^k) - 1 = -2^{k-1} - 1 \pmod{2^k}$.

When $k = 1$ we find that there is one solution of $x^2 \equiv 1 \pmod{2}$ namely $x \equiv 1 \pmod{2}$. When $k = 2$ we find that there are two solutions of $x^2 \equiv 1 \pmod{2^2}$ namely $x \equiv \pm \pmod{2^2}$.

4.2.17. To find the inverse of a modulo m, we must solve the Diophantine equation $ax + my = 1$, which can be done using the Euclidean algorithm. Using Corollary 2.5.1, we can find the greatest common divisor in $O(\log^3 m)$ bit operations. The back substitution to find x and y will take no more than $O(\log m)$ multiplications, each taking $O(\log^2 m)$ operations. Therefore the total number of operations is $O(\log^3 m) + O(\log m)O(\log^2 m) = O(\log^3 m)$.

4.2.18. (This is Lemma 9.1.) From Exercise 44 in Section 4.1, we know that the congruence has no more than two solutions, so we seek to show that it can not have exactly one solution. Let y be a solution. Then $y^2 \equiv (-y)^2 \equiv a \pmod{p}$, so $-y$ is also a solution. If $y \equiv -y \pmod{p}$, then $2y \equiv 0 \pmod{p}$, so either $p|2$ or $p|y$. But p is odd, so it can not divide 2, and if $p|y$, then $p|y^2$, and we have $a \equiv y^2 \equiv 0 \pmod{p}$ so that

$p|a$, a contradiction. Therefore y and $-y$ are incongruent.

4.3. The Chinese Remainder Theorem

4.3.1. The integers x that leave a remainder of one when divided by 2 or 3 are those integers x that are solutions of $x \equiv 1 \pmod 2$ and $x \equiv 1 \pmod 3$. The solutions of these two simultaneous congruences are those integers x such that $x \equiv 1 \pmod 6$. These integers are the integers leaving a remainder of 1 when divided by either 2 or 3.

4.3.2. We need an integer $x \equiv 1 \pmod 2, x \equiv 1 \pmod 5$, and $x \equiv 0 \pmod 3$. The integers x that satisfy this set of simultaneous congruences are given by $x \equiv 1 \cdot 15 \cdot 1 + 1 \cdot 6 \cdot \cdot 1 + 0 \cdot 10 \cdot 1 = 21 \pmod{30}$, since 1 is the inverse of 15 modulo 2, 1 is the inverse of 6 modulo 3, and 1 is the inverse of 10 modulo 3.

4.3.3. We want a solution to the congruences $x \equiv 2 \pmod 3, x \equiv 2 \pmod 5$, and $x \equiv 0 \pmod 4$. Using the iterative method described in the text (because our moduli aren't relatively prime!), $x = 4k$, and so $k \equiv 2 \pmod 5$. Thus $k = 3+5j \equiv 2 \pmod 3$. Finally, $j = 1+3m$. So $x = 4k = 4(3+5j) = 12+20(1+3m) = 32 + 60m$. The smallest possible such number is 32.

4.3.4. a. Using the Chinese remainder theorem, we have $M = 11 \cdot 17 = 187, M_1 = 17, M_2 = 11, y_1 = 2, y_2 = 14$, and so $x = 4 \cdot 17 \cdot 2 + 3 \cdot 11 \cdot 14 = 598 \equiv 37 \pmod{187}$.

b. we have $M = 30, M_1 = 15, M_2 = 10, M_3 = 6, y_1 = 1, y_2 = 1, y_3 = 1$ and so $x = 1 \cdot 15 \cdot 1 + 2 \cdot 10 \cdot 1 + 3 \cdot 6 \cdot 1 = 53 \equiv 23 \pmod{30}$.

c. The easiest way is to see that 6 works by, inspection.

d. We have $M = 554268, M_1 = 50388, M_2 = 46189, M_3 = 42636, M_4 = 32604, M_5 = 29172, y_1 = 7, y_2 = 1, y_3 = 3, y_4 = 8, y_5 = 11$, and $x = 4585143 \equiv 150999 \pmod{554268}$.

4.3.5. We have $m_1 = 2, m_2 = 3, m_3 = 5, m_4 = 7$, and $m_5 = 11$. Also $M_1 = 1115, M_2 = 770, M_3 = 462, M_4 = 330$, and $M_5 = 210$. By the Chinese remainder theorem, $x = M_1y_1 + 2M_2y_2 + 3M_3y_3 + 4M_4y_4 + 5M_5y_5$, where $M_iy_i \equiv \pmod{m_i}$. We find that solutions are $y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 1$, and $y_5 = 1$. So, $x \equiv 1523 \pmod{2310}$.

4.3.6. The solutions are the integers congruent to $326741466757708 \pmod{1014060069938916}$, found with the aid of computational software.

4.3.7. Let b be the number of bananas. Then $b \equiv 6 \pmod{11}$ and $b \equiv 0 \pmod{17}$. This implies that $b \equiv 6 \cdot 17 \cdot 2 + 0 \cdot 11 \cdot 14 \equiv 204 \equiv 17 \pmod{187}$. We also know that $b > 11 \cdot 7 + 6 = 83$ since the equal piles contain at least 7 bananas each. It follows that the least number of bananas in the pile is 204.

4.3.8. Let x be the number of miles the car has travelled. The odometer can only tell us that $x \equiv 49335 \pmod{100000}$. If we also know the value of c where $x \equiv c \pmod 7$ and $0 \leq c < 7$ by the Chinese remainder theorem we know the congruence satisfied by x modulo $100000 \cdot 7 = 700000$. As long as the car has been driven less that 700000 this uniquely determines the number of miles driven. In particular, we easily see that if $x \equiv 6 \pmod 7$ then the car was driven 49335 miles, if $x \equiv 4 \pmod 7$ then the car was driven 149335 miles, and if $x \equiv 2 \pmod 7$ then the car was driven 249335 miles.

4.3.9. The situation we have here is $0 \leq x \leq 1200, x \equiv 3 \pmod 5, x \equiv 3 \pmod 6, x \equiv 1 \pmod 7$, and $x \equiv \pmod{11}$. Using the iterative method described in the text, $x = 11x_0, 11x_0 \equiv 1 \pmod 7, x_0 = 2+7x_1, x = 11x_0 = 22 + 77x_1, 22 + 77x_1 \equiv 3 \pmod 6, x_1 = 1 + 6x_2, x = 99 + 462x_2$, $99 + 462x_2 \equiv 3 \pmod 5, x_2 = 2 + 5x_3, x = 1023 + 2310x_3$. The only solution satisfying $0 \leq x \leq 1200$ is $x = 1023$. It follows that 1023 troops remained.

4.3.10. We solve the system $x \equiv 9 \pmod{10}, x \equiv 9 \pmod{11}, x \equiv 0 \pmod{13}$ and get $x = 559$.

4.3.11. We solve the system $x \equiv 0 \pmod{11}, x \equiv 1 \pmod{2}, x \equiv 1 \pmod{3}, x \equiv 1 \pmod{5}, x \equiv 1 \pmod{7}$, to find that $x \equiv 2101 \pmod{2310}$.

4.3.12. We need to solve the system $x \equiv 1 \pmod{2}, x \equiv 2 \pmod{3}, x \equiv 3 \pmod{4}, x \equiv 4 \pmod{5}, x \equiv 5 \pmod{6}, x \equiv 0 \pmod{7}$, but the moduli are not mutually relatively prime. Note that if $x \equiv 5 \pmod{6}$ then it satisfies the first two congruences, so we can eliminate the 5th congruence. We solve the system consisting of the last 4 remaining congruences and get 119. Note that this also solves the first congruence, so we're done.

4.3.13. We can construct a sequence of k consecutive integers each divisible by a square as follows. Consider the system of congruences $x \equiv 0 \pmod{p_1^2}, x \equiv -1 \pmod{p_3^2}, x \equiv -4 \pmod{p_3^2}, \ldots, x \equiv -k+1 \pmod{p_k^2}$, where p_k is the kth prime. By the Chinese remainder theorem there is a solution to this simultaneous system of congruence since the moduli are relatively prime. It follows that there is a positive integer N that satisfies each of these congruences. Each of the k integers $n, N+1, \ldots, N+k-1$ is divisible by a square since p_j^2 divides $N+j-1$ for $j = 1, 2, \ldots, k$.

4.3.14. If every prime divisor of c divides b, then $(c, a) = 1$ and hence $(a+b, c) = 1$ and we have $n = 1$. Otherwise, let n be the product of all primes dividing c that do not divide b. Then if a prime p divides c, it divides exactly one of an and b, therefore, p doesn't divide $an + b$, and we have $(an + b, c) = 1$.

4.3.15. Suppose that x is a solution to the system of congruences. Then $x \equiv a_1 \pmod{m_1}$, so that $x = a_1 + km_1$ for some integer k. We substitute this into the second congruence to get $a_1 + km_1 \equiv a_1 \pmod{m_2}$ or $km_1 \equiv (a_2 - a_1) \pmod{m_2}$, which has a solution in k if and only if $(m_1, m_2) \mid (a_1, a_2)$. Now assume such a solution k_0 exists. Then all incongruent solutions are given by $k = k_0 + m_2 t/(m_1, m_2)$, where t is an integer. Then $x = a_1 + km_1 = a_1 + \left(k_0 + \dfrac{m_2 t}{(m_1, m_2)}\right) m_1 = a_1 + k_0 m_1 + \dfrac{m_1 m_2}{(m_1, m_2)} t$. Note that $m_1 m_2/(m_1, m_2) = [m_1, m_2]$ so that if we set $x_1 = a_1 + k_0 m_1$, we have $x = x_1 + [m_1, m_2]t \equiv x_1 \pmod{[m_1, m_2]}$, and so the solution is unique modulo $[m_1, m_2]$.

4.3.16. a. Since $x \equiv 4 \pmod{6}$, we let $x = 6k + 4$ where k is an integer. Since $x \equiv 13 \pmod{15}$, it follows that $6k + 4 \equiv 13 \pmod{15}$, so that $6k \equiv 9 \pmod{15}$. Dividing this congruence by 3 and since (3,15)=3 and $\frac{15}{3} = 5$, we see that $2k \equiv 3 \pmod{5}$, so that $k \equiv 4 \pmod{5}$, and $k = 5l + 4$, where l is an integer. Hence $x \equiv 6(5l + 4) + 4 = 30l + 28$. This implies that all solutions satisfy $x \equiv 28 \pmod{30}$ and it is easy to see that all x satisfying this congruence are solutions.

b. Since $x \equiv 7 \pmod{10}$, we let $x = 10k + 7$ where k is an integer. Since $x \equiv 4 \pmod{15}$, it follows that $10k + 7 \equiv 4 \pmod{15}$, so that $10k \equiv 12 \pmod{15}$. Since (2,15)=1 it follows that $5k \equiv 3 \pmod{15}$. Since (5,15)=5 and 5 does not divide 3, it follows that there are no solutions of this congruence and consequently no solutions of the original congruence.

4.3.17. a. Using Exercise 15, there is one solution modulo [60,350]=2100 because $(60, 350) = 10 \mid (80 - 10)$. Because $x \equiv 10 \pmod{60}$, we know that $x = 10 + 60k$, where k is an integer. Continuing onward, $x = 10 + 60k \equiv 80 \pmod{350}$, so $60k \equiv 70 \pmod{350}$ and so $k \equiv 7 \pmod{350}$; thus $k = 7 + (350/(350, 60))j$, where j is an integer. In conclusion, $x = 10 + 60k = 10 + 60(7 + 35j) = 430 + 2100j$.

b. Using Exercise 15, there is one solution modulo [910,1001]=2100 because $(910, 1001) = 91 \mid (93 - 2)$. Because $x \equiv 2 \pmod{910}$, we know that $x = 2 + 910k$, where k is an integer. Continuing onward, $x = 2 + 910k \equiv 93 \pmod{1001}$, so $910k \equiv 91 \pmod{1001}$ and so $k \equiv 10 \pmod{1001}$; thus $k = 10 + (1001/(1001, 910))j$, where j is an integer. In conclusion, $x = 2 + 910k = 2 + 910(10 + 11j) = 9102 + 10010j$.

4.3.18. No, the first congruence implies x is odd, while the last one implies x is even.

4.3.19. The basis step $r = 2$ is given by Exercise 15. Suppose that the system of the first k congruences has a unique solution A modulo $M = [m_1, \ldots, m_k]$ and $(m_i, m_j) | a_j - a_i$ for $1 \leq i < j \leq k$. Consider the system $x \equiv A \pmod{M}, x \equiv a_{r+1} \pmod{m_{r+1}}$. First suppose it has a solution B modulo

$[[m_1, m_2, \ldots, m_k], m_{k+1}]$. Then by Exercise 15, $([m_1, m_2, \ldots, m_k], m_{r+1})|B - a_{k+1}$. Since $m_i|[m_1, m_2, \ldots, m_k]$ for $1 \leq i \leq k$, we have $(m_i, m_{k+1})|B - a_{kr+1}$. That is, there exists an integer n such that $(m_i, m_{k+1})n = B - a_{k+1}$. If we reduce this equation modulo m_i, for $1 \leq i \leq k$ we have $(0, m_{k+1})n \equiv m_{k+1} = a_i - a_{k+1} \pmod{m_i}$. If we reduce modulo m_{k+1} we have $(m_i, 0)n = m_i n = 0 \pmod{m_{k+1}}$. In either case we have that $(m_i, m_j)|a_j - a_i$ for $1 \leq i < j \leq k+1$. Conversely, suppose we have the conditions $(m_i, m_j)|a_j - a_i$ for $1 \leq i < j \leq k+1$. Then as we have just shown, $([m_1, m_2, \ldots, m_k], m_{k+1})|A - a_{k+1}$. Therefore, by exercise 15, there is a unique solution B to the first $k+1$ congruences. This completes the induction step.

4.3.20. a. We use the iterative method of Example 3.17, as suggested in Exercises 15 and 19. We have $[6, 10, 15] = 30, (10, 6) = 2 \mid (3 - 5), (15, 6) = 3 \mid (8 - 5)$, and $(15, 10) = 5 \mid (8 - 3)$, so there exists a unique solution modulo 30, by Exercise 19. The first congruence gives us $x = 5 + 6t$. Plugging this in the second congruence gives $5 + 6t \equiv 3 \pmod{10}$ which has solution $t \equiv 3 \pmod{5}$. So $t = 3 + 5s$ and $x = 5 + 6(3 + 5s) = 23 + 30s$, which, as a congruence, is $x \equiv 23 \pmod{30}$.

b. We have $[14, 21, 30] = 210$, and the conditions of Exercise 19 are met, so a unique solution exists modulo 210. The first congruence gives $x = 2 + 14t$, so $2 + 14t \equiv 16 \pmod{21}$ or $t \equiv 1 \pmod{3}$ or $t = 1 + 3s$ and hence, $x = 16 + 42s$. Then $16 + 42s \equiv 10 \pmod{30}$ or $s = 2 + 5v$ and we have $x = 16 + 42(2 + 5v) = 100 + 210v \equiv 100 \pmod{210}$.

c. Since $(25, 15) = 5$ and $5 \nmid (10 - 8)$, there is no solution.

d. We have $x \equiv 44 \pmod{840}$.

e. Since $(9, 12) = 3$ and $3 \nmid (7 - 3)$ there is no solution.

4.3.21. This is equivalent to the system: $x \equiv 1 \pmod{2}, x \equiv 1 \pmod{3}, x \equiv 1 \pmod{5}, x \equiv 1 \pmod{7}, x \equiv 0 \pmod{11}$. So, using the iterative method described in the text, $x = 11k_1 \equiv 7 \pmod{7}$, and we see that $k_1 = 2 + 7k_2$. Now, $x = 11(2 + 7k_2) \equiv 1 \pmod{5}$ and $k_2 = 2 + 5k_3$. Now, $x = 176 + 385k_3 \equiv 1 \pmod{3}$ and $k_3 = 2 + 3k_4$. Now, $x = 946 + 1155k_4 \equiv 1 \pmod{2}$ and $k_4 = 1 + 2k_5$. So $x = 2101 + 2310k_5$. The smallest such number is 2101.

4.3.22. Let x be the number of coins. The problem yields the system of congruences $x \equiv 3 \pmod{17}, x \equiv 10 \pmod{16}, x \equiv 0 \pmod{15}$. By the Chinese remainder theorem, $x = 3930$.

4.3.23. Let x be the number of grams of rice each farmer took to market. The problem yields the system of congruences $x \equiv 32 \pmod{83}, x \equiv 70 \pmod{110}, x \equiv 30 \pmod{135}$. In order to apply the Chinese remainder theorem, we replace the modulus 110 by 22. The solution is then given by $x = 24600$. This solution remains consistent modulo 110. Thus the original amount of rice was $3 \cdot 24600 = 73800$.

4.3.24. Let $x = 784$ and $y = 813$. We choose $m_1 = 95, m_2 = 97$, and $m_3 = 99$ for our moduli, so that $M = 912285$. This leads to the systems

$$\begin{array}{ll} x \equiv 24 \pmod{95} & y \equiv 53 \pmod{95} \\ x \equiv 8 \pmod{97} & y \equiv 37 \pmod{97} \\ x \equiv 91 \pmod{99} & y \equiv 21 \pmod{99}. \end{array}$$

Using the Chinese remainder theorem to solve the systems

$$\begin{array}{ll} x + y \equiv 24 + 53 \equiv 77 \pmod{95} & xy \equiv 24 \cdot 53 \equiv 37 \pmod{95} \\ x + y \equiv 8 + 37 \equiv 45 \pmod{97} & xy \equiv 8 \cdot 37 \equiv 5 \pmod{97} \\ x + y \equiv 91 + 21 \equiv 13 \pmod{99} & xy \equiv 91 \cdot 21 \equiv 30 \pmod{99}. \end{array}$$

yields $x + y = 1597$ and $xy = 637392$ respectively.

4.3.25. Suppose that x is a base 10 automorph with four digits. Then $x^2 \equiv x \pmod{10^4}$ since the last four digits of x and x^2 must agree. It follows that $x^2 - x = x(x-1) \equiv 0 \pmod{10^4}$. This is equivalent to the two congruences $x(x-1) \equiv 0 \pmod{2^4}$ and $x(x-1) \equiv 0 \pmod{5^4}$. We can conclude that either $x \equiv 0 \pmod{2^4}$ or $x \equiv 1 \pmod{2^4}$, since 2^4 must divide either x or $x - 1$ since x and $x - 1$ have no common factors. Similarly, either $x \equiv 0 \pmod{2^4}$ or $x \equiv 1 \pmod{5^4}$. It follows that x satisfies one of four simultaneous congruences: $x \equiv 0 \pmod{2^4}$ and $x \equiv 0 \pmod{5^4}$; $x \equiv 0 \pmod{2^4}$ and $x \equiv 1 \pmod{5^4}$; $x \equiv 1 \pmod{2^4}$ and $x \equiv 0 \pmod{5^4}$; or $x \equiv 1 \pmod{2^4}$ and $x \equiv \pmod{5^4}$. Using the Chinese remainder

theorem for each of these sets of congruences gives $x \equiv 0 \pmod{10000}, x \equiv 625 \pmod{10000}, x \equiv 9376 \pmod{10000}$, and $x \equiv 1 \pmod{10000}$. The base 10 automorphs with four digits, allowing initial digits of 0 are 0000,0001,0625, and 9376.

4.3.26. Following the reasoning in the solution to Exercise 25, we have, for each prime dividing b, that $x \equiv 0 \pmod{p_i^{a_1}}$ or $x \equiv 1 \pmod{p_i^{a_1}}$. Thus a unique solution is given for each way of choosing a system of k congruences, that is, for each k, we choose whether $x \equiv 0$, or $1 \pmod{p_i^{a_1}}$. This gives us 2^k automorphs.

4.3.27. We need to solve the system $x \equiv 23 + 2 \pmod{4 \cdot 23}, x \equiv 28 + 1 \pmod{4 \cdot 28}, x \equiv 33 \pmod{4 \cdot 33}$, where we have added 2 and 1 to make the system solvable under the conditions of Exercise 19. The solution to this system is $x \equiv 4257 \pmod{85008}$.

4.3.28. We need to solve the system $x \equiv 3 \cdot 23 \pmod{4 \cdot 23}, x \equiv 3 \cdot 28 - 1 \pmod{4 \cdot 28}, x \equiv 3 \cdot 33 + 2 \pmod{4 \cdot 33}$, where we have added -1 and 2 to make the system solvable under the conditions of Exercise 19. The solution to this system is $x \equiv 16997 \pmod{85008}$.

4.3.29. We need to solve the system $x \equiv 0 \pmod{4 \cdot 23}, x \equiv 0 \pmod{4 \cdot 28}, x \equiv 0 \pmod{4 \cdot 33}$. The solution to this system is $x \equiv 0 \pmod{85008}$.Every 85008 quarter-days, starting at 0.

4.3.30. We have $x \equiv 0 \pmod 2$ if $x \equiv 0, 2, 4, 6, 8$ or $10 \pmod{12}, x \equiv 0 \pmod 3$ if $x \equiv 0, 3, 6,$ or $9 \pmod{12}, x \equiv 1 \pmod 4$ if $x \equiv 1, 5,$ or $9 \pmod{12}, x \equiv 1 \pmod 6$ if $x \equiv 1$ or $7 \pmod{12}$. Since the only integers not covered by these four congruences are those x with $x \equiv 11 \pmod{12}$, adding this congruence modulo 12 to the other four congruences gives a covering set of congruences.

4.3.31. If the set of distinct congruences cover the integers modulo the least common multiple of the moduli, then that set will cover all integers. Examine the integers modulo 210, the l.c.m. of the moduli in this set of congruences. The first four congruences take care of all numbers containing a prime divisor of 2, 3, 5, or 7. The remaining numbers can be examined one at a time, and each can be seen to satisfy one (or more) of the congruences.

4.3.32. The congruence $x^2 \equiv 1 \pmod m$ is equivalent to the system $x^2 \equiv 1 \pmod{2^{a_0}}, x^2 \equiv 1 \pmod{p_1^{a_1}}, \ldots, x^2 \equiv 1 \pmod{p_r^{a_r}}$. Each of the odd prime congruences has 2 solutions by Exercise 15 of Section 4.2. The first congruence has e solutions by Exercise 16 of Section 4.2. Therefore, there are 2^{r+e} systems of the form $x \equiv b_0 \pmod{2^{a_0}}, x \equiv b_1 \pmod{p_1^{a_1}}, \ldots, x \equiv b_r \pmod{p_r^{a_r}}$, where the b_i's are the solutions to the congruences above. Each of these systems has a unique solution modulo m, so we have a total of 2^{r+e} solutions.

4.3.33. Let x be the length in inches of the dining room. Then $x \equiv 3 \pmod 5, x \equiv 3 \pmod 7$, and $x \equiv 3 \pmod 9$. Since 5,7,and 9 are pairwise relatively prime, the Chinese remainder theorem tells us that there is a unique solution to this system of congruences modulo $5 \cdot 7 \cdot 9 = 315$. This solution is immediately seen to be $x \equiv 3 \pmod{315}$. Since x is a length it is positive. Hence possible values for x are 3,318,633,948, and so on. Since x is the length of a room in inches, the possibility that $x = 3$ is absurd, and it is most likely that $x = 318$, so that the room is 26 feet and 6 inches long. This is a big dining room. Of course, it is possible that the dining room is 633 inches, or 52 feet and 9 inches long. However, unless the house is huge this, and larger possible answers, are extremely unlikely.

4.3.34. Trying all the integers from 0 to 8 in the congruence $x^2 + 6x - 31 \equiv 0 \pmod 9$ yields $x \equiv 3$ or $8 \pmod 9$. Trying all the integers from 0 to 7 in the congruence $x^2 + 6x - 31 \equiv 0 \pmod 8$ yields $x \equiv 1$ or $5 \pmod 8$. The various combinations of congruences give us 4 systems to solve. $x \equiv 3 \pmod 9$ and $x \equiv 1 \pmod 8$ yields $x \equiv 49 \pmod{72}$. The other 3 solutions are 13, 17, and 53 modulo 72.

4.3.35. Examining $x^2 + 18x - 823 \equiv 0 \pmod{1800}$ modulo 8, we see that $x^2 + 18x - 823 \equiv x^2 + 2x + 1 = (x + 1)^2 \equiv 0 \pmod 8$ has solutions $x \equiv 3 \pmod 8$ and $x \equiv 7 \pmod 8$. Examining $x^2 + 18x - 823 \equiv 0 \pmod{1800}$ modulo 9, we see that $x^2 + 18x - 823 \equiv x^2 + 5 \equiv 0$ has solutions $x \equiv 2 \pmod 9$ and $x \equiv 7 \pmod 9$. Examining $x^2 + 18x - 823 \equiv 0 \pmod{1800}$ modulo 25, we see that $x^2 + 18x - 823 \equiv x^2 + 18x + 77 = (x + 11)(x + 7) \equiv 0 \pmod{25}$. This has solutions $x \equiv 18 \pmod{25}$, and $x \equiv 14 \pmod{14}$.

Thus there are $2^3 = 8$ systems to examine. We may find, by the iterative method discussed in the text, that the solutions are given by $x = 225a_1 + 1000a_2 + 576a_3 + 1800k$, where k is an integer and a_1 is 3 or 7, a_2 is 2 or 7, and a_3 is 14 or 18.

4.3.36. Let p_k represent the kth prime. Then the set $\{p_1, p_2, \ldots, p_R, p_{R+1}, \ldots, p_{2R}\}$ of numbers is mutually relatively prime, since all members are prime. Let P be the product of the elements in the set. Then by the Chinese remainder theorem, there is a unique solution x modulo P to the system of congruences $x \equiv 1 \pmod{p_1}, x \equiv 2 \pmod{p_2}, \ldots, x \equiv R \pmod{P_R}, x \equiv -1 \pmod{p_{R+1}}, x \equiv -2 \pmod{p_{R+2}}, \ldots, x \equiv -R \pmod{p_{2R}}$. Then for $j = 1, 2, \ldots, R$, we have $p_j | x - j$ and $p_{R+j} | x + j$, so if x is larger than p_{2R}, then all of the integers from $x - R$ to $x + R$ are composite, except perhaps for x itself. Now consider the arithmetic progression $x + Pn$. All of these integers satisfy the system of congruences above. For each $j = 1, 2, \ldots, R$, we have $(x, P) = 1$, since for each of the primes p_j dividing P, we have $x \equiv j \pmod{p_j}$ and $1 \leq j < p_j$, so $p_j \nmid x$ and $x \equiv -j \pmod{p_{R+j}}$ and $-p_{R+j} < -j \leq -1$, so $p_{R+j} \nmid x$, and hence x and P can have no common factors. Therefore, by Dirichlet's theorem on primes in arithmetic progression, there are infinitely many primes in the progression $x + Pn$, each of which satisfy the system of congruences, and hence are R-reclusive primes.

4.4. Solving Polynomial Congruences

4.4.1. a. By testing each of the integers $0, 1, \ldots, 6$, we see that $1^2 + 4(1) + 2 \equiv 0 \pmod{7}$ and $2^2 + 4(2) + 2 \equiv 0 \pmod{7}$. So the solutions are the integers $x \equiv 1$ or $2 \pmod{7}$.

b. Let $f(x) = x^2 + 4x + 2$. Then $f'(x) = 2x + 4$. Since $f'(1) \equiv 6 \not\equiv 0 \pmod{7}$, we can apply case (i) of Hensel's lemma. The solutions $x \equiv 1 \pmod{7}$ lift uniquely to solutions $x \equiv 1 + 7t \pmod{49}$, where $t \equiv -\overline{f'(1)} f(1)/7 \equiv -\overline{6} \cdot 7/7 \equiv 1 \pmod{7}$. So $x \equiv 8 \pmod{49}$. Similarly, since $f'(2) \equiv 1 \not\equiv 0 \pmod{7}$, the solutions $x \equiv 2 \pmod{7}$ lift uniquely to $x \equiv 2 + 7t \pmod{49}$, where $t = -\overline{8} f(2)/7 \equiv 5 \pmod{7}$. So $x \equiv 2 + 7(5) \equiv 37 \pmod{49}$. The solutions are the integers $x \equiv 8$ or $37 \pmod{39}$.

c. Since $f'(8) \equiv 6 \pmod{7}$, the solutions $x \equiv 8 \pmod{49}$ lift uniquely to solutions $x \equiv 8 + 49t$ where $t \equiv -\overline{6} f(8)/49 \equiv 2 \pmod{7}$. So $x \equiv 8 + 49(2) \equiv 106 \pmod{343}$. Similarly, since $f'(37) \equiv 1 \pmod{7}$, the solutions $x \equiv 37 \pmod{49}$ lift uniquely to solutions $x \equiv 37 + 49t$ where $t \equiv -\overline{f'(37)} f(37)/49 \equiv 4 \pmod{7}$. So $x \equiv 37 + 49(4) \equiv 233 \pmod{343}$. The solutions are the integers $x \equiv 106$ or $233 \pmod{343}$.

4.4.2. a. Let $f(x) = x^3 + 8x^2 - x - 1$. By inspection, we find that the only solutions to $f(x) \equiv 0 \pmod{11}$ are the integers $x \equiv 4$ or $5 \pmod{11}$.

b. From part (a), $f'(x) = 3x^2 + 16x - 1$, so $f'(4) = 111 \equiv 1 \pmod{11}$. Then $x \equiv 4 \pmod{11}$ lifts uniquely to a solution $r_2 \equiv 4 - f(4)\overline{f'(4)} \equiv 59 \pmod{121}$. On the other hand, $f'(5) = 154 \equiv 0 \pmod{11}$. Since $f(5) = 319 \not\equiv 0 \pmod{121}$, we know, by part (iii) of Hensel's lemma, that 5 does not lift to any solution modulo 121. Thus, the only solution is $x \equiv 59 \pmod{121}$.

c. From part b, $f'(4) = 111 \equiv 1 \pmod{11}$ and 4 lifted to the solution 59 modulo 121. This solution lifts to $r_3 \equiv 59 - f(59)\overline{f'(59)} \equiv 59 - 233167 \cdot 1 \equiv 1148 \pmod{1331}$. This is the only solution.

4.4.3. Let $f(x) = x^2 + x + 47$. By inspection, the solutions to $f(x) \equiv x^2 + x + 5 \equiv 0 \pmod{7}$ are $r \equiv 1$ or $5 \pmod{7}$. Since $f'(1) \equiv 3 \pmod{7}$, we know, by Corollary 4.14.1, that $r \equiv 1$ lifts successively to unique solutions modulo each power of 7. Note that $\overline{f'(1)} = \overline{3} \equiv 5 \pmod{7}$. Then, with notation as in Corollary 4.14.1, $r_2 = 1 - f(1) \cdot 5 \equiv 1 - 49 \cdot 5 \equiv 1 \pmod{49}$, and $r_3 = 1 - 49 \cdot 5 \equiv 99 \pmod{343}$, and finally, $r_4 = 99 - f(99) \cdot 5 \equiv 785 \pmod{2401}$. Similarly, since $f'(5) \equiv 4 \pmod{7}$, we know, by Corollary 4.14.1, that $r \equiv 5$ lifts successively to unique solutions modulo each power of 7. Note that $\overline{f'(5)} = \overline{4} \equiv 2 \pmod{7}$. Then, with notation as in Corollary 4.14.1, $r_2 = 5 - f(5) \cdot 2 \equiv 47 \pmod{49}$, and $r_3 = 47 - f(47) \cdot 2 \equiv 243 \pmod{343}$, and finally, $r_4 = 243 - f(243) \cdot 2 \equiv 1615 \pmod{2401}$. Therefore the solutions are $x \equiv 785$ or $1615 \pmod{2401}$.

4.4.4. Let $f(x) = x^2 + x + 34$. By inspection, the only solution of $f(x) \equiv 0 \pmod 3$ is $r \equiv 1 \pmod 3$. Since $f'(1) = 3 \equiv 0 \pmod 3$, we check that $f(1) \equiv 36 \not\equiv 0 \pmod{81}$, so by part (iii) of Hensel's lemma, there are no solutions to $f(x) \equiv 0 \pmod{81}$.

4.4.5. Let $f(x) = 13x^7 - 42x - 649$ and observe that $1323 = 3^3 7^2$. We start by solving the congruence modulo 3 and lifting to modulo 27. First $f(x) \equiv x^7 - 1 \equiv 0 \pmod 3$, which has only the solution $r \equiv 1 \pmod 3$. Since $f'(1) = 13 \cdot 7 \cdot 1^6 - 42 \equiv 1 \pmod 3$, this solution lifts to unique solutions modulo 9 and 27. Following Corollary 4.14.1, we have $r_2 = 1 + f(1)\overline{f'(1)} = 1 - (13 - 42 - 649)(1) \equiv 4 \pmod 9$, and $r_3 = 4 - f(4)(1) \equiv 22 \pmod{27}$. Next we solve the congruence modulo 7 and lift to 49. Then $f(x) \equiv -x^7 + 2 \equiv 0 \pmod 7$ has only the solution $r \equiv 2 \pmod 7$. Note that $f'(2) = 5782 \equiv 0 \pmod 7$ and that $f(2) = 931 \equiv 0 \pmod 7$, so $r = 2$ lifts to 7 solutions modulo 49, namely 2, 9, 16, 23, 30, 37, and 44. Finally, we pair the solution for 27 with each of the solutions for 49 to produce solutions for 1323. Solving the system $x \equiv 22 \pmod{27}, x \equiv 2 \pmod{49}$ yields $x \equiv 1129 \pmod{1323}$. Solving the system $x \equiv 22 \pmod{27}, x \equiv 9 \pmod{49}$ yields $x \equiv 940 \pmod{1323}$. Solving the system $x \equiv 22 \pmod{27}, x \equiv 16 \pmod{49}$ yields $x \equiv 751 \pmod{1323}$. Solving the system $x \equiv 22 \pmod{27}, x \equiv 23 \pmod{49}$ yields $x \equiv 562 \pmod{1323}$. Solving the system $x \equiv 22 \pmod{27}, x \equiv 30 \pmod{49}$ yields $x \equiv 373 \pmod{1323}$. Solving the system $x \equiv 22 \pmod{27}, x \equiv 37 \pmod{49}$ yields $x \equiv 184 \pmod{1323}$. Solving the system $x \equiv 22 \pmod{27}, x \equiv 44 \pmod{49}$ yields $x \equiv 1318 \pmod{1323}$. So the incongruent solutions are 184, 373, 562, 751, 940, 1129, and 1318.

4.4.6. Let $f(x) = x^8 - x^4 + 1001$ and note that $539 = 7^2 11$. Solving $f(x) \equiv x^8 - x^4 \equiv 0 \pmod 7$ yields $r \equiv 0, 1$, or $-1 \pmod 7$. Since $f'(0) \equiv 0$, but $f(0) \equiv 21 \pmod{49}$, we know that 0 doesn't lift to a solution of $f(x) \equiv 0 \pmod{49}$. On the other hand $f'(1) \equiv 4 \pmod 7$, so 1 lifts to $r_2 = 1 - f(1)\overline{f'(1)} \equiv 1 - 1001 \cdot 2 \equiv 8 \pmod{49}$. Next, note that $f'(-1) \equiv 3 \pmod 7$, so -1 lifts to $r_2 = -1 - f(-1)\overline{f'(-1)} \equiv -1 - 1001 \cdot 5 \equiv 41 \pmod{49}$. Now we turn to the prime 11. By inspection, the solutions to $f(x) \equiv 0 \pmod{11}$ are $x \equiv 0, 1$ or $-1 \pmod{11}$. We now pair each solution modulo 49 with each solution modulo 11 to obtain 6 systems of congruences. Solving the system $x \equiv 8 \pmod{49}, x \equiv 0 \pmod{11}$ yields $x \equiv 253 \pmod{539}$. Solving the system $x \equiv 8 \pmod{49}, x \equiv 1 \pmod{11}$ yields $x \equiv 155 \pmod{539}$. Solving the system $x \equiv 8 \pmod{49}, x \equiv -1 \pmod{11}$ yields $x \equiv 351 \pmod{539}$. Solving the system $x \equiv 41 \pmod{49}, x \equiv 0 \pmod{11}$ yields $x \equiv 286 \pmod{539}$. Solving the system $x \equiv 41 \pmod{49}, x \equiv 1 \pmod{11}$ yields $x \equiv 188 \pmod{539}$. Solving the system $x \equiv 41 \pmod{49}, x \equiv -1 \pmod{11}$ yields $x \equiv 384 \pmod{539}$. So the incongruent solutions modulo 539 are 155, 188, 253, 286, 351, 384.

4.4.7. Let $f(x) = x^4 + 2x + 36$ and note that $4375 = 5^4 7$. By inspection, the only solution to $f(x) \equiv x^4 + 2x + 1 \equiv 0 \pmod 5$ is $r \equiv -1 \pmod 5$. Since $f'(-1) = 4(-1)^3 + 2 \equiv 3 \not\equiv 0 \pmod 5$, we know that $r = -1$ lifts uniquely to solutions modulo 5^k. Applying Corollary 4.14.1, we have $r_2 = (-1) - f(-1) \cdot \overline{3} \equiv -1 - 35 \cdot 2 \equiv 4 \pmod{25}$, and $r_3 = 4 - f(4)2 \equiv 29 \pmod{125}$ and $r_4 = 29 - f(29)2 \equiv 279 \pmod{625}$. Again, by inspection, we solve $f(x) \equiv x^4 + 2x + 1 \equiv 0 \pmod 7$ and obtain the two solutions $x \equiv 2$ or $-1 \pmod 7$. Finally we solve the two systems $x \equiv 279 \pmod{625}, x \equiv 2 \pmod 7$ and $x \equiv 279 \pmod{625}, x \equiv -1 \pmod 7$ to get the two solutions 3404 and $279 \pmod{4375}$, respectively.

4.4.8. Let $f(x) = x^6 - 2x^5 - 35$ and note that $6125 = 5^3 7^2$. By inspection we solve $f(x) \equiv 0 \pmod 5$ and obtain the two solutions $x \equiv 0$ or $2 \pmod 5$. Since $f'(0) \equiv 0 \pmod 7$ and $f(0) = -35 \not\equiv 0 \pmod{25}$, we know that the solution $x \equiv 0 \pmod 5$ does not lift to solutions modulo 5^k. However, since $f'(2) \equiv 2 \pmod 5$, we know that $x \equiv 2$ lifts to a unique solution modulo 5^k. By Corollary 4.14.1, $r_2 = 2 - f(2) \cdot \overline{2} = 2 + 35 \cdot 3 \equiv 7 \pmod{25}$ and $r_3 = 7 - f(7)3 \equiv 7 \pmod{125}$. Again, by inspection we solve $f(x) \equiv 0 \pmod 7$ to obtain the solutions $x \equiv 0$ or $2 \pmod 7$. Since $f'(0) \equiv 0 \pmod 7$ and $f(0) = -35 \not\equiv 0 \pmod{49}$, we know that the solution $x \equiv 0 \pmod 7$ does not lift to any solutions modulo 49. On the other hand, since $f'(2) \equiv 4 \pmod 7$, we know that $x \equiv 2$ lifts to a unique solution modulo 49. By Corollary 4.14.1, $r_2 = 2 - f(2) \cdot \overline{4} = 2 + 35 \cdot 2 \equiv 23 \pmod{49}$. Solving the system $x \equiv 7 \pmod{125}, x \equiv 23 \pmod{49}$ yields the solution $x \equiv 3257 \pmod{6125}$.

4.4.9. Let $f(x) = 5x^3 + x^2 + x + 1$. By inspection, the solution of the congruence $f(x) \equiv 0 \pmod 2$ is $x \equiv 1 \pmod 2$. Note that $f'(x) = 15x^2 + 2x + 1$, so $f'(1) \equiv 0 \pmod 2$. Since $f(1) = 8 \equiv 0 \pmod 4$, we know that $x = 1$ lifts to two solutions $x \equiv 1$ or $3 \pmod 4$. Since $f(3) \equiv 4 \pmod 8$, but $f'(3) \equiv 0 \pmod 2$, we know that 3 does not lift to solutions modulo 8. However, since $f'(1) \equiv 0 \pmod 2$ and $f(1) \equiv 0$

$\pmod 8$, we know that 1 lifts to the two solutions 1 and 5 $\pmod 8$. Since $f(1) \equiv 8 \not\equiv 0 \pmod{16}$, we know that 1 does not lift further. Since $f(5) \equiv 0 \pmod{16}$, we know that 5 lifts to solutions 5 and 13 $\pmod{16}$. Since $f(5) \equiv 16 \not\equiv 0 \pmod{32}$, we know that 5 does not lift further. Since $f(13) \equiv 0 \pmod{32}$, we know that 13 lifts to solutions 13 and 29 $\pmod{32}$. Since $f(13) \equiv 32 \not\equiv 0 \pmod{64}$, we know that 13 does not lift further. Since $f(29) \equiv 0 \pmod{32}$, we know that 29 lifts to solutions 29 and 61 $\pmod{64}$. So there are only two incongruent solutions.

4.4.10. Let $f(x) = x^5 + x - 6$ and note that $144 = 2^4 3^2$. Both 0 and 1 are solutions to $f(x) \equiv 0 \pmod 2$. Since $f'(0) \equiv 1 \pmod 2$ we know that 0 lifts to a unique solution modulo 16. Since $f'(1) \equiv 0 \pmod 2$ and $f(1) = -4 \equiv 0 \pmod 4$, we know that 1 lifts to solutions 1 and 3 $\pmod 4$. Since $f(1) \equiv 4 \pmod 8$, 1 lifts no further. Since $f(3) \equiv 0 \pmod 8$, 3 lifts to solutions 3 and 7 $\pmod 8$. Since $f(7) \equiv 8 \pmod{16}$, we know that 7 lifts no further. Since $f(3) \equiv 0 \pmod{16}$ we know that 3 lifts to solutions 3 and 11 $\pmod{16}$. Thus there must be a total of 3 solutions modulo 16. By inspection, there is only one solution to $f(x) \equiv 0 \pmod 3$, namely $x \equiv 0$. Since $f'(0) \equiv 1 \pmod 3$, we know that 0 lifts uniquely to a solution modulo 9. Finally, since there are 3 solutions modulo 16 and 1 solution modulo 9, we must have $3 \cdot 1 = 3$ solutions modulo $16 \cdot 9 = 144$.

4.4.11. Since $(a, p) = 1$, we know that a has an inverse b modulo p. Let $f(x) = ax - 1$. Then $x \equiv b \pmod p$ is the unique solution to $f(x) \equiv 0 \pmod p$. Since $f'(x) = a \not\equiv 0 \pmod p$, we know that $r \equiv b$ lifts uniquely to solutions modulo p^k for all natural numbers k. By Corollary 4.14.1, we have that $r_k = r_{k-1} - f(r_{k-1})\overline{f'(b)} = r_{k-1} - (ar_{k-1} - 1)\overline{a} = r_{k-1} - (ar_{k-1} - 1)b = r_{k-1}(1 - ab) + b$. This gives a recursive formula for lifting b to a solution modulo p^k for any k.

4.4.12. a. Since $a \equiv b \pmod{p^{k-j}}$, $b = a + tp^{k-j}$ for some integer t. By Lemma 4.6, we have $f(b) = f(a + tp^{k-j}) = f(a) + f'(a)tp^{k-j} + (f''(a)/2)t^2p^{2k-2j} + \cdots$. So we have $f(b) \equiv f(a) + f'(a)tp^{k-j} \pmod{p^{2k-2j}}$. Since $2k - 2j > k$, and $p^k \mid f(a)$ and $p^j \mid f'(a)$, $f(b) \equiv 0 \equiv f(a) \pmod{p^k}$. Say that $f(a) = xp^k$, $f(b) = yp^k$, and $f'(a) = zp^j$, where $(z, p) = 1$. Then from the original congruence, $f(b) \equiv f(a) + f'(a)tp^{k-j} \pmod{p^{k+1}}$, so that dividing through by p^k yields $y - x \equiv zt \pmod p$. This last is a linear congruence with $(z, p) = 1$, so there is a unique solution modulo p for t. That is, there is a unique value of t modulo p such that $f(a + tp^{k-j}) \equiv 0 \pmod{p^{k+1}}$.) Again from the original congruence, we have $f(b) - f(a) \equiv f'(a)tp^{k-j} \pmod{p^{2k-2j}}$ and from a symmetrical argument we have $f(b) - f(a) \equiv f'(b)tp^{k-j} \pmod{p^{2k-2j}}$, whence $f'(a)tp^{k-j} \equiv f'(b)tp^{k-j} \pmod{p^{k+1}}$. Dividing through by p^k gives us $zt \equiv (f'(b)/p^j)t \pmod p$. Since $(x, p) = 1$, we must have $p^j \parallel f'(b)$.

b. From part (a), for each solution a of $f(x) \equiv 0 \pmod{p^k}$ There is a unique value of t modulo p such that $a + tp^k - j$ is a solution to $f(x) \equiv 0 \pmod{p^{k+1}}$. That is, each solution $a \pmod{p^k}$ lifts uniquely to a solution $b \pmod{p^{k+1}}$.

4.4.13. By inspection the only solution of $f(x) = x^2 + x + 223 \equiv 0 \pmod 3$ is $x \equiv 1 \pmod 3$. Since $f'(1) = 2 \cdot 1 + 1 = 3 \equiv 0 \pmod 3$ and $f(1) = 225 \equiv 0 \pmod 9$, we have by Theorem 4.14 that $1, 4,$ and 7 are the only solutions modulo 9. Since $f(1) = 225 \equiv 9 \pmod{27}$, this solutions doesn't lift. Since $f(4) = 243 \equiv 0 \pmod{27}$, this solution lifts to three solutions $4, 13,$ and $22 \pmod{27}$. Since $f(7) = 279 \equiv 9 \pmod 7$, this solution doesn't lift. So the only solutions modulo 27 are 4, 13, and 22. Next $f(4) \equiv f(13) \equiv f(22) \equiv 0 \pmod{81}$, so each of these solutions lifts to three solutions modulo 81, namely 4, 31, 58, 13, 40, 67, 22, 49, and 76. Of these, $f(13) \equiv f(40) \equiv f(67) \equiv 163 \not\equiv 0 \pmod{3^5}$ and so these do not lift to solutions. But $f(4) \equiv f(31) \equiv f(58) \equiv f(22) \equiv f(49) \equiv f(76) \equiv 0 \pmod{3^5}$. Therefore, each of these 6 solutions lifts to three solutions modulo 3^5, namely $x \equiv 166, 112, 238, 193, 85, 130, 58, 103, 211, 31, 76, 157, 184, 49, 4, 139, 220$ or $22 \pmod{3^5}$. It is easy to check that each of these solutions satisfies the hypotheses of Exercise 12 with $p = 3$, $k = 5$ and $j = 2$. E.g., $f(166) \equiv 0 \pmod{3^5}$ and $3^2 \parallel f'(166) = 333 = 3^2 \cdot 37$. Therefore each of these solutions lifts uniquely to solutions modulo 3^n for $n \geq 5$. So there are exactly 18 solutions modulo 3^n for $n \geq 5$.

4.5. Systems of Linear Congruences

4.5.1. a. Multiplying the first congruence by 2 gives $2x + 4y = 2 \pmod 5$. Subtracting the second congruence $2x + y \equiv 1 \pmod 5$ from this gives $3y \equiv 1 \pmod 5$. Since 2 is the inverse of 3 modulo 5 we have $y \equiv 2 \pmod 5$. Inserting this into the congruence $x+2y \equiv 1 \pmod 5$ gives $x+4 \equiv 1 \pmod 5$. Hence $x \equiv -3 \equiv 2 \pmod 5$. The unique solution modulo 5 is $x \equiv 2 \pmod 5$ and $x \equiv 2 \pmod 5$.

b. Multiplying the first congruence by 3 gives $3x + 9y \equiv 3 \pmod 5$. Subtracting the second congruence $3x + 4y \equiv 2 \pmod 5$ from this gives $5y \equiv 1 \pmod 5$ which is impossible. Hence this system has no solutions.

c. Multiplying the second congruence by 2 gives $4x + 6y \equiv 2 \pmod 5$. Subtracting the first congruence from this gives $5y \equiv 0 \pmod 5$. The solutions to this are all values of y, that is, $y \equiv 0, 1, 2, 3,$ or $4 \pmod 5$. This implies that $4x \equiv 2, 1, 0, 4,$ or $3 \pmod 5$, respectively, or that $x \equiv 3, 4, 0, 1$ or $2 \pmod 5$, respectively. The solutions are $x \equiv 3 \pmod 5, y \equiv 0 \pmod 5; x \equiv 4 \pmod 5, y \equiv 1 \pmod 5; x \equiv 0 \pmod 5, y \equiv 2 \pmod 5; x \equiv 1 \pmod 5, y \equiv 3 \pmod 5$; and $x \equiv 2 \pmod 5, y \equiv 4 \pmod 5$.

4.5.2. a. Subtracting twice the second congruence from the first gives us $-7y \equiv -7 \pmod 7$, which is $0y \equiv 0 \pmod 7$. Therefore, y can take on any residue modulo 7. When $y = 0$, we have $x \equiv 6 \pmod 7$, from the second congruence, so the first solution is $(6, 0)$. When $y = 1$, the second congruence gives us $x + 5 \equiv 6 \pmod 7$, so $(1, 1)$ is another solution. Continuing in this fashion, get the seven solutions: $(6, 0), (1, 1), (3, 2), (5, 3), (0, 4), (2, 5)$, and $(4, 6)$.

b. Subtracting twice the first congruence from the second yields $-7x \equiv -6 \pmod 7$, which reduces to $0 \equiv 1 \pmod 7$, which is false. Therefore there is no solution.

4.5.3. If we use one congruence to eliminate a variable from the other congruence, we are left with linear congruence of the form $ax \equiv b \pmod p$. If $(a, p) = 1$, then this congruence has a unique solution, but if $p \mid a$, we have $0 \equiv b \pmod p$, which has 0 solutions if b is not 0 modulo p and p solutions if b is 0 modulo p. So there are $0, 1,$ or p solutions for this variable. Similarly, There are $0, 1,$ or p solutions for the other variable. Multiplying all the possible combinations gives us 0, 1, p, or p^2 solutions for the system.

4.5.4. Multiplying the matrices in the usual fashion and reducing each entry modulo 5 gives $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$

4.5.5. The basis step, where $k = 1$, is clear by assumption. For the inductive hypothesis assume that $\mathbf{A} \equiv \mathbf{B} \pmod m$ and $\mathbf{A}^k \equiv \mathbf{B}^k \pmod m$. Then, $\mathbf{A} \cdot \mathbf{A}^k \equiv \mathbf{A} \cdot \mathbf{B}^k \pmod m$ by Theorem 4.16. Further, $\mathbf{A}^{k+1} = \mathbf{A} \cdot \mathbf{A}^k \equiv \mathbf{A} \cdot \mathbf{B}^k \equiv \mathbf{B} \cdot \mathbf{B}^k = \mathbf{B}^{k+1} \pmod m$ by simple substitution. This completes the inductive proof.

4.5.6. We have $\begin{pmatrix} 4 & 11 \\ 1 & 22 \end{pmatrix}^2 = \begin{pmatrix} 27 & 26 \\ 26 & 495 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{26}$. Hence this matrix is involutory $\pmod{26}$.

4.5.7. Note that $1 = \det(\mathbf{I}) = \det(\mathbf{A}^2) = (\det(\mathbf{A}))^2 \pmod m$. So, $(\det(\mathbf{A}))^2 - 1 = (\det(\mathbf{A}) + 1)(\det(\mathbf{A}) - 1) \equiv 0 \pmod m$. It follows that $\det(\mathbf{A}) = \pm 1$.

4.5.8. a. (We use Theorem 4.17 in each part to find the inverse of a 2x2 matrix modulo 5.)
Since the determinant of this matrix is -1, and -1 is an inverse of -1 modulo 5, an inverse of this matrix modulo 5 is $-1 \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b. Since the determinant of this matrix is -2, and 2 is an inverse of -2 modulo 5, an inverse of this matrix modulo 5 is $2 \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

c. Since the determinant of this matrix is 2, and 3 is an inverse of 2 modulo 5, an inverse of this matrix modulo 5 is $3 \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$.

4.5.9. a. Let $\mathbf{A}$ be the matrix. We have $\det \mathbf{A} = -2$ which has inverse 3 modulo 7. Then

$$\overline{\mathbf{A}} = 3 \cdot \text{adj } \mathbf{A} = 3 \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 3 \\ 4 & 3 & 4 \\ 3 & 4 & 4 \end{pmatrix}.$$

b. Let $\mathbf{A}$ be the matrix. We have $\det \mathbf{A} = 3$ which has inverse 5 modulo 7. Then

$$\overline{\mathbf{A}} = 5 \cdot \text{adj } \mathbf{A} = 5 \begin{pmatrix} -1 & 0 & 4 \\ -1 & 3 & -2 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 2 & 1 & 4 \\ 3 & 4 & 0 \end{pmatrix}.$$

c. Let $\mathbf{A}$ be the matrix. We have $\det \mathbf{A} = 4$ which has inverse 2 modulo 7. Then

$$\overline{\mathbf{A}} = 2 \cdot \text{adj } \mathbf{A} = 2 \begin{pmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 & 4 \\ 5 & 5 & 4 & 5 \\ 5 & 4 & 5 & 5 \\ 4 & 5 & 5 & 5 \end{pmatrix}.$$

4.5.10. a. Using the inverse from Problem 9(a) we have $\begin{pmatrix} 4 & 4 & 3 \\ 4 & 3 & 4 \\ 3 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \pmod 7$.

b. Using the inverse from Problem 9(b) we have $\begin{pmatrix} 2 & 0 & 6 \\ 2 & 1 & 4 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \pmod 7$.

c. Using the inverse from Problem 9(c) we have $\begin{pmatrix} 5 & 5 & 5 & 4 \\ 5 & 5 & 4 & 5 \\ 5 & 4 & 5 & 5 \\ 4 & 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix} \pmod 7$.

4.5.11. a. Multiplying the first congruence by 2 gives $2x + 2y + 2z \equiv 2 \pmod 5$. Subtracting this from the second congruence gives $2y + z \equiv 4 \pmod 5$. There are five possible values for z modulo 5, and since $(2, 5) = 1$, each of these leads to a unique value of y modulo 5, and substituting these values of y and z modulo 5 into the first congruence we obtain a unique value of x modulo 5. Hence there are exactly 5 incongruent solutions modulo 5. There are $x \equiv 4 \pmod 5, y \equiv 2 \pmod 5, z \equiv 0 \pmod 5$; $x \equiv 1 \pmod 5, y \equiv 4 \pmod 5, z \equiv 1 \pmod 5$; $x \equiv 3 \pmod 5, y \equiv 1 \pmod 5, z \equiv 2 \pmod 5$; $x \equiv 0 \pmod 5, y \equiv 3 \pmod 5, z \equiv 3 \pmod 5$; and $x \equiv 2 \pmod 5, y \equiv 0 \pmod 5, z \equiv 4 \pmod 5$.

b. Subtracting the last congruence from the first gives $3y \equiv 3 \pmod 5$, so $y \equiv 4 \pmod 5$. Let z take on the values 0, 1, 2, 3, and 4 and solve the last congruence for x to get 3, 0, 2, 4, and 1, respectively. This represents the 5 incongruent solutions.

c. Since the coefficient matrix for x and y is $\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$, which had determinant $2 \not\equiv 0 \pmod 5$, we can find a unique solution in x and y for any of the 5 possible values for z. Therefore, there are 5 incongruent solutions.

d. Since the determinant of the coefficient matrix is $4 \not\equiv 0 \pmod 5$ there is a unique solution to the system.

4.5.12. Cramer's rule will work for congruences just like for systems of equations. The determinant of the coefficient matrix must be relatively prime to the modulus.

4.5.13. In Gaussian elimination, the chief operation is to subtract a multiple of one equation or row from another, in order to put a 0 in a desirable place. Given that an entry a must be changed to 0 by subtracting

a multiple of b, we proceed as follows: Let $\overline{b}$ be the inverse for $b \pmod k$. Then $a-(a\overline{b})b=0$, and elimination proceeds as for real numbers. If $\overline{b}$ doesn't exist, and one cannot swap rows to get an invertible b, then the system is underdetermined.

4.5.14. Let k and l be integers in the range $0,1,\ldots,n^2-1$, and suppose that they are put into the same position (i,j). Then $a+ck+e[k/n] \equiv a+cl+e[l/n] \pmod n$ and $b+dk+f[k/n] \equiv b+dl+f[l/n] \pmod n$. This system reduces to $c(k-l)+e([k/n]-[l/n]) \equiv 0 \pmod n, c(k-l)+e([k/n]-[l/n]) \equiv 0 \pmod n$, which we can solve for $k-l$ and $[k/n]-[l/n]$. The coefficient matrix is $\begin{pmatrix} c & e \\ d & f \end{pmatrix}$, with determinant $cf-de$, which is relatively prime to n. Therefore the system has a unique solution modulo n, and this solution is obviously $(0,0)$. Thus we have $k \equiv l \pmod n$ and $[k/m] \equiv [l/n] \pmod n$. This last congruence, along with the fact that $0 \le k,l \le n^2-1$, implies that $\mid k-l \mid < n$. Then, since $k \equiv l \pmod n$, we have that $k=l$, as desired.

4.5.15. Consider summing the ith row. Let $k=xn+y$, where $0 \le y < n$. Then x and y must satisfy the Diophantine equation $i \equiv a+cy+ex \pmod n$, if k is in the ith row. Then $x-ct$ and $y+et$ is also a solution for any integer t. By Exercise 14, there must be n positive solutions which yield n numbers k between 0 and n^2. Let $s, s+1, \ldots, s+n-1$ be the values for t that give these solutions. Then the sum of the ith row is $\sum_{r=0}^{n-1}(n(x-c(s+r))+y+e(s+r)) = n(n+1)$, which is independent of i.

4.5.16. If an integer l from the range $0,1,\ldots,n^2-1$ is entered according to Exercise 14, and it is in a particular positive diagonal, then we must have $c+cl+e[l/n]+b+dl+f[l/n] \equiv k \pmod n$, or $(c+d)l+(e+f)[l/n] \equiv (a+b-k) \pmod n$. Let $l=x, y=[l/x]$, and $s=(a+b-k)$. Since $(c+d,n)=(e+f,n)=1$, if we choose x from the range $0,1,\ldots,n-1$, there will be a unique solution for y modulo n, namely $y \equiv (s-(c+d)x)\overline{(e+f)}$. Then $l=yn+x$. Let x run through it's possible values, and sum to get $\sum_{x=0}^{n-1} yn+x = \sum_{x=0}^{n-1}(s-(c+d)x)\overline{(e+f)}n+x = \sum_{x=0}^{n-1} s\overline{(e+f)}n+(1-(c+d))\overline{(e+f)}x = n^2 s\overline{(e+f)}+(1-(c+d))\overline{(e+f)}(n-2)(n-1)/2$. This is the sum of one positive diagonal. Since it does not depend on l, the sum must be the same for all diagonals. The argument for negative diagonals is identical.

4.6. Factoring Using the Pollard Rho Method

4.6.1. a. We compute $x_1=2^2+1=5$ and $x_2=5^2+1=26$. Then $(26-5,133)=(21,133)=7$, so we have $133=7\cdot 19$.

b. $x_1=5, x_2=26, x_3=677, x_4=565, x_5=574, x_6=124, x_7=1109, x_8=456, x_9=1051, x_{10}=21, x_{11}=442, x_{12}=369, x_{13}=616$, and $x_{14}=166$. Then $(x_{2i}-x_i,1189)=1$ for $i=1,2,\ldots,6$, but $(x_{14}-x_7,1189)=41$, and we have $1189=29\cdot 41$.

c. We need to compute up to $x_7=1273$ and $x_{14}=535$. Then we have $(535-1273,1927)=41$, and so $1927=41\cdot 47$.

d. We need to compute up to $x_4=2994$ and $x_8=6973$. Then we have $(6973-2994,8131)=173$, and so $8131=47\cdot 173$.

e. We need to compute up to $x_7=24380$ and $x_{14}=12066$. Then we have $(12066-24380,36287)=131$, and so $36287=131\cdot 277$.

f. We need to compute up to $x_8=18842$ and $x_{16}=7329$. Then we have $(7329-18842,48227)=29$, and so $48227=29\cdot 1663$.

4.6.2. a. We have $x_0=2, x_1=2^2+1=5, x_2=5^2+1=26, x_3=26^2+1=677 x_4=677^2+1=458330 \equiv 620 \pmod{1387}, x_5=202, x_6=582$, and so on. Then $(x_2-x_1,1387)=(26-5,1387)=1, (x_4-x_2,1387)=(620-26,1387)=1, (x_6-x_3,1387)=(582-677,1387)=19$, so $19 \mid 1387$.

b. We have $x_0 = 3, x_1 = 10, x_2 = 101, x_3 = 493, x_4 = 325, x_5 = 214, x_6 = 26, x_7 = 677, x_8 = 620, x_9 = 202, x_{10} = 582, x_{11} = 297, x_{12} = 829$, and so on. Then $(x_2 - x_1, 1387) = (101 - 10, 1387) = 1, (x_4 - x_2, 1387) = (325 - 101, 1387) = 1, (x_6 - x_3, 1387) = (26 - 493, 1387) = 1, (x_8 - x_4, 1387) = (620 - 325, 1387) = 1, (x_{10} - x_5, 1387) = (582 - 214, 1387) = 1, (x_{12} - x_6, 1387) = (829 - 26, 1387) = 73$,, so $73 \mid 1387$.

c. We have $x_0 = 2, x_1 = 3, x_2 = 8, x_3 = 63, x_4 = 1194, x_5 = 1186, x_6 = 177$, and so on. Then $(x_2 - x_1, 1387) = (8 - 3, 1387) = 1, (x_4 - x_2, 1387) = (1194 - 8, 1387) = 1, (x_6 - x_3, 1387) = (177 - 63, 1387) = 19$, so $19 \mid 1387$.

d. We have $x_0 = 2, x_1 = 11, x_2 = 1343, x_3 = 767, x_4 = 978$, and so on. Then $(x_2 - x_1, 1387) = (1343 - 11, 1387) = 1, (x_4 - x_2, 1387) = (978 - 1343, 1387) = 73$, so $73 \mid 1387$.

4.6.3. Numbers generated by linear functions where $a > 1$ will not be random in the sense that $x_{2s} - x_k = ax_{2s-1} + b - (ax_{s-1} + b) = a(x_{2s-1} - x_{s-1})$ is a multiple of a for all s. If $a = 1$, then $x_{2s} - x_s = x_0 + sb$. In this case, if $x_0 \neq 0$, then we will not notice if a factor of b that is not a factor of x_0 is a divisor of n.

CHAPTER 5

Applications of Congruences

5.1. Divisibility Tests

5.1.1. a. Since $2 \mid 4, 4 \mid 84, 8 \mid 984, 16 \mid 1984, 64 \mid 201984, 128 \mid 201984, 256 \mid 201984$, but 512 does not divide 201984, it follows that $256 = 2^8$ is the highest power of 2 that divides 201984.

b. Since $2 \mid 8, 4 \mid 8, 8 \mid 408, 16 \mid 3408$, but 32 does not divide 23408, it follows that $16 = 2^4$ is the highest power of 2 that divides 1423408.

c. Since $2 \mid 4, 4 \mid 44, 8 \mid 744, 16 \mid 5744, 32 \mid 75744, 64 \mid 375744, 128 \mid 9375744, 256 \mid 89375744, 512 \mid 89375744, 1024 \mid 89375744$, but 2048 does not divide 89375744, it follows that $1024 = 2^{10}$ is the highest power of 2 that divides 8937544.

d. Since $2 \mid 6$ but 4 does not divide 46, it follows that $2 = 2^1$ is the highest power of 2 that divides 41578912246.

5.1.2. a. Since $5 \mid 0, 25 \mid 50, 125 \mid 250$ but 625 does not divide 2250, it follows that $125 = 5^3$ is the highest power of 5 that divides 112250.

b. Since $5 \mid 5, 25 \mid 25, 125 \mid 625, 625 \mid 625$, but 3125 does not divide 60625, it follows that $625 = 5^4$ is the highest power of 5 that divides 4860625.

c. Since $5 \mid 0$ but 25 does not divide 90, it follows that $5 = 5^1$ is the highest power of 5 that divides 235555790.

d. Since $5 \mid 5, 25 \mid 25, 125 \mid 125, 625 \mid 3125, 3125 \mid 53125, 15625 \mid 953125, 78125 \mid 6953125, 390625 \mid 26953125, 1953125 \mid 126953125$, but 9765625 does not divide 8126953125 it follows that $1953125 = 5^9$ is the highest power of 5 that divides 48126953125.

5.1.3. a. The sum of the digits of 18381 is $1 + 8 + 3 + 8 + 1 = 21$. Since this sum is divisible by 3 but not by 9, 18381 is divisible by 3, but not by 9.

b. The sum of the digits of 65412351 is $6 + 5 + 4 + 1 + 2 + 3 + 5 + 1 = 27$. Since this sum is divisible by 3 and by 9 it follows that 65412351 is divisible by both 3 and 9.

c. The sum of the digits of 987654321 is $9+8+7+6+5+4+3+2+1 = 45$. since this sum is divisible by 3 and by 9 it follows that 987654321 is divisible by both 3 and 9.

d. The sum of the digits of 78918239735 is $7+8+9+1+8+2+3+9+7+3+5 = 62$. Since this sum is not divisible by 3, 78918239735 is divisible by neither 3 nor 9.

5.1.4. a. We have $1 - 0 + 7 - 6 + 3 - 7 + 3 - 2 = -1$, so $11 \nmid 10763732$.

b. We have $1 - 0 + 8 - 6 + 3 - 2 + 0 - 0 + 1 - 5 = 0$, so $11 \mid 1086320015$.

c. We have $6 - 7 + 4 - 3 + 1 - 0 + 9 - 7 + 6 - 3 + 7 - 5 = 8$, so $11 \nmid 674310976375$.

d. We have $8-9+2-4+3-1+0-0+6-4+5-3+7=10$, so $11 \nmid 8924310064537$.

5.1.5. By Theorem 5.1, the power of 2 dividing a number is equal to the number of zeros at the end of its binary expression. **a.** $2^1=2$ **b.** $2^0=1$ **c.** $2^6=64$ **d.** $2^0=1$

5.1.6. a. Since $3 \mid (2+1)$, we use Theorem 5.3. We have $1-0+1-1+1-1+1-1+0=1$, so $3 \nmid (101111110)_2$.

b. We have $1-0+1-0+0-0+0-0+1-1=(10)_2$, so $3 \nmid (1010000011)_2$.

c. We have $1-1+1-0+0-0+0-0+0=1$, so $3 \nmid (111000000)_2$.

d. We have $1-0+1-1+0-1+1-1+0-1=-1$, so $3 \nmid (1011011101)_2$.

5.1.7. a. Using Theorem 5.2, we need only examine the sum of the digits. We have $1+2+1+0+1+2+2=9$. As 2 does not divide 9, 2 does not divide $(1210122)_3$.

b. Since 2 does not divide $2+1+1+1+0+2+1+0+1=9$, 2 does not divide $(211102101)_3$.

c. Since 2 divides $1+1+1+2+2+0+1+1+1+2=12$, then $2 \mid (1112201112)_3$.

d. Since 2 divides $1+0+1+2+2+2+2+2+0+1+1+1+0+1=16$, then $2 \mid (10122222011101)_3$.

5.1.8. a. Since $4 \mid (3+1)$, we use Theorem 5.2. We have $1+2+1+0+1+2+2=9$, so $4 \nmid (1210122)_3$.

b. We have $2+1+1+1+0+2+1+0+1=9$, so $4 \nmid (211102101)_3$.

c. We have $1+1+1+2+2+0+1+1+1+2=12$, so $4 \mid (1112201112)_3$.

d. We have $1+0+1+2+2+2+2+2+0+1+1+1+0+1=16$, so $4 \mid (10122222011101)_3$.

5.1.9. a. As both 3 and 5 divide $16-1$, Theorem 5.2 tells us that we need only examine the sum of the base 16 digits.
$3+E+A+2+3+5=3+14+10+2+3+5=37$. As neither 3 nor 5 divides 7, neither 3 nor 5 divides $(3EA235)_{16}$.

b. Since $A+B+C+D+E+F=10+11+12+13+14+15=75$ is divisible by 3 and 5, we see that both 3 and 5 divide $(ABCDEF)_{16}$.

c. Since neither 3 nor 5 divides $15+1+1+7+9+2+1+1+7+3=47$, neither 3 nor 5 divides $(F117921173)_{16}$.

d. Since 5 divides $1+0+10+11+9+8+7+3+0+1+15=65$, but 3 does not, we have that 5 divides $(10AB987301F)_{16}$, but 3 does not.

5.1.10. a. Since $17 \mid (16+1)$, we use Theorem 5.3. We have $3-E+A-2+3-5=-5$, so $17 \nmid (3EA235)_{16}$.

b. We have $A-B+C-D+E-F=-3$, so $17 \nmid (ABCDEF)_{16}$.

c. We have $F-1+1-7+9-2+1-1+7-3=19$, so $17 \nmid (f117921173)_{16}$.

d. We have $1+0-A+B-9+8-7+3-0+1=-2$, so $17 \nmid (10AB987301)_{16}$.

5.1.11. The sum of the digits of a repunit with n 1's in its decimal expansion is n. This repunit is divisible by 3 if and only if n is divisible by 3 and is divisible by 9 if and only if n is divisible by 9.

5.1.12. The alternating sum of the digits of a repunit with n digits is 0 if n is even and 1 if n is odd. Hence the repunit with n digits is divisible by 11 if and only if n is even.

5.1.13. The alternating sum of blocks of three digits of an n-digit repunit is 0 if $n \equiv 0 \pmod 6$, 1 if $n \equiv 1 \pmod 6$, 11 if $n \equiv 2 \pmod 6$, 111 if $n \equiv 3 \pmod 6$, 110 if $n \equiv 4 \pmod 6$, 100 if $n \equiv 5 \pmod 6$. Hence a repunit with n decimal digits is divisible by 1001 if and only if $n \equiv 0 \pmod 6$. Since 7 divides this alternating sum if and only if $n \equiv 0 \pmod 6$, these are exactly the values of n for which this requnit is divisible by 7. Exactly the same reasoning and conclusion holds for divisibility by 13.

5.1.14. The repunit with 2 digits, 11, is prime, while the repunit with 1 digit, 1, is not prime. By Exercise 11 we know that the repunits with 3,6, and 9 digits are divisible by 3. By Exercise 12 we know that the repunits with 4,6,and 8 digits are divisible by 11. this leaves the repunits with 5 digits and 7 digits. But we find that $41 \mid 11111$ and $239 \mid 1111111$. Hence 11 is the only repunit with less that 10 digits that is prime.

5.1.15. Let d be a divisor of $b-1$. By Theorem 5.2, a number is divisible by d if and only if the sum of its digits is a multiple of d. Since the sum of the digits of a repunit is equal to the number of digits it has, a repunit is divisible by d if and only if it has a multiple of d digits.

5.1.16. Let $d \mid (b+1)$. by Theorem 5.3, d will divide a repunit of n digits if and only if the alternating sum of the digits is divisible by d. But the only possible alternating sums of repunits are 0 if n is even and 1 if n is odd. So a factor of $b+1$ divides a repunit if and only if the repunit has an even number of digits.

5.1.17. A palindromic integer with $2k$ digits has the form $(a_k a_{k-1} \dots a_1 a_1 a_2 \dots a_k)_{10}$. Using the test for divisibility by 11 developed in this section, we find that $a_k - a_{k-1} + \cdots \pm a_1 \mp a_1 \pm a_2 \mp \cdots - a_k = 0 \equiv 0 \pmod{11}$ and so $(a_k a_{k-1} \dots a_1 a_1 a_2 \dots a_k)_{10}$ is divisible by 11.

5.1.18. Let $a = (a_1 a_2 \dots a_n a_n \dots a_1)_7$ be a base 7 palindromic integer with an even number of digits. Since $8 \mid (7+1)$, by Theorem 5.3, 8 will divide a if and only if 8 divides $a_1 - a_2 + \cdots + (-1)^n a_n + (-1)^{n+1} a_n + (-1)^{n+2} a_{n-1} + \ldots - a_1 = 0$ which it does.

5.1.19. Let $a_k a_{k-1} \dots a_1 a_0$ be the decimal representation of an integer. Then $a_k a_{k-1} \dots a_1 a_0 = a_0 a_1 a_2 + 10^3 a_3 a_4 a_5 + 10^3(10^3 a_6 a_7 a_8) + \cdots$. So, $a_k a_{k-1} \dots a_1 a_0 \equiv a_0 a_1 a_2 + a_3 a_4 a_5 + a_6 a_7 a_8 + \cdots \pmod{37}$. Thus $a_k a_{k-1} \dots a_1 a_0$ is divisible by 37 if and only if $a_0 a_1 a_2 + a_3 a_4 a_5 + a_6 a_7 a_8 + \cdots$ is also. Hence, 443692 is divisible by 37 if and only if $443 + 692 = 1135$ is. And 1134 is divisible by 37 if and only if $1 + 135 = 136$ is. But 136 is not, and so 37 does not divide 443692. Further, 11092785 is divisible by 37 if and only if $11 + 092 + 785 = 888$ is. We know that $888 = 24 \cdot 37$, so 11092785 is a multiple of 37.

5.1.20. Group the digits of the integer into blocks of 2, starting at right. Now consider the number as a base b^2 integer, with each block of 2 representing a digit. Then by Theorem 5.3, n will divide the integer if and only if n divides the alternating sum of the blocks of 2.

5.1.21. a. Applying Exercise 20, we have $(1)_2 - (01)_2 + (11)_2 - (01)_2 + (10)_2 = (100)_2 = 4$, Since 4 is not divisible by $5 = 2^2 + 1$, neither is $(101110110)_2$.

b. Applying Exercise 20, we have $-(12)_3 + (10)_3 - (01)_3 + (22)_3 = (12)_3 = 5$. Since $2 \nmid 12$ but $5 \mid 12$, only 5 divides $(12100122)_3$.

c. Applying Exercise 20, we have $(3)_8 - (64)_8 + (70)_8 - (12)_8 + (44)_8 = (41)_8 = 33$ which is divisible by neither 5 nor 13. Hence neither divides the number.

d. Applying Exercise 20, we have $5 - 83 + 70 - 41 + 32 - 02 + 19 = 0$ which is divisible by 101, and therefore, so is $(5837041320219)_{10}$.

5.1.22. We have that $88 \mid (x42y)_{10}$, so $8 \mid (x42y)$. Then we must have $8 \mid 42y$, so $y = 4$. Also, $11 \mid (x424)$ and so $11 \mid (x - 4 + 2 - 4) = x - 6$. Therefore, $x = 6$, and the price of each chicken was $\$64.24/88 = \$0.73$.

5.1.23. First, note that $89878 \equiv 8+9+8+7+8 \equiv 4 \pmod 9$, $58965 \equiv 5+8+9+6+5 \equiv 6 \pmod 9$, and $5299?56270 \equiv 5+2+9+9+?+5+6+2+7+0 \equiv ? \pmod 9$. So, $89878 \cdot 58965 \equiv 4 \cdot 6 \equiv 6 \equiv ? \pmod 9$. Thus, as the question mark represents a single decimal digit congruent to 6, the question mark represents the digit 6.

5.1.24. a. We have $8+7+5+9+6+1 = 33, 2+7+5+3 = 17$, and $2+4+1+0+5+2+0+6+3+3 = 26$, but $37 \not\equiv 26 \pmod 9$, so there is an error in the multiplication.

b. We have $1+4+7+9+8 \equiv 2 \pmod 9, 2+3+5+6+7 \equiv 5 \pmod 9$, and $3+4+8+5+3+2+3+6+7 \equiv 5 \pmod 9$, but $2 \cdot 5 \not\equiv 5 \pmod 9$, so there is an error.

c. We have $2+4+7+8+9 \equiv 3 \pmod 9, 4+3+7+1+7 \equiv 4 \pmod 9$, and $1+0+9+2+7+0+0+7+1+30 \equiv 3 \pmod 9$, and $3 \cdot 4 \equiv 12 \equiv 3 \pmod 9$ so the multiplication may be correct. (Actually it's not! See Exercises 25 and 26.)

5.1.25. Casting out nines is not infallible. To see this, note that $19 \equiv 2 \cdot 5 \pmod 9$, but $19 \neq 2 \cdot 5$. The cause of this problem is that $0 \equiv 9 \pmod 9$, and so any 0 may be replace by a 9, or vice versa, and the congruence $c \equiv ab \pmod 9$ will still hold, whereas in general, the equality $c = ab$ will not hold.

5.1.26. a. The sum of blocks of two digits, starting at the right, are congruent to the integer modulo 99. Then we have $87+59+61 \equiv 9 \pmod{99}, 27+53 \equiv 80 \pmod{99}$, and $24+10+52+06+33 \equiv 26 \pmod{99}$, but $9 \cdot 80 \equiv 27 \not\equiv 26 \pmod{99}$, so the error is detected.

b. We have $01+47+89 \equiv 38 \pmod{99}, 02+35+67 \equiv 5 \pmod{99}$, and $03+48+53+23+67 \equiv 95 \pmod{99}$ but $38 \cdot 5 \equiv 91 \not\equiv 95 \pmod{99}$, so the error is detected.

c. We have $2+47+89 \equiv 39 \pmod{99}, 04+37+17 \equiv 58 \pmod{99}$, and $10+92+70+07+13 \equiv 93 \pmod{99}$, but $39 \cdot 58 \equiv 84 \not\equiv 93 \pmod{99}$, so the error is detected.

5.2. The Perpetual Calendar

5.2.1. Happy Birthday!

5.2.2. a. October 12, 1492 would be October 2, 1492 in the Gregorian Calendar. So $k = 2, m = 8, C = 14, Y = 92$. Then $W = 2 + [2.6 \cdot 8 + 0.2] - 2 \cdot 14 + 92 + [92/4] + [14/4] \equiv 0 \pmod 7$. Hence October 12, 1492 was a Sunday.

b. May 6, 1692 would be April 26, 1692 in the Gregorian Calendar. So $k = 26, m = 2, C = 16, Y = 92$. Then $W = 26 + [2.6 \cdot 2 - 0.2] - 2 \cdot 16 + 92 + [92/4] + [16/4] \equiv 6 \pmod 7$. Hence May 6, 1692 was a Saturday.

c. June 15,1752 would be June 5,1752 in the Gregorian Calendar. So $k = 15, m = 2, C = 17, Y = 52$. Then $W = 15 + [2.6 \cdot 4 - 0.2] - 2 \cdot 17 + 52 + [52/4] + [17/4] \equiv 4 \pmod 7$. Hence June 15,1752 was a Thursday.

d. For July 4, 1776 we have $k = 4, m = 5, C = 17$, and $Y = 76$. This implies that $W = 4 + [2.6 \cdot 5 - 0.2] - 2 \cdot 17 + 76 + [76/4] + [19/4] = 4 + 12 - 34 + 76 + 19 + 4 = 81 \equiv 4 \pmod 7$. Hence July 4, 1776 was a Thursday.

e. For March 30, 1867 we have $k = 30, m = 1, C = 18$, and $Y = 67$. This implies that $W \equiv 30 + [2.6 \cdot 1 - 0.2] - 2 \cdot 18 + 67 + [18/4] + [67/4] = 30 + 2 - 36 + 67 + 4 + 16 = 83 \equiv 6 \pmod 7$. Hence March 30, 1867 was a Saturday.

f. For March 17, 1888 we have $k = 17, m = 1, C = 18$, and $Y = 88$. This implies that $W \equiv 17 + [2.6 \cdot 1 - 0.2] - 2 \cdot 18 + 88 + [18/4] + [88/4] = 17 + 2 - 36 + 88 + 4 + 22 = 97 \equiv 6 \pmod 7$. Hence March

17, 1888 was a Saturday.

g. For February 15, 1898 we have $k = 15, m = 12, C = 18$, and $Y = 97$. This implies that $W \equiv 15 + [2.6 \cdot 12 - 0.2] - 2 \cdot 18 + 97 + [18/4] + [98/4] = 15 + 31 - 36 + 97 + 4 + 24 = 135 \equiv 2 \pmod 7$. Hence February 15, 1898 was a Tuesday.

h. For July 2, 1925 we have $k = 2, m = 5, C = 19$, and $Y = 25$. This implies that $W \equiv 2 + [2.6 \cdot 5 - 0.2] - 2 \cdot 19 + 25 + [19/4] + [25/4] = 2 + 12 - 38 + 25 + 4 + 6 = 11 \equiv 4 \pmod 7$. Hence July 2, 1925 was a Thursday.

i. For July 16, 1945 we have $k = 16, m = 5, C = 19$, and $Y = 45$. This implies that $W \equiv 16 + [2.6 \cdot 5 - 0.2] - 2 \cdot 19 + 45 + [19/4] + [45/4] = 16 + 12 - 38 + 45 + 4 + 11 = 50 \equiv 1 \pmod 7$. Hence July 16, 1945 was a Monday.

j. For July 20, 1969 we have $k = 20, m = 5, C = 19$, and $Y = 69$. this implies that $W \equiv 20 + [2.6 \cdot 5 - 0.2] - 2 \cdot 19 + 69 + [19/4] + [69/4] = 20 + 12 - 38 + 69 + 4 + 17 = 84 \equiv 0 \pmod 7$. Hence July 20, 1969 was a Sunday.

k. For August 9, 1974 we have $k = 9, m = 6, C = 19$, and $Y = 74$. This implies that $W \equiv 9 + [2.6 \cdot 6 - 0.2] - 2 \cdot 19 + 74 + [19/4] + [74/4] = 9 + 15 - 38 + 74 + 4 + 18 = 82 \equiv 5 \pmod 7$. Hence August 9, 1974 was a Friday.

l. For March 28, 1979 we have $k = 28, m = 1, C = 19$, and $Y = 79$. This implies that $W \equiv 28 + [2.6 \cdot 1 - 0.2] - 2 \cdot 19 + 79 + [19/4] + [79/4] = 28 + 2 - 38 + 79 + 4 + 19 = 94 \equiv 3 \pmod 7$. Hence March 28, 1979 was a Wednesday.

m. For June 5, 2013 we have $k = 5, m = 4, C = 20$, and $Y = 13$. this implies that $W \equiv 5 + [2.6 \cdot 4 - 0.2] - 2 \cdot 20 + 13 + [20/4] + [13/4] = 5 + 10 - 40 + 13 + 5 + 3 = -4 \equiv 3 \pmod 7$. Hence June 5, 2013 will be a Wednesday.

n. For December 25, 1991, we have $k = 25, m = 10, C = 19, Y = 91$. Then $W = 25 + [2.6 \cdot 10 - 0.2] - 2 \cdot 19 + 91 + [91/4] + [19/4] \equiv 3 \pmod 7$. So December 25, 1991 was a Wednesday.

o. For June 5, 2013 we have $k = 5, m = 4, C = 20$, and $Y = 27$. this implies that $W \equiv 5 + [2.6 \cdot 4 - 0.2] - 2 \cdot 20 + 27 + [27/4] + [20/4] = 5 + 10 - 40 + 13 + 6 + 5 = 6 \pmod 7$. Hence June 5, 2027 will be a Saturday.

5.2.3. For this problem, we let $k = 13, C = 20, Y = 20$, and $W = 5$. Now, $[2.6m - 0.2] \equiv W - k + 2C - Y - [\frac{Y}{4}] - [\frac{C}{4}] \equiv 2 \pmod 7$. And since $[2.6m - 0.2] \equiv 2 \pmod 7$ with $0 < m \leq 10$ only for $m = 1$ and 9, we see that March and November have Friday the 13th. But we have only checked the months after (and including March). To check January and February, let $W = 5, k = 13, C = 20$, and $Y = 19$. Now, $[2.6m - 0.2] \equiv W - k + 2C - Y - [\frac{Y}{4}] - [\frac{C}{4}] \equiv 4 \pmod 7$. But $[2.6 \cdot 11 - 0.2] \equiv 0 \pmod 7$ and $[2.6 \cdot 12 - 0.2] \equiv 3 \pmod 7$, so neither January nor February have Friday the 13th. So the 13th will fall on Friday only twice in the year 2020.

5.2.4. There are $[10,000/4] = 2500$ years divisible by 4, which are candidates for leap years. But there are $[10,000/100] = 100$ centuries which are not leap years, except for the $[10,000/400] = 25$ centuries divisible by 4, which are leap years. This gives $2500 - 100 + 25 = 2425$ leap years between the year 1 and the year 10,000.

5.2.5. For each 4000 years, we need to subtract one day from the total number of days before reducing modulo 7. Therefore, we subtract $[N/4000] = [C/40]$ from the right hand side of the formula, giving $W \equiv k + [2.6m - 0.2] - 2C + Y + [Y/4] + [C/4] - [C/40] \pmod 7$.

5.2.6. Let the later date be in the year $100C + Y_1$ and the earlier date be in the year $100C + Y_2$, where $Y_1 - Y, 2 = 28, 56$, or 84. Then $W_1 - W_2 \equiv (Y_1 - Y_2) + [Y_1/4] - [Y_2/4] \equiv [Y_1/4] - [Y_2/4] \pmod 7$ since $7 \mid 28, 56$, and 84. Since $Y_1 \equiv Y_2 \pmod 4$, we have $Y_1 \equiv 4n_1 + r$ and $Y_2 \equiv 4n_2 + r$, for integers n_1, n_2, r, with $0 \leq$

$r < 4$. Then $7 \mid (Y_1 - Y_2) \equiv 4(n_1 - n_2)$, so $7 \mid (n_1 - n_2)$. Then we have $W_1 - W_2 \equiv n_1 - n_2 \equiv 0 \pmod 7$. Therefore, the two days fall on the same day of the week.

5.2.7. If B is the number of the day of the week you were born, $0 \leq B < 7$, and M is the month and K is the day, then we need to solve the congruence $B \equiv K + [2.6M - 0.2] - 2C + Y + [Y/4] + [C/4] \pmod 7$ for C and Y. There are two cases. If $C = 19$, then the congruence reduces to $B \equiv K + [2.6M - 0.2] + Y + [Y/4] + 1 \pmod 7$. If $C = 20$ then the congruence reduces to $B \equiv K + [2.6M - 0.2] + Y + [Y/4] \pmod 7$. In both cases, there are 4 subcases depending on the residue of Y modulo 4. Restrict Y to only those years between your birth and your 100th birthday.

5.2.8. This is the sequence of years that are not divisible by 4, so the next term is 2005. These are all non-leap years.

5.2.9. This is the sequence of years divisible by 100, but not by 400, so the next term is 2500. These are all the century years that are not leap years.

5.2.10. In any 400 consecutive years, there will be exactly 100 multiples of 4, exactly 4 multiples of 100 and exactly 1 multiple of 400, so there will be exactly $100 - 4 + 1 = 97$ leap years in that time span.

5.2.11. If the 13th falls on the same day of the week on two consecutive months, then the number of days in the first month must be congruent to 0 modulo 7, and the only such month is February during non-leap year. If February 13th is a Friday, then January 1st is $31 + 13 - 1 \equiv 1 \pmod 7$ week days earlier, that is, Thursday.

5.2.12. 12 Years in the International Fixed Calendar match exactly with years in the Gregorian Calendar. Since June gets the extra day for leap year, we number the months as follows: Sol = 1, July = 2, August = 3, September = 4, October = 5, November = 6, December = 7, January = 8, February = 9, March = 10, April = 11, May = 12, and June = 13. Other notation is the same as for the Gregorian Calendar. Since January 1, 1600 Gregorian = January 1, 1600 IFC, we can compute that Sol 1, 1600 was a Monday. We compute the number of leap years since then in the same way as in the Gregorian. Each normal year shifts the day of the week by one, and each leap year shifts the day of the week by two, just as in the Gregorian Calendar. Thus, if d_N is the day of the week of Sol 1 in year N, then $d_N \equiv 1 - 2C + Y + [C/4] + [Y/4] \pmod 7$. Since each month has 28 days, which is divisible by 7, the first day of each of the months of Sol through December are the same day of the week, for months January through June, (after year end day), the first day of the month is shifted one. Then $[m/8] + 1$ gives the shift for the change in months. The day of the week is given by $W \equiv k + [m/8] + 1 - 2C + Y + [Y/4] + [C/4] \pmod 7$.

5.2.13. In the perpetual calendar formula we let $W = 5$ and $k = 13$ to get $5 \equiv 13 + [2.6m - 0.2] - 2C + Y + [Y/4] + [C/4] \pmod 7$. Then $[2.6m - 0.2] \equiv 6 + 2C - Y - [Y/4] - [C/4] \pmod 7$. We note that as the month varies from March to December, the expression $[2.6m - 0.2]$ takes on every residue class modulo 7. So regardless of the year, there is always an m which makes the left side of the last congruence congruent to the right side.

5.2.14. Note that as the month runs from March to December, the expression $[2.6m - 0.2] \pmod 7$ runs through the sequence $2, 5, 0, 3, 5, 1, 4, 6, 2, 4$, so all residue classes modulo 7 are covered and all of these months have at least 30 days. The perpetual calendar formula gives us $W \equiv k + [2.6m - 0.2] - 2C + Y + [Y/4] + [C/4] \pmod 7$. The only think not fixed on the right hand side is the expression $[2.6m - 0.2]$, and since it runs through all residue classes, so does W.

5.2.15. The months with 31 days are March, May, July, August, October, December and January, which is considered in the previous year. The corresponding numbers for these months are $1, 3, 5, 6, 8, 10$, and 12. Given Y and C, we let $k = 31$ in the perpetual calendar formula and get $W \equiv 31 + [2.6m - 0.2] - 2C + Y + [Y/4] + [C/4] \equiv 3 + [2.6m - 0.2] - 2C + Y + [Y/4] + [C/4] \pmod 7$. To see which days of the week the 31st will fall on, we let m take on the values $1, 3, 5, 6, 8, 10$ and reduce. Finally, we decrease the year by one (which may require decreasing the century by one) and let m take on the value 12 and

reduce modulo 7. The collection of values of W tells us the days of the week on which the 31st will fall.

5.2.16. For February to have 5 Sundays, we must have the 29th be a Sunday. So in the perpetual calendar formula, we let $W = 0$, $k = 29$ and $m = 12$, giving us $0 \equiv 29 + [2.6(12) - 0.2] - 2C + Y + [Y/4] + [C/4] \pmod 7$. Since C is fixed, this linear congruence has a unique solution modulo 7. And since February has 29 days only in leap years, we have $Y \equiv 0 \pmod 4$. By the Chinese remainder theorem, these two congruences have a unique solution modulo 28. Therefore February has 5 Sundays every 28 years during a given century. Therefore this could happen at most 4 times during one century. To show that this happens at least 4 times, we seek a century in which February has 5 Sundays for a very small value of Y. Since $Y + 1$ must be a multiple of 4, we set $Y = 4b - 1$, so that $[Y/4] = b - 1$. Then the perpetual calendar formula reduces to $0 \equiv 29 + [2.6(12) - 0.2] - 2C + (4b - 1) + (b - 1) + [C/4] \pmod 7$, or $5b \equiv -2 + 2C - [C/4] \pmod 7$. Multiplying through by 3 gives us $b \equiv 1 - C - 3[C/4] \pmod 7$. We seek a value of C which will make $b \equiv 1 \pmod 7$ so that Y will be small. We note that for $C = 20$, we have $b = 1$ so that $y = 3$ which corresponds to February of 2004, which had 5 Sundays. Then in that century the other years with 5 Sundays will be 2032, 2060 and 2088.

5.3. Round-Robin Tournaments

5.3.1. a. Teams i and j are paired in round k if and only if $i + j \equiv k \pmod 7$ with team i drawing a bye if $2i \equiv k \pmod 7$. The result is shown in the following table.

	Team						
Round	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	7	6	5	bye	3h	2h	1h
2	bye	7h	6h	5h	4	3	2
3	2h	1	7h	6h	bye	4	3
4	3h	bye	1	7	6	5h	4h
5	4	3	2h	1h	7h	bye	5
6	5h	4h	bye	2	1	7	6h
7	6	5	4	3h	2h	1h	bye

b. Teams i and j are paired in round k if and only if $i + j \equiv k \pmod 7$. Team i draws a Team 8 if $2i \equiv k \pmod 7$. The result is shown in the following table.

	Team							
Round	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	7	6	5	8	3	2	1	4
2	8	7	6	5	4	3	2	1
3	2	1	7	6	8	4	3	5
4	3	8	1	7	6	5	4	2
5	4	3	2	1	7	8	5	6
6	5	4	8	2	1	7	6	3
7	6	5	4	3	2	1	8	7

c. Teams i and j are paired in round k if and only if $i + j \equiv k \pmod 9$. Team i draws a bye if $2i \equiv k \pmod 9$. The result is shown in the following table.

	Team								
Round	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	9h	8h	7h	6h	bye	4	3	2	1
2	bye	9	8	7	6	5h	4h	3h	2h
3	2	1h	9h	8h	7h	bye	5	4	3
4	3h	bye	1	9	8	7	6h	5h	4h
5	4	3	2h	1h	9h	8h	bye	6	5
6	5h	4h	bye	2	1	9	8	7h	6h
7	6	5	4	3h	2h	1h	9	bye	7h
8	7h	6h	5h	bye	3	2	1	9	8h
9	8	7	6	5	4h	3h	2h	1h	bye

d. Teams i and j are paired in round k if and only if $i + j \equiv k \pmod 9$. Team i draws Team 10 if $2i \equiv k \pmod 9$. The result is shown in the following table.

	Team									
Round	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	9	8	7	6	10	4	3	2	1	5
2	10	9	8	7	6	5	4	3	2	1
3	2	1	9	8	7	10	5	4	3	6
4	3	10	1	9	8	7	6	5	4	2
5	4	3	2	1	9	8	10	6	5	7
6	5	4	10	2	1	9	8	7	6	3
7	6	5	4	3	2	1	9	10	7	8
8	7	6	5	10	3	2	1	9	8	4
9	8	7	6	5	4	3	2	1	10	9

5.3.2. Let n be an odd positive integer. First suppose that i is odd. Then $i+j$ is even for $j = 1, 3, 5, \ldots, i, \ldots, n$. Team i is the home team in its game with team j where j is odd if and only if $i > j$, and this occurs $(i - 1)/2$ times. Furthermore, $i + j$ is odd for $j = 2, 4, 6, \ldots, n - 1$. Team i is the home team in its game with a team j where j is even if and only if $i < j$, and this occurs $[(n - 1) - (i - 1)]/2 = (n - i)/2$ times. Hence team i is the home team $(i - 1)/2 + (n - 1)/2 = (n - 1)/2$ times. Since this team plays $n - 1$ games, it is the away team $n - (n - 1)/2 = (n - 1)/2$ times. Now suppose that i is even. Then $i + j$ is even for $j = 2, 4, 6, \ldots, i, \ldots, n - 1$. Team i is the home team in its game with team j where j is even if and only if $i > j$, and this occurs $(i - 2)/2$ times. Furthermore, $i + j$ is odd for $j = 1, 3, 5, \ldots, n$. Team i is the home team in its game with a team j where j is odd if and only if $i > j$, and this occurs $[n - 1(i - 1)]/2 = (n - 1 + 1)/2$ times. Hence team i is the home team $(i - 2)/2 + (n - i + 1)/2 = (n - 1)/2$ times. Since this team plays $n - 1$ games, it is the away team $n - (n - 1)/2 = (n - 1)/2$ times. We conclude that each team plays an equal number of home and away games.

5.3.3. a. For round 1, teams i and j are paired if $i + j \equiv 1 \pmod 5$. Teams 1 and 5 are paired, and since $1 + 5 = 6$ is even, team 5 is the home team. Teams 2 and 4 are paired, and since $2 + 4 = 6$ is even, team 4 is the home team. Finally, in round 1 team 3 draws a bye.

For round 2, teams i and j are paired if $i + j = 2 \pmod 5$. Team 1 draws a bye. Teams 2 and 5 are paired, and since $2 + 5 = 7$ is odd, team 2 is the home team. Teams 3 and 4 are paired, and since $3 + 4 = 7$ is odd, team 3 is the home team.

For round 3, teams i and j are paired if $i + j = 3 \pmod 5$. Teams 1 and 2 are paired, and since $1 + 2 = 3$ is odd, team 1 is the home team. Teams 3 and 5 are paired, and since $3 + 5 = 8$ is even, team 5 is the home team. Team 4 draws a bye.

For round 4, teams i and j are paired if $i + j = 4 \pmod 5$. Teams 1 and 3 are paired, and since $1+3 = 4$ is even, team 3 is the home team. Team 2 draws a bye. Teams 4 and 5 are paired, and since $4 + 5 = 9$ is odd, team 4 is the home team.

For round 5, teams i and j are paired if $i + j = 5 \pmod 5$. Teams 1 and 4 are paired, and since $1 + 4 = 5$ is odd, team 1 is the home team. Teams 2 and 3 are paired, and since $2 + 3 = 5$ is odd, team 2 is the home team. Team 4 draws a bye.

We see that each team plays 2 home and 2 away games.

b. In the table in Exercise 1 part (a), the teams who play at home are marked with an "h."

c. In the table in Exercise 1 part (c), the teams who play at home are marked with an "h."

5.4. Hashing Functions

5.4.1. Let k be the six-digit number on the license plate of a car. We can assign this car the space numbered $h(k) \equiv k \pmod{101}$ where the spaces are numbered $0, 1, 2, \ldots, 100$. When a car is assigned the same space as another car we can assign it to the space $h(k) + g(k)$ where $g(k) \equiv k + 1 \pmod{99}$ and $0 < g(k) \leq 98$. When this space is occupied we next try $h(k)+2g(k)$, then $h(k)+3g(k)$, and so on. All spaces are examined since $(g(k), 101) = 1$.

5.4.2. a. For example, suppose a student was born on the 23rd of the month. Then $K = 23$, and $h(23) \equiv 23 \equiv 4 \pmod{19}$. so we would assign the 4th memory location to this student if it is free. If it's not, then we would try $h_1(23) \equiv 4 + 1 \equiv 5 \pmod{19}$. If this one is not free, then we would try $h_2(23) \equiv 4 + 2 \equiv 6 \pmod{19}$, and so on until we found an empty location.

b. For example, suppose $K = 23$. As in part (a), $h(23) \equiv 4 \pmod{19}$. We compute $g(23) \equiv 1 + 23 \equiv 7 \pmod{17}$. If there is a collision, we try $h_1(23) \equiv 4 + 1 \cdot 7 \equiv 11 \pmod{19}$ In case of a collision here, we try $h_2(23) \equiv 4 + 2 \cdot 7 \equiv 18 \pmod{19}$, and so on.

5.4.3. a. It is clear that m memory locations will be probed as $j = 0, 1, 2, \ldots, m - 1$. To see that they are all distinct, and hence every memory location is probed, assume that $h_i(K) \equiv h_j(K) \pmod m$. Then $h(K) + iq \equiv h(K) + jq \pmod m$. From this it follows that $iq \equiv jq \pmod m$, and as $(q, m) = 1$, $i \equiv j \pmod m$ by corollary 3.4.1. And so $i = j$ since i and j are both less than m.

b. It is clear that m memory locations will be probed as $j = 0, 1, 2, \ldots, m - 1$. To see that they are all distinct, and hence every memory location is probed, assume that $h_i(K) \equiv h_j(K) \pmod m$. Then $h(K) + iq \equiv h(K) + jq \pmod m$. From this it follows that $iq \equiv jq \pmod m$, and as $(q, m) = 1$, $i \equiv j \pmod m$ by corollary 3.4.1. And so $i = j$ since i and j are both less than m.

5.4.4. a. Let l represent some memory location. Then we seek to solve $l \equiv h(K) + j(2h(K) + 1) \pmod m$, or $l - h(K) - j \equiv 2h(K)j \pmod m$. Since m is prime, and $1 < 2, h(K) < m$, then 2 and $h(K)$ have inverses modulo m. Therefore, we can solve the congruence for j, and hence the location l is proved at this value for j.

b. If we the definition into the congruence $h_{j+r}(K_1) \equiv h - j + r(K_2) \pmod m$, and rearrange, we get $h(K_1)(1 + 2(j + r)) \equiv h(K_2)(1 + (j + r)) \pmod m$. Since this must be true for all r, many of which cause $(1 + 2(j + r))$ to be invertible, we must have $h(K_1) \equiv h(K_2) \pmod m$.

5.4.5. We have $k_{11} = 137612044 \equiv 558 \pmod{4969}$ so that the files of the student with this social security number are assigned to location $h(k_{11}) = 558$. We find that $k_{12} = 505576452 \equiv 578 \pmod{4969}$, but location $h(k_{12}) = 578$ is taken, so we continue with the probing sequence $h_1(k_{12}) = h(k_{12}) + g(k_{12})$, where $g(k_{12}) \equiv 505576452 + 1 \equiv 424 \pmod{4967}$, so that $g(k_{12}) = 424$. We have $h_1(k_{12}) \equiv 578 + 424 = 1002 \pmod{4969}$. Since location 1002 is not occupied, we assign the files of the student with this social security number to location 1002. We find that $k_{13} = 157170996 \equiv 1526 \pmod{4969}$ but location 1526 is taken. We find that $g(k_{13}) \equiv 157170996 + 1 \equiv 216$. We probe locations $h_1(k_{13}) = h(k_{13}) + g(k_{13}) = 1742$ and $h_2(k_{13}) = h(k_{13}) + 2g(k_{13}) = 1958$, but they are taken. Finally, we probe once more and find that

we can place the files of this student in location $h_3(k_{13}) = h(k_{13}) + 3g(k_{13}) = 2174$. Finally, we see that $k_{14} = 131220418 \equiv 4 \pmod{4969}$ so that we can place the files of this last student in location 4 which is not already taken.

5.5. Check Digits

5.5.1. a. Since $1+1+1+1+1+1 \equiv 0 \pmod 2$, the check bit is 0.

b. Since $0+0+0+0+0+0 \equiv 0 \pmod 2$, the check bit is 0.

c. Since $1+0+1+0+1+0 \equiv 1 \pmod 2$, the check bit is 1.

d. Since $1+0+0+0+0+0 \equiv 1 \pmod 2$, the check bit is 1.

e. Since $1+1+1+1+1+1+1+1 \equiv 0 \pmod 2$, the check bit is 0.

f. Since $1+1+0+0+1+0+1+1 \equiv 1 \pmod 2$, the check bit is 1.

5.5.2. a. Since $1+1+1+1+1+1+1+1+1+1 \equiv 1 \pmod 2$, we know there is an error.

b. Since $0+1+0+1+0+1+0+1+0+1+0+1 \equiv 0 \pmod 2$, we don't know whether there is an error.

c. Since $1+1+1+1+0+1+0+1+0+1+0+1+0+1+0+1 \equiv 0 \pmod 2$, we don't know whether there is an error.

5.5.3. a. The sum of the known digits is even, so the keep the sum of all digits even, we must have $? = 0$.

b. The sum of the known digits is odd, so $? = 1$.

c. The sum of the known digits is even, so $? = 0$.

5.5.4. An error changes a 0 into a 1 or a one into a 0, so one error must change the sum of the digits (including the parity check bit) from even to odd. A second error will change the sum back to even, and so on.

5.5.5. a. We have $7 \cdot 1 + 3 \cdot 3 + 2 + 7 \cdot 9 + 3 \cdot 9 + 9 \equiv 7 \pmod{10}$, so the check digit is 7.

b. We have $7 \cdot 8 + 3 \cdot 0 + 5 + 7 \cdot 2 + 3 \cdot 3 + 7 \equiv 1 \pmod{10}$, so the check digit is 1.

c. We have $7 \cdot 6 + 3 \cdot 4 + 5 + 7 \cdot 1 + 3 \cdot 5 + 3 \equiv 4 \pmod{10}$, so the check digit is 4.

5.5.6. a. We apply the congruence to the first 6 digits and get $7 \cdot 3 + 3 \cdot 3 + 0 + 7 \cdot 0 + 3 \cdot 0 + 0 \equiv 4 \not\equiv 8 \pmod{10}$, so the number is invalid.

b. We have $7 \cdot 4 + 3 \cdot 5 + 0 + 7 \cdot 1 + 3 \cdot 8 + 2 \equiv 6 \not\equiv 4 \pmod{10}$, so the number is invalid.

c. We have $7 \cdot 1 + 3 \cdot 8 + 7 + 7 \cdot 3 + 3 \cdot 3 + 3 \equiv 1 \not\equiv 6 \pmod{10}$, so the number is invalid.

5.5.7. Here, transposition means that adjacent digits are in the wrong order. Suppose, first, that the first two digits, x_1 and x_2, or equivalently, the fourth and fifth digits are exchanged, and the error is not detected. Then $x_7 \equiv 7x_1 + 3x_2 + x_3 + 7x_4 + 3x_5 + x_6 \equiv 7x_2 + 3x_1 + x_3 + 7x_4 + 3x_5 + x_6 \pmod{10}$. It follows that $7x_1 + 3x_2 \equiv 7x_2 + 3x_1 \pmod{10}$ or $4x_1 \equiv 4x_2 \pmod{10}$. By Corollary 3.4.1, we see that $x_1 \equiv x_2 \pmod 5$. This is equivalent to $\mid x_1 - x_2 \mid = 5$, as x_1 and x_2 are single digits. Similarly, if the second and third (or fifth and sixth) digits are transposed, we find that $2x_2 \equiv 2x_3 \pmod{10}$, which again reduces to $x_2 \equiv x_3 \pmod 5$ by Corollary 3.4.1. Also, if the third and fourth digits are transposed, we find that $6x_3 \equiv 6x_4 \pmod{10}$ and $x_3 \equiv x_4 \pmod 5$, similarly as before. The reverse argument will complete the proof.

5.5.8. a. We have $7 \cdot 0 + 3 \cdot 0 + 9 \cdot 1 + 7 \cdot 8 + 3 \cdot 5 + 9 \cdot 4 + 7 \cdot 0 + 3 \cdot 3 \equiv 5 \pmod{10}$, so the check digit is 5.

b. Suppose x_i is replaced by y_i. Denote the check digit of this new number by y_9. Then $x_9 - y_9 \equiv ax_i - ay_i \pmod{10}$, where a is 7, 3 or 9. So if this replacement produces no change in the check digit, we have $a(x_i - y_i) \equiv 0 \pmod{10}$, or $x_i \equiv y_i \pmod{10}$, since 7, 3, and 9 have inverses modulo 10. Therefore, all single errors are detected.

c. If two digits x_i and x_j are switched, the difference in the check digits will be $a_i x_i + a_j x_j - a_i x_j - a_j x_i \equiv (a_i - a_j)(x_i - x_j) \pmod{10}$, where a_i and a_j are 3, 7, or 9. The transposition will go undetected if and only if $(a_i - a_j)(x_i - x_j) \equiv 0 \pmod{10}$. Since $a_i - a_j$ is even, if either $x_i \equiv x_j \pmod{5}$, or $a_i = a_j$, then the transposition will go undetected.

5.5.9. a. We have $x_{10} \equiv 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 + 3 \cdot 4 + 5 \cdot 5 + 4 \cdot 6 + 0 \cdot 7 + 0 \cdot 8 + 1 \cdot 9 \equiv 0 \pmod{11}$.

b. We have $x_{10} \equiv 0 \cdot 1 + 1 \cdot 2 + 9 \cdot 3 + 0 \cdot 4 + 8 \cdot 5 + 1 \cdot 6 + 0 \cdot 7 + 8 \cdot 8 + 2 \cdot 9 \equiv 3 \pmod{11}$.

c. We have $x_{10} \equiv 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 9 \cdot 6 + 9 \cdot 7 + 4 \cdot 8 + 0 \cdot 9 \equiv 4 \pmod{11}$.

d. We have $x_{10} \equiv 0 \cdot 1 + 0 \cdot 2 + 7 \cdot 3 + 0 \cdot 4 + 3 \cdot 5 + 8 \cdot 6 + 1 \cdot 7 + 3 \cdot 8 + 3 \cdot 9 \equiv 10 (= X) \pmod{11}$.

5.5.10. Since $\sum_{i=1}^{10} ix_i \equiv 0 \pmod{11}$, we have $\sum_{i=1}^{9} ix_i \equiv -10x_{10} \equiv -(1)x_{10} \equiv x_{10} \pmod{11}$, as desired.

5.5.11. a. We have $x_{10} \equiv 0 \cdot 1 + 3 \cdot 2 + 9 \cdot 3 + 4 \cdot 4 + 3 \cdot 5 + 8 \cdot 6 + 0 \cdot 7 + 4 \cdot 8 + 9 \cdot 9 \equiv 5 \pmod{11}$, which matches the check digit, so the ISBN is valid.

b. We have $x_{10} \equiv 1 \cdot 1 + 0 \cdot 2 + 9 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 1 \cdot 6 + 2 \cdot 7 + 2 \cdot 8 + 1 \cdot 9 \equiv 8 \pmod{11}$, so the ISBN is not valid.

c. We have $x_{10} \equiv 0 \cdot 1 + 8 \cdot 2 + 2 \cdot 3 + 1 \cdot 4 + 8 \cdot 5 + 0 \cdot 6 + 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \equiv 6 \pmod{11}$, so the ISBN is valid.

d. We have $x_{10} \equiv 0 \cdot 1 + 4 \cdot 2 + 0 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 + 0 \cdot 6 + 8 \cdot 7 + 7 \cdot 8 + 6 \cdot 9 \equiv 10 \pmod{11}$, so the ISBN is valid.

e. We have $x_{10} \equiv 9 \cdot 1 + 0 \cdot 2 + 6 \cdot 3 + 1 \cdot 4 + 9 \cdot 5 + 1 \cdot 6 + 7 \cdot 7 + 0 \cdot 8 + 5 \cdot 9 \equiv 0 \pmod{11}$, so the ISBN is not valid.

5.5.12. a. We have $1 \cdot 0 + 2 \cdot 1 + 3 \cdot 9 + 4 \cdot 8 + 5x + 6 \cdot 3 + 7 \cdot 8 + 8 \cdot 0 + 9 \cdot 4 \equiv 9 \pmod{11}$, or $5x \equiv 2 \pmod{11}$ which has solution $x \equiv 7 \pmod{11}$. So the missing digit is 7.

b. We have $1 \cdot 9 + 2 \cdot 1 + 3 \cdot 5 + 4 \cdot 5 + 5 \cdot 4 + 6 \cdot 2 + 7 \cdot 1 + 8 \cdot 2 + 9x \equiv 6 \pmod{11}$, or $9x \equiv 4 \pmod{11}$ which has solution $x \equiv 9 \pmod{11}$. So the missing digit is 9.

c. We have $1x + 2 \cdot 2 + 3 \cdot 6 + 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 5 + 7 \cdot 0 + 8 \cdot 7 + 9 \cdot 3 \equiv 10 \pmod{11}$, or $x \equiv 3 \pmod{11}$. So the missing digit is 7.

5.5.13. Computing the check digit for the incorrect ISBN yields $y_{10} \equiv 0 \cdot 1 + 0 \cdot 2 + 7 \cdot 3 + 2 \cdot 4 + 8 \cdot 5 + 9 \cdot 6 + 0 \cdot 7 + 9 \cdot 8 + 5 \cdot 9 \equiv 9 \pmod{11}$. Then using the notation in the text, we have $(j - k)(x_k - x_j) \equiv 9 \pmod{11}$. Without loss of generality, assume that $j > k$. There are a number of possibilities to check. Let's suppose first that $j - k = 3$ and $x_k - x_j = 3$. We search for two digits which are three places apart and such that the second digit is 3 more than the first. Finding none, we suppose $j - k = 1$ and $x_k - x_j = 9$. We search for two consecutive digits such that the second is 9 more than the first. We find that the 7th and 8th digits satisfy these conditions and conclude that the correct ISBN is 0-07-289905-0. Had we been unsuccessful, we might have tried $j - k = 9$ or we might have replaced 9 by -2 or 20 or some other integer congruent to 9 modulo 11.

5.5.14. Let $x_1, x_2, \ldots, x_{11}$ be the first eleven digits of the UPC. Then $x_{12} \equiv -3(x_1 + x_3 + x_5 + x_7 + x_9 + x_{11}) - (x_2 + x_4 + x_6 + x_8 + x_{10}) \pmod{10}$, where x_{12} is taken to be the least non-negative residue.

5.5.15. a. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(0+7+0+0+1+3) - (4+0+0+0+8) \equiv 5 \pmod{10}$. Since 5 is not the check digit for this UPC, the code is invalid.

b. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(3+1+0+0+0+8) - (1+0+0+1+3) \equiv 9 \pmod{10}$. Since 9 is the check digit for this UPC, the code is valid.

c. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(0+8+0+0+1+7) - (5+0+0+0+2) \equiv 5 \pmod{10}$. Since 5 is the check digit for this UPC, the code is valid.

d. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(2+6+0+0+1+9) - (2+5+0+1+7) \equiv 1 \pmod{10}$. Since 1 is not the check digit for this UPC, the code is invalid.

5.5.16. a. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(3+1+7+0+9+8) - (8+3+0+2+1) \equiv 2 \pmod{10}$. So 2 is the check digit for this UPC.

b. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(5+1+7+0+5+7) - (0+1+5+0+5) \equiv 4 \pmod{10}$. So 4 is the check digit for this UPC.

c. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(0+3+0+3+4+9) - (3+0+3+1+3) \equiv 3 \pmod{10}$. So 3 is the check digit for this UPC.

d. Using the congruence from Exercise 14, we compute $x_{12} \equiv -3(4+1+0+0+0+8) - (1+0+0+1+2) \equiv 7 \pmod{10}$. So 7 is the check digit for this UPC.

5.5.17. Let $x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}$ be a correct UPC. Suppose that when the product is scanned, the numbers are read as $y_1y_2y_3y_4y_5y_6y_7y_8y_9y_{10}y_{11}y_{12}$, where $x_i = y_i$ if $i \neq k$, but $x_k \neq y_k$, for some k. Then, from the congruence in Exercise 14, $0 \equiv x_{12} - y_{12} \equiv -3(x_1+x_3+x_5+x_7+x_9+x_{11}) - (x_2+x_4+x_6+x_8+x_{10}) + 3(y_1+y_3+y_5+y_7+y_9+y_{11}) + (y_2+y_4+y_6+y_8+y_{10}) \equiv a \cdot (y_k - x_k) \pmod{10}$, where $a = 3$ or 1 according as k is odd or even. In either case, $(a, 10) = 1$, so we can divide the congruence by a to obtain $(y_k - x_k) \equiv 0 \pmod{10}$, which contradicts the assumption that $x_k \neq y_k$. We conclude that this code will always detect a single error.

5.5.18. From the congruence in Exercise 14, we see that transposing any two digits in odd numbered places, or any two digits in even numbered place leaves the check digit unchanged. Therefore, this code cannot detect every transposition. However, if the transposition occurs between adjacent digits, then the check digit is changed by $2|x_j - x_{j+1}|$ which will be detected if $x_j - x_{j+1}$ is not divisible by 5.

5.5.19. a. Yes. If x_i is entered as y_i, then for both codewords to be valid, $x_i \equiv y_i \pmod{11}$. As x_i and y_i are single digits (less than 11), $x_i = y_i$.

b. No. We cannot detect any transpositions, as addition is commutative.

5.5.20. a. Solving the first congruence for x_{10} and substituting into the second gives $\sum_{i=1}^{10} ix_i \equiv \sum_{i=1}^{9} ix_i + 10\left(\sum_{i=1}^{9} -x_i\right) \equiv \sum_{i=1}^{9} ix_i + \left(\sum_{i=1}^{9} x_i\right) \equiv \sum_{i=1}^{9}(i+1)x_i \equiv 0 \pmod{11}$. Solving this last congruence for x_9 gives $\sum_{i=1}^{8}(i+1)x_i \equiv -(9+1)x_9 \equiv x_9 \pmod{11}$, as desired. We also have $x_{10} \equiv -\sum_{i=1}^{9} x_i \equiv -\sum_{i=1}^{8} + x_9 \equiv -\sum_{i=1}^{8} x_i - \sum_{i=1}^{8}(i+1)x_i \equiv -\sum_{i=1}^{8}(i+2)x_i \equiv -\sum_{i=1}^{8}(9-i)x_i$, as desired.

b. By part (a), we can freely choose the digits x_1 through x_8, and these determine x_9 and x_{10}. Since there are 10 choices for the first 8 digits, we have 10^8 valid codewords.

c. Suppose x_k is changed to y_k, and recompute the 9th and 10 digits using the formulae from part (a). Call the new 9th and 10th digits y_9 and y_{10}. Then we have $y_9 \equiv x_9 + (k+1)(y_k - x_k) \pmod{11}$ and $y_{10} \equiv x_{10} + (9-k)(y_k - x_k)$. We can solve this linear system for k and $y_k - x_k$ and thereby correct the error.

d. Suppose x_n and x_n are transposed, but the error goes undetected. Then from the first congruence in part (a), we have $(m+1)x_m + (n+1)x_n \equiv (m+1)x_n + (n+1)x_m \pmod{11}$. This reduces to $(x_m - x_n)(m-n) \equiv 0 \pmod{m}$, but since m, n, x_m, and x_n are all digits between 0 and 9, we must have $m = n$ or $x_m = x_n$.

5.5.21. a. $x_{10} \equiv 8\cdot 1 + 4\cdot 1 + 5\cdot 0 + 10\cdot 4 + 3\cdot 9 + 2\cdot 1 + 7\cdot 2 + 6\cdot 3 + 9\cdot 8 \equiv 9 \pmod{11}$. $x_{11} \equiv 6\cdot 1 + 7\cdot 1 + 8\cdot 0 + 9\cdot 4 + 4\cdot 9 + 5\cdot 1 + 6\cdot 2 + 7\cdot 3 + 8\cdot 8 + 9\cdot 9 \equiv 4 \pmod{11}$.

b. If x_i is misentered as y_i, then if the congruence defining x_{10} holds, we see that $ax_i \equiv ay_i \pmod{11}$ by setting the two definitions of x_{10} congruent. From this, it follows by Corollary 3.4.1 that $x_i \equiv y_i \pmod{11}$ and so $x_i = y_i$. If the last digit, x_{11} is misentered as y_{11}, then the congruence defining x_{11} will hold if and only if $x_{11} = y_{11}$.

c. Suppose that x_i is misentered as y_i and x_j is misentered as y_j, with $i < j < 10$. Suppose both of the congruences defining x_{10} and x_{11} hold. Then by setting the two versions of each congruence congruent to each other we obtain $ax_i + bx_j \equiv ay_i + by_j \pmod{11}$ and $cx_i + dx_j \equiv cy_i + dy_j \pmod{11}$ where $a \neq b$ and $c \neq d$. If it is the case that $ad - bc \not\equiv 0 \pmod{11}$, then the coefficient matrix is invertible and we can multiply both sides of this system of congruences by the inverse to obtain $x_i = y_i$ and $x_j = y_j$. Indeed, after (tediously) checking each possible choice of a, b, c, and d, we find that all the matrices are invertible modulo 11.

5.5.22. a. We have 4 linear congruences which may be solved for 4 of the variables, leaving 6 to be freely chosen. Therefore, there are 10^6 valid codewords.

b. If x_m and x_n are changed to y_m and y_n, respectively, then we can solve the 4 congruences for $m, n, (y_m - x_m)$, and $(y_n - x_n)$, as in Exercise 14. With this information, we can correct the errors.

c. If x_m and x_n are changed to y_m and y_n, let $d_m = (y_m - x_m)$ and $d_n = (y_n - x_n)$. The procedure in part (b) gives the system $d_m + d_n \equiv 7, md_m + n_d n \equiv 7, m^2 d_m + n^2 d_n \equiv 9$, and $m^3 d_m + n^3 d_n \equiv 2 \pmod{11}$. Solving the first congruence for d_m and substituting into the others gives us the system $7m - md_n + nd_n \equiv 7, 7m^2 - m^2 d_n + n^2 d_n \equiv 9$, and $7m^3 - m^3 d_n + n^3 d_n \equiv 2 \pmod{11}$. Solving the first of these, substituting into the others and simplifying gives us the system $n + m - mn \equiv 6$ and $n^2 + nm + m^2 - mn^2 - nm^2 \equiv 5 \pmod{11}$. We rewrite this last one as $n(n + m - mn) + m(n + m - nm) - mn \equiv 5 \pmod{11}$. Then using the first congruence we have $6n + 6m - mn \equiv 5 \pmod{11}$. We subtract the first congruence to get $5n + 5m \equiv 10 \pmod{11}$ or $n \equiv 2 - m \pmod{11}$. Then we may write $2 - m + m - m(2 - m) \equiv 6 \pmod{11}$, or $(m-1)^2 \equiv 5 \pmod{11}$. Trial and error gives us the solutions $m \equiv 5$ or $8 \pmod{11}$, which gives $n \equiv 8$ or $5 \pmod{11}$. The problem is symmetric in m and n so we need only consider $m \equiv 5, n \equiv 8 \pmod{11}$. Then $y_8 - x_8 \equiv d_8 \equiv 9 \pmod{11}$, and hence $x_8 = 9$ instead of 7. Also, $y_5 - x_5 \equiv d_5 \equiv 7 - d_8 \equiv 9 \pmod{11}$, so $x_5 = 0$ instead of 9. The correct codeword is 0204006910.

5.5.23. a. When we divide 00032781811224 by 7 we get a remainder of 1, so the check digit is $a_{15} = 1$.

b. When we divide 10238544122339 by 7 we get a remainder of 1, so the check digit is $a_{15} = 1$.

c. When we divide 00611133123278 by 7 we get a remainder of 6, so the check digit is $a_{15} = 6$.

5.5.24. a. When we take the first 14 digits 10228471103312 and divide by 7 we get a remainder of 0, so the check digit should be 0 and not 2 as printed. This is not a valid ticket number.

b. When we take the first 14 digits 00411371131124 and divide by 7 we get a remainder of 2, so the check digit should be 2 and not 0 as printed. This is not a valid ticket number.

c. When we take the first 14 digits 10026141300153 and divide by 7 we get a remainder of 6, so the check digit should be 6 and not 3 as printed. This is not a valid ticket number.

5.5.25. Suppose an undetectable error is made in the ith digit, so that the incorrect digit b is written in place of the correct digit a_i. Then we must have $a_1a_2\cdots a_{14} \equiv a_1a_2\cdots b\cdots a_{14} \pmod 7$ which reduces to $a_i10^i \equiv b10^i \pmod 7$. Since $(7,10)=1$, we can divide out the power of 10 and we have $a_i \equiv b \pmod 7$. Since $0 \le b \le 9$, the only undetectable errors are when we have one of the following substitutions: 0 for 7, 1 for 8, 2 for 9 or vice versa.

5.5.26. Suppose the digits a_i and a_{i+1} are transposed and the error is undetected. Then $a_1a_2\cdots a_{14} \equiv a_1a_2\cdots a_{i+1}a_i\cdots a_{14} \pmod{14}$, which reduces to $a_i10^i + a_{i+1}10^{i+1} \equiv a_{i+1}10^i + a_i10^{i+1} \pmod 7$. Since $(7,10)=1$, we can divide both sides by 10^i to get $a_i + 10a_{i+1} \equiv a_{i+1} + 10a_i \pmod 7$, which reduces to $2a_{i+1} \equiv 2a_i \pmod 7$. Since $(2,7)=1$, we can divide by 2 and get $a_i \equiv a_{i+1} \pmod 7$. So the only undetectable transpositions of adjacent digits are when we have one of the following substitutions: 0 for 7, 1 for 8, 2 for 9 or vice versa.

5.5.27. a. Since $3\cdot 0 + 4\cdot 3 + 5\cdot 1 + 6\cdot 7 + 7\cdot 8 + 8\cdot 4 + 9\cdot 7 = 210 \equiv 1 \pmod{11}$, the check digit is 1.

b. Since $3\cdot 0 + 4\cdot 4 + 5\cdot 2 + 6\cdot 3 + 7\cdot 5 + 8\cdot 5 + 9\cdot 5 = 164 \equiv 10 \pmod{11}$, the check digit is X.

c. Since $3\cdot 1 + 4\cdot 0 + 5\cdot 6 + 6\cdot 3 + 7\cdot 6 + 8\cdot 6 + 9\cdot 9 = 222 \equiv 2 \pmod{11}$, the check digit is 2.

d. Since $3\cdot 1 + 4\cdot 3 + 5\cdot 6 + 6\cdot 3 + 7\cdot 8 + 8\cdot 3 + 9\cdot 7 = 206 \equiv 8 \pmod{11}$, the check digit is 8.

5.5.28. Suppose one digit d_i is replaced by the digit b, and that this error is undetected. Then $3d_1 + 4d_2 + \cdots + 9d_7 \equiv 3d_1 + \cdots + (i+2)b + \cdots + 9d_7 \pmod{11}$, which reduces to $(i+2)d_i \equiv (i+2)b \pmod{11}$. Since $3 \le i+2 \le 9$, we have $(i+2, 11) = 1$ and so we can divide through by $i+2$ and get $d_i \equiv b \pmod{11}$, but since both d_i and b are digits from 0 to 9, they must be the same, so there must not have been an error. We conclude that all single-digit errors will be detected.

5.5.29. Suppose two consecutive digits are transposed and the error is undetected. Then $3d_1 + 4d_2 + \cdots + 9d_7 \equiv 3d_1 + \cdots + (i+2)d_{i+1} + (i+3)d_i + \cdots + 9d_7 \pmod{11}$, which reduces to $(i+2)d_i + (i+3)d_{i+1} \equiv (i+2)d_{i+1} + (i+3)d_i \pmod{11}$, which in turn simplifies to $d_{i+1} \equiv d_i \pmod{11}$. So no error in fact existed. We conclude that all single transpositions of consecutive digits are detectable. (In fact all single transpositions are detectable.)

CHAPTER 6

Some Special Congruences

6.1. Wilson's Theorem and Fermat's Little Theorem

6.1.1. Note that $10!+1 = 1(2\cdot 6)(3\cdot 4)(5\cdot 9)(7\cdot 8)10+1 = 1\cdot 12\cdot 12\cdot 45\cdot 56\cdot 10+1 \equiv 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot(-1)+1 \equiv 0 \pmod{11}$. Therefore 11 divides $10!+1$.

6.1.2. Note that $12!+1 = (1)(2\cdot 7)(3\cdot 9)(4\cdot 10)(5\cdot 8)(6\cdot 11)(12)+1 \equiv (1)(1)(1)(1)(1)(1)(-1)+1 \equiv 0 \pmod{13}$. Therefore 13 divides $12!+1$.

6.1.3. By Wilson's theorem, we have $18 \equiv 18! \equiv 16!(17)(18) \equiv 16!(-2)(-1) \equiv 16!2 \pmod{19}$. Since $(2,19) = 1$, we can divide both sides by 2 and get $9 \equiv 16! \pmod{19}$.

6.1.4. We compute $5!25! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 25! \equiv (-30)(-29)(-28)(-27)(-26)25! \equiv (-1)^5 30! \equiv (-1)^5(-1) \equiv 1 \pmod{31}$, by Wilson's theorem. So the remainder is 1.

6.1.5. We see that $8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13 \equiv 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 6! \equiv -1 \pmod{7}$, using Wilson's theorem for the last congruence.

6.1.6. We compute $7\cdot 8\cdot 9\cdot 15\cdot 16\cdot 17\cdot 23\cdot 24\cdot 25\cdot 43 \equiv 7\cdot 8\cdot 9\cdot 4\cdot 5\cdot 6\cdot 1\cdot 2\cdot 3\cdot 10 \equiv 10! \equiv -1 \pmod{11}$. So the remainder is 10.

6.1.7. Note that $437 = 19\cdot 23$. From Wilson's theorem we have $18! \equiv -1 \pmod{19}$ and $22 \equiv 22! \pmod{23}$. Then $22 \equiv 22! \equiv 18!(-4)(-3)(-2)(-1) \equiv 18!(1) \pmod{23}$ Hence, $18! \equiv 22 \pmod{23}$. Now applying the Chinese remainder theorem to the system $x \equiv -1 \pmod{19}, x \equiv 22 \pmod{23}$ yields $x \equiv 436 \equiv -1 \pmod{437}$.

6.1.8. Note that $1763 = 41\cdot 43$. By Wilson's theorem, $40! \equiv -1 \pmod{41}$. Further $-1 \equiv 42! \equiv 40!(-2)(-1) \equiv 40!2 \pmod{43}$. We multiply both sides by 22, which is an inverse for 2 modulo 43. This yields $40! \equiv -22 \equiv 21 \pmod{43}$. Applying the Chinese remainder theorem to the system $x \equiv 40! \equiv -1 \pmod{41}$ and $x \equiv 40! \equiv 21 \pmod{43}$ gives us $x \equiv 40! \equiv 1311 \pmod{1763}$.

6.1.9. By the Division algorithm, we have $100 = 6\cdot 16+4$. Then by Fermat's little theorem, $5^{100} \equiv 5^{6\cdot 16+4} \equiv (5^6)^1 6\cdot 5^4 \equiv 1^1 6\cdot 5^4 \equiv 25^2 \equiv 4^2 \equiv 16 \equiv 2 \pmod{7}$.

6.1.10. From Fermat's little theorem, we know that $6^{10} \equiv 1 \pmod{11}$. Then $6^{2000} \equiv (6^{10})^{200} \equiv 1^{200} \equiv 1 \pmod{11}$. Therefore the remainder is 1.

6.1.11. Since 999999999 is an odd multiple of 3, we know it is congruent to 3 modulo 6. So by Fermat's Little Theorem, we have $3^{999999999} \equiv 3^3 \equiv 27 \equiv -1 \pmod{7}$.

6.1.12. We have $2^{1000000} \equiv (2^{16})^{62500} \equiv 1^{62500} \equiv 1 \pmod{17}$.

6.1.13. We have $(3^5)^2 \equiv 243^2 \equiv 1^2 \equiv 1 \pmod{11^2}$.

6.1.14. We have $3^{100} \equiv (3^6)^{16} 3^4 \equiv 1^{16}\cdot 9\cdot 9 \equiv 2\cdot 2 \equiv 4 \pmod{7}$.

6.1.15. a. Multiply both sides of $7x \equiv 12 \pmod{17}$ by 7^{15} to obtain $7^{16}x \equiv 7^{15} \cdot 12 \pmod{17}$. Since $7^{16} \equiv 1 \pmod{17}$ this gives $x \equiv 7^{15} \cdot 12 \equiv (7^3)^5 \cdot 12 \equiv 343^5 \cdot 12 \equiv 3^5 \cdot 12 \equiv 243 \cdot 12 \equiv 5 \cdot 12 \equiv 60 \equiv 9 \pmod{17}$.

b. Multiply both sides of $4x \equiv 11 \pmod{19}$ by 4^{17} to obtain $4^{18}x \equiv 4^{17} \cdot 11 \pmod{19}$. Since $4^{18} \equiv 1 \pmod{19}$, this gives $x \equiv 4^{17} \cdot 11 \equiv (4^2)^8 \cdot 4 \cdot 11 \equiv (-3)^8 \cdot 4 \cdot 11 \equiv ((-3)^4)^2 \cdot 4 \cdot 11 \equiv 81^2 \cdot 44 \equiv 5^2 \cdot 44 \equiv 6 \cdot 6 \equiv 17 \pmod{19}$.

6.1.16. If n is composite, then n has a divisor d less than or equal to $\sqrt{n}$. Then $1 < n/d < n$ and the factors d and n/d both appear among the factors of $(n-1)! = 1 \cdot 2 \cdots (n-1)$, and so if $d \neq n/d$, then $n \mid (n-1)!$ If $d = n/d$, then $2d < n$, so $2d^2 \mid (n-1)!$. In either case $(n-1)! \equiv 0 \pmod{n}$.

6.1.17. Suppose that p is an odd prime. Then Wilson's theorem tells us that $(p-1)! \equiv -1 \pmod{p}$. Since $(p-1)! = (p-3)!(p-1)(p-2) \equiv (p-3)!(-1)(-2) \equiv 2 \cdot (p-3)! \pmod{p}$ this implies that $2 \cdot (p-3)! \equiv -1 \pmod{p}$.

6.1.18. Since $(3, n) = 1$, we have $n^2 \equiv 1 \pmod{3}$ by Fermat's Little Theorem, so $3 \mid (n^2 - 1)$. Since n is odd, $n = 2k + 1$ for some integer k. then $n^2 - 1 = 4(k^2 + k) = 8l$, since k^k and k have the same parity. Therefore $8 \mid n^2 - 1$. Since $(3, 8) = 1, 3 \cdot 8 = 24 \mid (n^2 - 1)$, so $n^2 \equiv 1 \pmod{24}$.

6.1.19. Since $(a, 35) = 1$, we have $(a, 7) = (a, 5) = 1$, so we may apply Fermat's little theorem to get $a^{12} - 1 \equiv (a^6)^2 - 1 \equiv 1^2 - 1 \equiv 0 \pmod{7}$, and $a^{12} - 1 \equiv (a^4)^3 - 1 \equiv 1^3 - 1 \equiv 0 \pmod{5}$. Since both 5 and 7 divide $a^{12} - 1$, then 35 must also divide it.

6.1.20. Note that $168 = 8 \cdot 3 \cdot 7$. Since $(a, 42) = 1$, a must be odd, so $a^6 \equiv 1 \pmod{8}$. By Fermat's Little Theorem, $a^6 \equiv (a^2)^3 \equiv 1 \pmod{3}$ and $a^6 \equiv 1 \pmod{7}$. Therefore a^6 and 1 are solutions to the system of congruences $x \equiv 1 \pmod{8}$, $x \equiv 1 \pmod{3}$, and $x \equiv 1 \pmod{7}$. Therefore $a^6 \equiv 1 \pmod{168}$. Hence 168 divides $a^6 - 1$.

6.1.21. When n is even, so is n^7, and when n is odd, so is n^7. It follows that $n^7 \equiv n \pmod{2}$. Furthermore, since $n^3 \equiv n \pmod{3}$, it follows that $n^7 = (n^3)^2 \cdot n \equiv n^2 \cdot n \equiv n^3 \equiv n \pmod{3}$. We also know by Fermat's little theorem that $n^7 \equiv n \pmod{7}$. since $42 = 2 \cdot 3 \cdot 7$, it follows that $n^7 \equiv n \pmod{42}$.

6.1.22. By Theorem 6.4 , we have $n^9 - n \equiv (n^3)^3 - n \equiv n^3 - n \equiv 0 \pmod{3}$, and $n^9 - n \equiv n^5 n^4 - n \equiv n^5 - n \equiv 0 \pmod{5}$. Since n^9 and n have the same parity, $n^9 - n \equiv 0 \pmod{2}$. By the Chinese remainder theorem, since both $n^9 - n$ and 0 are solutions to the system $x \equiv 0 \pmod{2}$, $x \equiv 0 \pmod{3}$, and $x \equiv 0 \pmod{5}$, we have $0 \equiv n^9 - n \pmod{2 \cdot 3 \cdot 5}$. Therefore 30 divides $n^9 - n$.

6.1.23. By Fermat's little theorem, $\sum_{k=1}^{p-1} k^{p-1} \equiv \sum_{k=1}^{p-1} 1 \equiv p - 1 \pmod{p}$.

6.1.24. For $k = 1, 2, \ldots, p-1$, we have, by Fermat's little theorem, that $k^p \equiv k \pmod{p}$. Then we have $1^p + 2^p + \cdots + (p-1)^p \equiv 1 + 2 + \cdots + (p-1) \equiv p(p-1)/2 \equiv 0 \pmod{p}$ since $p - 1$ is even.

6.1.25. By Fermat's little theorem we have $a \equiv a^p \equiv b^p \equiv b \pmod{p}$, hence $b = a + kp$ for some integer k. Then by the binomial theorem, $b^p = (a + kp)^p = a^p + \binom{p}{1} a^{p-1} kp + p^2 N$ where N is some integer. Then $b^p \equiv a^p + p^2 ak + p^2 N \equiv a^2 \pmod{p^2}$, as desired.

6.1.26. We find $r_2 = 4, r_3 = 64, r_4 \equiv 66 \pmod{689}$. Then $(3, 689) = 1, (63, 689) = 1$, but $(65, 689) = 13$ which is a factor of 689.

6.1.27. Using computational software, we find $r_2 = 4, r_3 = 64, r_4 \equiv 2114982 \pmod{7331117}, r_5 = 2937380 \pmod{7331117}, r_6 = 6924877 \pmod{7331117}, r_7 = 3828539 \pmod{7331117}$, and $r_8 = 4446618 \pmod{7331117}$. We have $(r_i - 1, 7331117) = 1$, for $i = 1, 2, \ldots, 7$, but $(r_8 - 1, 7331117) = 641$, so this is a factor of 7331117.

6.1.28. By Fermat's little theorem since $(p, q) = 1$ we know that $p^{q-1} \equiv \pmod{q}$. Hence $p^{q-1} + q^{p-1} \equiv 1 \pmod{q}$. similarly, by Fermat's little theorem since $(q, p) = 1$ we know that $q^{p-1} \equiv \pmod{p}$. Hence

$p^{q-1} + q^{p-1} \equiv 1 \pmod{p}$. It follows that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

6.1.29. Suppose that p is prime. Then by Fermat's little theorem for every integer $a, a^p \equiv a \pmod{p}$ and by Wilson's theorem $(p-1)! \equiv -1 \pmod{p}$ so that $a(p-1)! \equiv -a \pmod{p}$. It follows that $a^p + (p-1)!a \equiv a + (-a) \equiv 0 \pmod{p}$. Consequently $p \mid [a^p + (p-1)!a]$.

6.1.30. Note that $1^2 3^2 \cdots (p-4)^2 (p-2)^2 \equiv (-1)^{(p-1)/2} \cdot 1 \cdot (-1) \cdot 2 \cdot (-2) \cdots (p-4) \cdot (4-p) \cdot (p-2) \cdot (2-p) \equiv (-1)^{(p-1)/2} \cdot 1 \cdot (p-1) \cdot 2 \cdot (p-2) \cdots (p-4) \cdot 4 \cdot 2 \equiv (-1)^{(p-1)/2} \cdot (p-1)! \equiv (-1)^{(p-1)/2}(-1) \equiv (-1)^{(p-1)/2} \pmod{p}$, where we have used Wilson's theorem to replace $(p-1)!$ by -1 in the congruence.

6.1.31. Since $p-1 \equiv -1, p-2 \equiv -2, \ldots, (p-1)/2 \equiv (p-1)/2 \pmod{p}$, we have $((p-1)/2)!^2 \equiv -(p-1)! \equiv 1 \pmod{p}$, (since $p \equiv 3 \pmod{4}$ the minus signs work out.) If $x^2 \equiv 1 \pmod{p}$, then $p \mid x^2 - 1 = (x-1)(x+1)$, so $x \equiv \pm 1 \pmod{p}$.

6.1.32. a. We use $(-1)^r r! \equiv (-1)$ to show that $(p-r-1)! \equiv -1$. Then by Wilson's theorem, we have $-1 \equiv (p-1)! \equiv 1 \cdot 2 \cdots (p-(r-1))(p-r)(p-(r-1) \cdots (p-2)(p-1) \equiv (p-r-1)!(-r)(-(r-1)) \cdots (-2)(-1) \equiv (p-r-1)!(-1)^r r! \equiv (p-r-1)! \pmod{p}$.

b. Note that $(-1)^7 7! \equiv (-1)^9 9! \equiv 1 \pmod{71}$. Then by part (a) we have $(71-7-1)! \equiv 63! \equiv -1 \pmod{71}$, and $(71-9-1)! \equiv 61! \equiv -1 \pmod{71}$.

6.1.33. Suppose that $p \equiv 1 \pmod{4}$. Let $y = \pm[(p-1)/2]!$. Then $y^2 \equiv [(p-1)/2]!^2 \equiv [(p-1)/2]!^2 (-1)^{(p-1)/2} \equiv (1 \cdot 2 \cdot 3 \cdots (p-1)/2)(-1 \cdot (-2) \cdot (-3) \cdots (-(p-1)/2)) \equiv 1 \cdot 2 \cdot 3 \cdots (p-1)/2 \cdot (p+1)/2 \cdots (p-3)(p-2)(p-1) = p! \equiv -1 \pmod{p}$, where we have used Wilson's theorem. Now suppose that $x^2 \equiv -1 \pmod{p}$. Then $x^2 \equiv y^2 \pmod{p}$ where $y = [(p-1)/2]!$. Hence $(x^2 - y^2) = (x-y)(x+y) \pmod{p}$. It follows that $p \mid (x-y)$ or $p \mid (x+y)$ so that $x \equiv \pm \pmod{p}$.

6.1.34. We have $(p-k)!(k-1)! \equiv (-k)(-(k+1)) \cdots (-(p-1))(k-1)! \equiv (-1)^{p-k}(p-1)! \equiv (-1)^{p+1-k} \equiv (-1)^k \pmod{p}$, by Wilson's theorem, and where we have used the fact that $p+1$ is even.

6.1.35. If n is composite and $n \neq 4$, then Exercise 16 shows that $(n-1)!/n$ is an integer, so $[((n-1)!+1)/n - [(n-1)!/n]] = [(n-1)!/n + 1/n - (n-1)!/n] = [1/n] = 0$ and if $n = 4$, then the same expression is also equal to 0. But if n is prime, then by Wilson's Theorem $(n-1)! = Kn - 1$ for some integer K. So $[((n-1)!+1)/n - [(n-1)!/n]] = [(Kn - 1 + 1)/n - [(Kn-1)/n]] = [K - (K-1)] = 1$. Therefore, the sum increases by 1 exactly when n is prime, so it must be equivalent to $\pi(x)$.

6.1.36. If n is even, $n^4 + 4^n$ is even and greater than two so it is not prime. If n is odd, note that $n^4 + 4^n = n^4 + 2n^2 2^n + 2^{2n} - 2n^2 2^n = (n^2 + 2^n)^2 - (n \cdot 2^{(n+1)/2})^{2 = (n^2 + 2^n + n \cdot 2^{(n+1)/2})}(n^2 + 2^n - n \cdot 2^{(n+1)/2})$. It is easy to see that both of these factors are greater than one if $n > 1$. Hence $n^4 + 4^n$ is prime if and only if $n = 1$, so that $n = 1^4 + 4^1 = 5$.

6.1.37. Suppose that n and $n+2$ are twin primes. Then since n is prime by Wilson's theorem we know that $(n-1)! = -1 \pmod{n}$. Hence $4[(n-1)!+1] + n \equiv 4 \cdot 0 + n \equiv 0 \pmod{n}$. Also, since $n+2$ is prime by Wilson's theorem it follows that $(n+1)! \equiv -1 \pmod{n+2}$, so that $(n+1)n \cdot (n-1)! \equiv (-1)(-2)(n-1)! \equiv 2(n-1)! \equiv -1 \pmod{n+2}$. Hence $4[(n-1)!+1] + n \equiv 2(2 \cdot (n-1)!) + 4 + n \equiv 2 \cdot (-1) + 4 + n = n + 2 \equiv 0 \pmod{n+2}$. Since $(n, n+2) = 1$ it follows that $4[(n-1)!+1] + n \equiv 0 \pmod{n(n+2)}$.

6.1.38. Suppose n and $n+k$ are prime. Then, by Wilson's theorem $(n-1)! + 1 \equiv 0 \pmod{n}$ and by Exercise 34, $(n-1)!k! \equiv (-1)^{k+1} \pmod{n+k}$. Then $(k!)^2((n-1)!+1) + n(k!-1)(k-1)! \equiv (k!)^2 \cdot 0 + 0 \cdot (k!-1) \equiv 0 \pmod{n}$, and $(k!)^2((n-1)!+1) + n(k!-1)(k-1)! \equiv (-1)^{k+1}(k!) + (k!)^2 + (-k)(k!-1)(k-1)! \equiv (-1)^{k+1}(k!) + (k!)^2 - k!(k!) + k! \equiv -k! + k! \equiv 0 \pmod{n+k}$, where we have used the fact that $k+1$ must be odd. By the Chinese Remainder Theorem, there is a unique solution to this system modulo $n(n+k)$, therefore, $(k!)^2((n-1)!+1) + n(k!-1)(k-1)! \equiv 0 \pmod{n(n+k)}$. The converse is false. $n = 9, k = 8$ provides a counterexample.

6.1.39. We have $1 \cdot 2 \cdots (p-1) \equiv (p+1)(p+2)\cdots(2p-1) \pmod{p}$. Each factor is prime to p, so $1 \equiv ((p+1)(p+2)\cdots(2p-1))/(1\cdot 2\cdots(p-1)) \pmod{p}$. Thus $2 \equiv ((p+1)(p+2)\cdots(2p-1)2p)/(1\cdot 2\cdots(p-1)p) \equiv \binom{2p}{p} \pmod{p}$.

6.1.40. We have $(a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^k b^{p-k} \equiv a^0 b^p + 0 + 0 + \cdots + a^p b^0 = b^p + a^p$ since $\binom{p}{k} \equiv 0 \pmod{p}$ when $1 \le k \le p-1$.

6.1.41. We first note that $1^p \equiv 1 \pmod{p}$. Now suppose that $a^p \equiv a \pmod{p}$. then by Exercise 40 we see that $(a+1)^p \equiv a^p + 1 \pmod{p}$. But by the inductive hypothesis $a^p \equiv a \pmod{p}$ we see that $a^p + 1 \equiv a + 1 \pmod{p}$. Hence $(a+1)^p \equiv a+1 \pmod{p}$. This completes the inductive step of the proof.

6.1.42. Let x be an integer less than and relatively prime to m. Then x has an inverse $\overline{x}$, which is also relatively prime to m. If $x \neq \overline{x}$, then both appear in the product, so we group them together and have $x\overline{x} \equiv 1 \pmod{m}$, and they contribute nothing to the product. If $x = \overline{x}$, then we have $x^2 \equiv 1 \pmod{m}$ and $(-x)^2 \equiv 1 \pmod{m}$. Then x and $-x$ appear in the product, so we group them together and get a factor of -1 in the product for every two solutions to $x^2 \equiv 1 \pmod{m}$. The numbers of such solutions are given by Exercise 32 of Section 4.3.

6.1.43. a. If $c < 26$ then c cards are put into the deck above the card, so it ends up in the $2c$th position and $2c < 52$, so $b = 2c$. If $c \ge 26$ then $c-1$ cards are put into the deck above the card, but 26 cards are taken away above it, so it ends up in the $b = (c-26+c-1)$th place. Then $b = 2c - 25 \equiv 2c \pmod{53}$.

b. 52.

6.1.44. We compute $q_p(ab) - q_p(a) - q_p(b) = ((ab)^{p-1} - 1)/p - (a^{p-1} - 1)/p - (b^{p-1} - 1)/p = (a^{p-1} - 1)(b^{p-1} - 1)/p \equiv a^{p-1} - 1) \cdot 0 \equiv 0 \pmod{p}$, as desired.

6.1.45. Assume without loss of generality that $a_p \equiv b_p \equiv 0 \pmod{p}$. Then, by Wilson's theorem, $a_1 a_2 \cdots a_{p-1} \equiv b_1 b_2 \cdots b_{p-1} \equiv -1 \pmod{p}$. Then $a_1 b_1 \cdots a_{p-1} b_{p-1} \equiv (-1)^2 \equiv 1 \pmod{p}$. If the set were a complete system, the last product would be $\equiv -1 \pmod{p}$.

6.1.46. If n is even, the proposition is clear. If n is odd and $n \mid 2^n - 1$, then let p be the smallest prime dividing n. Then $(n, p-1) = 1$ and there exist integers a and b such that $an + b(p-1) = 1$. Since $2^n \equiv 1 \pmod{n}$, we have $2^n \equiv 1 \pmod{p}$. Then $2^{an} \equiv 1 \pmod{p}$. By Fermat's little theorem, $2^{b(p-1)} \equiv 1^b \equiv 1 \pmod{p}$. Multiplying these last two congruences gives us $2 \equiv 2^{an+b(p-1)} \equiv 1 \cdot 1 \pmod{p}$, a contradiction.

6.1.47. The basis step is omitted. Assume $(p-1)^{p^{k-1}} \equiv -1 \pmod{p^k}$. Then, $(p-1)^{p^k} \equiv ((p-1)^{p^{k-1}})^p \equiv (-1 + mp^k)^p \equiv -1 + \binom{p}{1} mp^k + \cdots + (mp^k)^p \equiv -1 \pmod{p^{k+1}}$, where we have used the fact that $p \mid \binom{p}{j}$ for $j \neq 0$ or p.

6.1.48. We need to show that for $p > 5$, $(p-1)! + 1$ is not a power of a prime. Suppose $(p-1)! + 1 = q^k$ for some prime q and positive integer k. By Wilson's theorem, $p \mid (p-1)! + 1 = q^k$, so we must have $q = p$, that is, $(p-1)! + 1 = p^k$. From this we have $p^k = (p-1)! + 1 < (p-1)^{p-1} < p^{p-1}$, so $k < p-1$. Now since p is a prime greater than 5, $p-1$ is a composite number greater than 4, so by Exercise 16, we have $0 \equiv (p-2)! \pmod{p-1}$. Also, we have $(p-1)! = p^k - 1 = (p-1)(p^{k-1} + p^{k-2} + \cdots + p + 1)$, so $0 \equiv (p-2)! = p^{k-1} + p^{k-2} + \cdots + p + 1 \equiv 1 + 1 + \cdots + 1 \equiv k \pmod{p-1}$. So we have $k < p-1$ and $p-1 \mid k$, so $k = 0$, which is impossible. Therefore $(p-1)! + 1$ has at least two distinct prime divisors.

6.2. Pseudoprimes

6.2.1. We find that $3^{90} = (3^4)^{22} \cdot 3^2 = 81^4 \cdot 9 \equiv (-10) \cdot 9 = -90 \equiv 1 \pmod{91}$. Hence 91 is a pseudoprime modulo 3.

6.2.2. Note that $17^4 \equiv 19^2 \equiv 1 \pmod{45}$. Then, $17^{45} \equiv 17^{4\cdot 11}17 \equiv 1^{11}17 \equiv 17 \pmod{45}$, and $1945 \equiv 19^{2\cdot 22}19 \equiv 1^{22}19 \equiv 19 \pmod{45}$. So 45 is a pseudoprime to the bases 17 and 19.

6.2.3. Note that $2^{262} \equiv 2 \pmod{161038}$. Then $2^{161038} \equiv 2^{262\cdot 614+170} \equiv 2^{614+170} \equiv 2 \pmod{161038}$.

6.2.4. Suppose that n is an odd composite integer. Then $1^n \equiv 1 \pmod{n}$ and $(-1)^n = -1 \pmod{n}$. Hence n is a pseudoprime to the bases 1 and -1.

6.2.5. From the Binomial Theorem, $(n-a)^n \equiv (-a)^n \equiv -(a^n) \equiv -a \equiv (n-a) \pmod{n}$, where we used $a^n \equiv a \pmod{n}$.

6.2.6. Since $n-1 = a^2(a^{2p-2}-1)/(a^2-1)$ and $(a^2)^{p-1} \equiv 1 \pmod{p}$, we get $n-1 \equiv 0 \pmod{n}$, since $p \nmid (a^2-1)$. Writing $n-1 = a^2(1+a^2+\cdots+(a^2)^{p-2})$, we get $n-1 \equiv 0 \pmod{2}$ since if a is odd, the sum has an even number of odd terms. So $2p \mid (n-1)$. Now $a^{2p}-1 = n(a^2-1) \equiv 0 \pmod{n}$, so $a^{n-1} \equiv a^{2pk} \equiv 1^k \equiv 1 \pmod{n}$, where k is an integer.

6.2.7. Raise the congruence $2^{2^n} \equiv -1 \pmod{F_n}$ to the 2^{2^n-n}th power.

6.2.8. Note that $p \mid 2^{p-1}-1$ by Fermat's Theorem. Let $k = (2^{p-1}-1)/p$. Then we have $2^p \equiv 1 \pmod{2^p-1}$. We raise both sides to the k power to get $2^{2^{p-1}-1} \equiv 1^k \equiv 1 \pmod{2^p-1}$. Squaring both sides gives us $2^{2^p-2} \equiv 1^k \equiv 1 \pmod{2^p-1}$, and multiplying both sides by 2 gives the result.

6.2.9. Suppose that n is a pseudoprime to the bases a and b. Then $b^n \equiv b \pmod{n}$ and $a_n \equiv a \pmod{n}$. It follows that $(ab)^n \equiv a^n b^n \equiv ab \pmod{n}$. Hence n is a pseudoprime to the base ab.

6.2.10. We have $1 \equiv \overline{a^n}a^n \equiv \overline{a}^n a^n \equiv \overline{a}^n a \pmod{n}$, since n is a pseudoprime to the base a. But then $\overline{a} \equiv \overline{a}^n \pmod{n}$, so n is also a pseudoprime to the base $\overline{a}$.

6.2.11. a. If $(ab)^{n-1} \equiv 1 \pmod{n}$, then, $1 \equiv a^{n-1}b^{n-1} \equiv 1 \cdot b^n \pmod{n}$ which implies that n is a pseudoprime to the base b.

b. Let $a_1, a_2, \ldots, a_r$ be the bases to which n is a pseudoprime and for which $(a_i, n) = 1$ for each i. Then, by part (a), we know that, for each i, n is not a pseudoprime to the base ba_i. Thus, we have $2r$ different elements relatively prime to n. Then by the definition of $\phi(n)$, we have $r \leq \phi(n)/2$.

6.2.12. We have $25-1 = 2^3 \cdot 3$. First note that $7^{2^2\cdot 3} = 7^{12} = (7^2)^6 \equiv (-1)^6 = 1 \pmod{25}$. Next note that $7^{2\cdot 3} = 7^6 = (7^2)^3 \equiv (-1)^3 = -1 \pmod{25}$. Hence 25 is a strong pseudoprime to the base 7.

6.2.13. From $2^{18} \equiv 1 \pmod{1387}$ we get $2^{1387} \equiv 2 \pmod{1387}$ so 1387 is a pseudoprime. But $1387-1 = 2\cdot 693$ and $2^{693} \equiv 512 \pmod{1387}$, which is all that must be checked, since $s = 1$. Thus 1387 fails Miller's test and hence is not a strong pseudoprime.

6.2.14. For $n = 1373653$, we have $n-1 = 2^2 343413$, and we have $2^{343413} \equiv 890592 \pmod{1373653}$ but $2^{2\cdot 343413} \equiv 1 \pmod{1373653}$. So n passes Miller's test to the base 2, and so n is a strong pseudoprime to the base 2. Further we have $3^{343413} \equiv -1 \pmod{1373653}$, so n passes Miller's test to the base 3, and so n is a strong pseudoprime to the base 3.

6.2.15. $25326001 = 2^4 1582875 = 2^s t$ and with this value of t, $2^t \equiv -1 \pmod{25326001}$, $3^t \equiv -1 \pmod{25326001}$, and $5^t \equiv 1 \pmod{25326001}$.

6.2.16. a. Since $(7-1) = 6 \mid (2821-1) = 2820$, $(13-1) = 12 \mid (2821-1) = 2820$, $(31-1) = 30 \mid (2821-1) = 2820$. Theorem 6.7 shows that 2821 is a Carmichael number.

b. Since $(5-1) = 4 \mid (10585-1) = 10584$, $(29-1) = 28 \mid (10585-1) = 10584$, and $(73-1) = 72 \mid (10585-1) = 10584$. Theorem 6.7 shows that 10585 is a Carmichael number.

c. Since $(13-1)=12 \mid (29341-1)=29340, (37-1)=36 \mid (29341-1)=29340$, and $(61-1)=60 \mid (29341-1)=29340$. Theorem 6.7 shows that 29341 is a Carmichael number.

d. Since $(13-1)=12 \mid (314821-1)=314820, (61-1)=60 \mid (314821-1)=314820$, and $(397-1)=396 \mid (314821-1)=314820$. Theorem 6.7 shows that 314820 is a Carmichael number.

e. Since $(5-1)=4 \mid (278545-1)=278544, (17-1)=16 \mid (278545-1)=278544, (29-1)=28 \mid (278545-1)=278544$, and $(113-1)=112 \mid (278545-1)=278544$ Theorem 6.7 shows that 278544 is a Carmichael number.

f. Since $(7-1)=6 \mid (172081-1)=172080, (13-1)=12 \mid (172081-1)=172080, (31-1)=30 \mid (172081-1)=172080$, and $(61-1)=60 \mid (172081-1)=172080$, Theorem 6.7 shows that 172081 is a Carmichael number.

g. Since $(43-1)=42 \mid (564651361-1)=564651360, (3361-1)=3360 \mid (564651361-1)=564651360$, and $(3907-1)=3096 \mid (564651361-1)=564651360$, we see that 564651361 is a Carmichael number.

6.2.17. Suppose $c=7\cdot 23\cdot q$, with q and odd prime, is a Carmichael number. Then by Theorem 6.7 we must have $(7-1)|(c-1)$, so $c=\equiv 7\cdot 23\cdot q\equiv 1 \pmod 6$. Solving this yields $q\equiv 5 \pmod 6$. Also, we must have $(23-1)|(c-1)$, so $c=\equiv 7\cdot 23\cdot q\equiv 1 \pmod{22}$. Solving this yields $q\equiv 19 \pmod{22}$ If we apply the Chinese remainder theorem to these two congruences we obtain $q\equiv 41 \pmod{66}$, that is $q=41+66k$. Then we must have $(q-1)|(c-1)$, which is $(40+66k)|(7\cdot 23\cdot(41+66k)-1$. So there is an integer m such that $m(40+66k)=6600+10626k=160+6440+10626k=160+161(40+66k)$. Therefore 160 must be a multiple of $40+66k$, which happens only when $k=$. Therefore $q=41$ is the only such prime.

6.2.18. a. Suppose that $6m+1, 12m+1$, and $18m+1$ are primes. Let $N=(6m+1)(12m+1)(18m+1)$. It follows that $N-1=6\cdot 12\cdot 18m^3+(6\cdot 12+6\cdot 18+12\cdot 18)m^2+(6+12+18)m+1=1296m^3+396m^2+36m$. We see that $[(6m+1)-1]=6m \mid (N-1)=6m(216m^2+66m+6)$, $[(12m+1)-1]=12m \mid (N-1)=12m(108m^2+33m+3)$, and $[(18m+1)-1]=18m \mid (N-1)=18m(72m^2+22m+2)$. Hence N is a Carmichael number.

b. We have $7=6\cdot 1+1, 13=12\cdot 1+1$ and $19=18\cdot 1+1$, so by part (a), $7\cdot 13\cdot 19=1729$ is a Carmichael number. We have $37=6\cdot 6+1, 73=12\cdot 6+1$ and $109=18\cdot 6+1$, so by part (a), $37\cdot 73\cdot 109=294409$ is a Carmichael number. We have $211=6\cdot 35+1, 421=12\cdot 35+1$ and $631=18\cdot 35+1$, so by part (a), $211\cdot 421\cdot 631=56052361$ is a Carmichael number. We have $271=6\cdot 45+1, 541=12\cdot 45+1$ and $811=18\cdot 45+1$, so by part (a), $271\cdot 541\cdot 811=118901521$ is a Carmichael number. We have $307=6\cdot 51+1, 613=12\cdot 51+1$ and $919=18\cdot 51+1$, so by part (a), $307\cdot 613\cdot 919=172947529$ is a Carmichael number.

6.2.19. We have $321197185-1=321197184=4\cdot 80299296=18\cdot 17844288=22\cdot 14599872=28\cdot 11471328=36\cdot 8922144=136\cdot 2361744$, so $p-1|321197185-1$ for every prime p which divides 321197185. Therefore, by Theorem 6.7, 321197185 is a Carmichael number.

6.2.20. Let n be a Carmichael number and suppose there is a prime p such that $n=p^t m$, with $(p,m)=1$ and $t\geq 2$. Let $x=b$ be a solution to the system of congruences $x\equiv p^{t-1}+1 \pmod{p^t}, x\equiv 1 \pmod m$. Then since $(b,p)=1$ and $(b,m)=1$, we have that $(b,n)=1$. If it were the case that $b\equiv 1 \pmod n$, then we would have $b\equiv 1 \pmod{p^t}$, a contradiction. Therefore $b\not\equiv 1 \pmod n$. On the other hand, note that $b^n\equiv(p^{t-1}+1)^n\equiv(p^{t-1})^n+n(p^{t-1})^{n-1}+\cdots+np^{t-1}+1\equiv 1 \pmod{p^t}$, by the binomial theorem and the fact that $p\mid n$, so p^t divides every term but the last. Also $b^n\equiv 1 \pmod m$, so that by the Chinese remainder theorem, we must have $b^n\equiv 1 \pmod n$. Since $(b,n)=1$ and $b\not\equiv 1\equiv b^n \pmod n$, n is not a Carmichael number. Therefore n must be squarefree.

6.2.21. We can assume that $b<n$. Then b has fewer than $\log_2 n$ bits. Also, $t<n$ so it has fewer than $\log_2 n$ bits. It takes at most $\log_2 n$ multiplications to calculate b^{2^s} so it takes $O(\log_2 n)$ multiplications to calculate $b^{2^{\log_2 t}}=b^t$. Each multiplication is of two $\log_2 n$ bit numbers, and so takes $O((\log_2 n)^2)$ operations.

So all together we have $O((\log_2 n)^3)$ operations.

6.3. Euler's Theorem

6.3.1. a. The set $1, 5$ is a reduced residue set modulo 6.

b. The set $1, 2, 4, 5, 7, 8$ is a reduced residue set modulo 9.

c. The set $1, 3, 7, 9$ is a reduced residue set modulo 10.

d. The set $1, 3, 5, 9, 11, 13$ is a reduced residue set modulo 14.

e. The set $1, 3, 5, 7, 9, 11, 13, 15$ is a reduced residue set modulo 16.

f. The set $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16$ is a reduced residue set modulo 17.

6.3.2. Since the integers relatively prime to 2^m are the odd integers, $1, 3, 5, \ldots, 2^{m-1}$ is a reduced residue system modulo 2^m.

6.3.3. If $(a, m) = 1$, then $(-a, m) = 1$, so $-c_i$ must appear among the c_j. Also $c_i \not\equiv -c_i \pmod{m}$, else $2c_i \equiv 0 \pmod{m}$ and so $(c_i, m) \neq 1$.

6.3.4. We have $(a-1)(1 + \cdots + a^{\phi(m)-1}) = a^{\phi(m)} - 1 \equiv 0 \pmod{m}$, since $(a, m) = 1$. But $(a-1, m) = 1$ so $m \mid (1 + \cdots + a^{\phi(m)-1})$, as desired.

6.3.5. Since $\phi(10) = 4$, we have, by Euler's theorem, $3^{1000} \equiv (3^4)^{250} \equiv 1^2 50 \equiv 1 \pmod{10}$. Therefore the last decimal digit of 3^{1000} is 1.

6.3.6. Since $\phi(10) = 4$ and $999999 = 4(249999) + 3$, we have, by Euler's theorem, $7^{999999} \equiv (7^4)^{249999} 7^3 \equiv 1^{249999} 7^3 \equiv 49 \cdot 7 \equiv 9 \cdot 7 \equiv 63 \equiv 3 \pmod{10}$. Therefore the last decimal digit is 3.

6.3.7. By Euler's theorem $3^{\phi}(35) = 3^{28} \equiv 1 \pmod{35}$. Since $100000 = 2857 \cdot 35 + 5$, it follows that $3^{100000} \equiv (3^{28})^{2857} \cdot 3^5 \equiv 1 \cdot 3^5 = 81 \equiv 11 \pmod{35}$.

6.3.8. By Fermat's little theorem, $a^7 \equiv a \pmod{7}$, so we need to show that $a^7 \equiv a \pmod{9}$. If $9 \mid a$ this reduces to $0 \equiv 0 \pmod{9}$ which is true. If $3 \nmid a$ then $(a, 9) = 1$. Then, since $\phi(9) = 6$, by Euler's theorem, we have $a^6 \equiv 1 \pmod{9}$ or $a^7 \equiv a \pmod{9}$. Therefore $a^7 \equiv a \pmod{63}$.

6.3.9. Since $a^2 \equiv 1 \pmod{8}$ whenever a is odd, it follows that $a^{12} \equiv 1 \pmod{8}$ whenever $(a, 32760) = 1$. Euler's theorem tells us that $a^{\phi(9)} = a^6 \equiv 1 \pmod{9}$ whenever $(a, 9) = 1$, so that $a^{12} = (a^6)^2 \equiv 1 \pmod{9}$ whenever $(a, 32760) = 1$. Furthermore, Fermat's little theorem tells us that $a^4 \equiv 1 \pmod{5}$ whenever $(a, 5) = 1, a^6 \equiv 1 \pmod{7}$ whenever $(a, 7) = 1$, and $a^{12} \equiv 1 \pmod{13}$ whenever $(a, 13) = 1$. It follows that $a^{12} \equiv (a^4)^3 \equiv 1 \pmod{5}, a^{12} \equiv (a^6)^2 \equiv 1 \pmod{7}$, and $a^{12} \equiv 1 \pmod{13}$ whenever $(a, 32760) = 1$. Since $32760 = 2^3 3^2 \cdot 5 \cdot 7 \cdot 13$ and the moduli $8, 9, 5, 7$, and 13 are pairwise relatively prime, we see that $a^{12} \equiv 1 \pmod{32760}$.

6.3.10. Suppose that a and b are relatively prime positive integers. Then by Euler's theorem $a^{\phi(b)} \equiv 1 \pmod{b}$ and $b^{\phi(a)} \equiv 1 \pmod{a}$. Since $a^{\phi(b)} \equiv 0 \pmod{a}$ and $b^{\phi(a)} \equiv 0 \pmod{b}$ it follows that $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{a}$ and $\pmod{b}$. By the Chinese remainder theorem, since a and b are relatively prime it follows that $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{ab}$.

6.3.11. a. We multiply both sides of the congruence $5x \equiv 3 \pmod{14}$ by $5^{\phi(14)-1} = 5^5$ to obtain $5^6 x \equiv 5^5 \cdot 3 \pmod{14}$. Since $5^6 \equiv 1 \pmod{14}$ by Euler's theorem, it follows that $x \equiv 5^5 \cdot 3 \equiv (5^2)^2 \cdot 5 \cdot 3 \equiv 11^2 \cdot 15 \equiv 9 \cdot 1 \equiv 9 \pmod{14}$.

b. We multiply both sides of the congruence $4x \equiv 7 \pmod{15}$ by $4^{\phi(15)-1} = 4^7$ to obtain $4^8 x \equiv 4^8 \cdot 7 \pmod{15}$. since $4^8 \equiv 1 \pmod{15}$ by Euler's theorem, it follows that $x \equiv 4^7 \cdot 7 \equiv (4^2)^3 \cdot 4 \cdot 7 \equiv 1 \cdot 28 \equiv 13 \pmod{15}$.

c. We multiply both sides of the congruence $3x \equiv 5 \pmod{16}$ by $3^{\phi(16)-1} = 3^7$ to obtain $3^8 x \equiv 3^7 \cdot 5 \pmod{16}$. since $3^8 \equiv 1 \pmod{16}$ by Euler's theorem, it follows that $x \equiv 3^7 \cdot 5 \equiv 3^4 \cdot 3^3 \cdot 5 \equiv 1 \cdot 27 \cdot 5 \equiv 11 \cdot 5 \equiv 7 \pmod{16}$.

6.3.12. Since the m_j are pairwise relatively prime, we have $(M_j, m_j) = 1$ for all j, and $(M_i, m_j) = m_j$ for $i \neq j$. Then, by Euler's theorem, we have $M_j^{\phi(m_j)} \equiv 1 \pmod{m_j}$. Therefore, for any j we have $x \equiv a_1 M_1^{\phi(m_1)} + \cdots + a_j M_j^{\phi(m_j)} + \cdots + a_r M_r^{\phi(m_r)} \equiv 0 + 0 + \cdots + a_j(1) + 0 + \cdots + 0 \equiv a_j \pmod{m_j}$. Therefore, x satisfies the system, and by the Chinese remainder Theorem, it must be the unique solution modulo M.

6.3.13. a. We have $x \equiv 4 \cdot 17^{10} + 3 \cdot 11^{16} \equiv 27 \pmod{187}$.

b. We have $x \equiv 1 \cdot 15^1 + 2 \cdot 10^2 + 3 \cdot 6^4 \equiv 23 \pmod{30}$.

c. We have $x \equiv 0 \cdot 105^1 + 0 \cdot 70^2 + 1 \cdot 42^4 + 6 \cdot 30^6 \equiv 6 \pmod{210}$.

d. We have $x \equiv 2 \cdot 50388^{10} + 3 \cdot 46189^4 + 4 \cdot 42636^{12} + 5 \cdot 32604^1 6 + 6 \cdot 29172^{18} \equiv 150999 \pmod{554268}$.

6.3.14. We have $M = 2310, M_1 = 1155, M_2 = 770, M_3 = 462, M_4 = 330$, and $M_5 = 210$. Then $x \equiv 1 \cdot 1155^1 + 2 \cdot 770^2 + 3 \cdot 462^4 + 4 \cdot 330^6 + 5 \cdot 210^{10} \equiv 1523 \pmod{2310}$.

6.3.15. We have $\phi(10) = 4$, so $7^4 \equiv 1 \pmod{10}$ and $7^{1000} \equiv (7^4)^{250} \equiv 1^{250} \equiv 1 \pmod{10}$.

6.3.16. We have $\phi(16) = 8$ and $5^{1000000} \equiv 5^{8 \cdot 125000} \equiv 1^{125000} \equiv 1 \pmod{16}$. Therefore the last digit is 1, in hexadecimal notation.

6.3.17. We note that $\phi(p) = p - 1$ if p is prime, so $\phi(13) = 12, \phi(17) = 16$, and $\phi(19) = 18$. Since the integers relatively prime to 16 are the odd integers, we see that $\phi(16) = 8$. The integers relatively prime to 14 are the odd integers not divisible by 7. We see that $\phi(14) = 6$. The integers relatively prime to 15 are those not divisible by either 3 or 5. we see that $\phi(15) = 8$. The integers relatively prime to 18 are the odd integers not divisible by 3. It follows that $\phi(18) = 6$. the integers relatively prime to 20 are the odd integers not divisible by 5. It follows that $\phi(20) = 8$.

6.3.18. Let a be an integer with $(a, 10) = 1$, and let $n = 9k\phi(a)$, where k is a positive integer. By Euler's theorem, $10^n \equiv (10^{\phi(a)})^{9k} \equiv 1 \pmod{a}$. By Fermat's Theorem, $10^n - 1 \equiv ((9+1)^{9k\phi(a)}) - 1 \equiv (9^{9k\phi(a)} + \cdots + \binom{9}{1}9 + 1) - 1 \equiv 0 \pmod{9^2}$. Then $(10^n - 1)$ is divisible by 81, and by a, and so $(10^n - 1)/9$ is divisible by 9 and by a.

6.3.19. If $(a, b) = 1$ and $(a, b-1) = 1$ then $a \mid (b^{k\phi(a)} - 1)/(b-1)$ which is a base b repunit. If $(a, b-1) = d > 1$, then d divides any repunit of length $k(b-1)$, and $(a/d) \mid (b^{k\phi(a/d)} - 1)/(b-1)$ and these sets intersect infinitely often.

6.3.20. Let $m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. If $(a, p_i) = 1$ for some integer i then by Euler's theorem we see that $a^{\phi(p_i^{a_i})} \equiv 1 \pmod{p_i^{a_i}}$. Since $\phi(p_i^{a_i}) \mid \phi(m)$ it follows $p_i^{a_i} \mid (a^{\phi(m)} - 1)$ if $(a, p_i) = 1$.

Since for $i = 1, 2, \ldots, m$ we know that $p_i^{a_i - 1} \mid \phi(m)$, it follows that $p_i^{a_i - 1} \mid (m - \phi(m))$. since $m - \phi(m) \geq 1$, it follows that $m - \phi(m) \geq p_i^{a_i - 1} \geq a_i$ (since $q^{a-1} \geq a$ for positive integers q and a with $q \geq 2$). Hence, if $(a, p_i) > 1$, so that $p_i \mid a$, we have $p_i^{a_i} \mid p_i^{m - \phi(m)}$, which implies that $p_i^{a_i} \mid a^{m - \phi(m)}$.

We conclude that for every integer a we have $p_i^{a_i} \mid a^{m-\phi(m)}(a^{\phi(m)} - 1) = a^m - a^{m-\phi(m)}$ for $i = 1, 2, \ldots, r$. It follows that $m \mid (a^m - a^{m-\phi(m)})$, which implies that $a^m \equiv a^{m-\phi(m)} \pmod{m}$.

CHAPTER 7

Multiplicative Functions

7.1. The Euler Phi-Function

7.1.1. a. Since for all positive integers m and n, $f(mn) = 0 = 0 \cdot 0 = f(m) \cdot f(n)$, f is completely multiplicative.

b. Since $f(6) = 2$, but $f(2) \cdot f(3) = 2 \cdot 2 = 4$, f is not completely multiplicative.

c. Since $f(6) = 3$, but $f(2) \cdot f(3) = \frac{2}{2} \cdot \frac{3}{2} = \frac{3}{2}$, f is not completely multiplicative.

d. Since $f(4) = \log(4) > 1$, but $f(2) \cdot f(2) = \log(2) \cdot \log(2) < 1$, f is not completely multiplicative.

e. Since for any positive integers m and n, $f(mn) = (mn)^2 = m^2n^2 = f(m) \cdot f(n)$, f is completely multiplicative.

f. Since $f(4) = 4! = 24$, but $f(2) \cdot f(2) = 2!2! = 4$, f is not completely multiplicative.

g. Since $f(6) = 7$, but $f(2) \cdot f(3) = 4 \cdot 3 = 12$, f is not completely multiplicative

h. Since $f(4) = 4^4 = 256$, but $f(2) \cdot f(2) = 2^2 2^2 = 16$, f is not completely multiplicative.

i. Since for any positive integers m and n, $f(mn) = \sqrt{mn} = \sqrt{m}\sqrt{n} = f(m) \cdot f(n)$, f is completely multiplicative.

7.1.2. a. We have $100 = 2^2 5^2$, so $\phi(100) = 100(1 - 1/2)(1 - 1/5) = 40$.

b. We have $256 = 2^8$, so $\phi(256) = 2^8 - 2^7 = 128$.

c. We have $1001 = 7 \cdot 11 \cdot 13$, so $\phi(1001) = (7 - 1)(11 - 1)(13 - 1) = 720$.

d. We have $\phi(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) = (2 - 1)(3 - 1)(5 - 1)(7 - 1)(11 - 1)(13 - 1) = 5760$.

e. The primes which divide 10! are 2,3,5 and 7, so $\phi(10!) = 10!(1 - 1/2)(1 - 1/3)(1 - 1/5)(1 - 1/7) = 829,440$.

f. The primes which divide 20! are 2, 3, 5, 7, 11, 13, 17 and 19, so $\phi(20!) = 20!(1 - 1/2)(1 - 1/3)(1 - 1/5)(1 - 1/7)(1 - 1/11)(1 - 1/13)(1 - 1/17)(1 - 1/19) =$ 416,084,687,585,280,000.

7.1.3. We have the following prime factorizations of 5186, 5187, and 5188: $5186 = 2 \cdot 2593$, $5187 = 3 \cdot 7 \cdot 13 \cdot 19$, and $5188 = 2^2 1297$. Hence $\phi(5186) = \phi(2)\phi(2593) = 1 \cdot 2592 = 2592$, $\phi(5187) = \phi(3)\phi(7)\phi(13)\phi(19) = 2 \cdot 6 \cdot 12 \cdot 18 = 2592$, and $\phi(5188) = \phi(2^2)\phi(1297) = 2 \cdot 1296 = 2592$. It follows that $\phi(5186) = \phi(5187) = \phi(5188)$.

7.1.4. a. If $n > 1$, let $n = 2^k p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime factorization of n. If $k > 0$ then $\phi(n) = 2^{k-1}(p_1^{a_1} - p_1^{a_1 - 1}) \cdots (p_r^{a_r} - p_r^{a_r - 1})$ and if $k = 0$ then $\phi(n) = (p_1^{a_1} - p_1^{a_1 - 1}) \cdots (p_r^{a_r} - p_r^{a_r - 1})$. If $\phi(n) = 1$, then either $n = 1$; or $k = 1$ and $n = 2$.

b. Using the notation developed in part (a), if $\phi(n) = 2$, then either $k = 2$ and $n = 4$; or $k = 1$ and $p_1^{a_1} - p_1^{a_1-1} = 2$ so $p_1^{a_1} = 3$ and $n = 6$; or $k = 0$ and $p_1^{a_1} - p_1^{a_1-1} = 2$, so $p_1^{a_1} = 3$ and $n = 3$.

c. Using the notation developed in part (a), if $\phi(n) = 3$, then $p_1^{a_1} - p_1^{a_1-1} = 3$, which is impossible, so there are no solutions.

d. Using the notation developed in part (a), if $p^t \mid n$, then $p^{t-1}(p-1) \mid \phi(n) = 4$. Therefore, no odd prime can appear in the factorization of n to a power higher than 1. Further, $p-1$ must be a divisor of 4, so p must be one of $2, 3$, or 5. Say $n = 2^k 3^a 5^b$, where a and b are 0 or 1. Note that $\phi(2^k) \geq 2^{k-1}$ which must divide 4, so k is either $0, 1, 2$, or 3. If $k = 3$, then $a = b = 0$, and so one solution is $n = 8$. If $k = 2$, then $\phi(2^2) = 2$ which forces $a = 1$ and $b = 0$, so a second solution is $n = 12$. If $k = 0$ or 1, then $\phi(2^k) = 1$. This forces $a = 0$ and $b = 1$. This gives us two more solutions $n = 5$ and $n = 10$. Having exhausted all possibilities, we have the complete set of solutions: 5, 8, 10, and 12.

7.1.5. If $\phi(n) = 6$, and suppose k distinct primes divide n. Then either $k = 2$ and $p_1^{a_1} - p_1^{a_1-1} = 3$, which is impossible, or $k = 1$ and $p_1^{a_1} - p_1^{a_1-1} = 6$, so $p_1^{a_1} = 9$ and $n = 18$ or $p_1^{a_1} = 7$ and $n = 14$, or $k = 0$ and $p_1^{a_1} - p_1^{a_1-1} = 6$ and $p_1^{a_1} = 9 = n$ or $p_1^{a_1} = 7$ and $n = 7$. So the only solutions are $n = 7, 9, 14$, or 18.

7.1.6. Suppose a prime p divides n. Then $p - 1$ is a divisor of 12. So $p - 1 = 1, 2, 3, 4, 6$, or 12, that is $p = 2, 3, 5, 7$, or 13. If $p^2 \mid n$, then $p \mid 12$, and so only 2 and 3 can divide n to a power higher than 1. If $3^3 \mid n$ then $\phi(n) \geq \phi(3^3) \geq 18 > 12$, a contradiction. If 3^2 divides n, say $n = 9k$ with $3 \nmid k$, then $12 = \phi(n) = \phi(9)\phi(k) = 6\phi(k)$, which forces $\phi(k) = 2$. Since $3 \nmid k$, Exercise 4(b) shows that $k = 4$, yielding the solution $n = 36$. Likewise, if $5 \mid n$, say $n = 5k$ with $5 \nmid k$, then $12 = \phi(n) = 4\phi(k)$, and so $\phi(k) = 3$, which is impossible. If $2^t \parallel n$, say $n = 2^t k$, then $\phi(2^t) = 2^{t-1} \mid 12$, and so $t \leq 3$. If $t = 3$, then $\phi(k) = 3$ which is still impossible, so $t \leq 2$. So $n = 2^t 3^a 7^b 13^c$, where $t = 0, 1$, or 2, and a, b, and c are either 0 or 1. If $t = 2$, then $n = 4k$ with k odd and $12 = \phi(n) = 2\phi(k)$, so k is an odd solution to $\phi(k) = 6$, and from Exercise 5, we know $k = 7$ or 9 and therefore $n = 28$ and $n = 36$ are two solutions. If $t = 1$ or 0, then $n = 2^t k$ with k odd and $12 = \phi(n) = \phi(k)$. If $13 \mid k$, then $\phi(2^t 13) = 12$ and there can be no other factors of n. So $n = 13$ and $n = 26$ are two more solutions. The only other possibilities for k are $3, 7$, and 21. But only $\phi(21) = 12$, so $n = 21$ and $n = 42$ are the last two solutions. This gives us $13, 21, 26, 28, 36$, and 42 as the only solutions.

7.1.7. If $\phi(n) = 24$, we have 5 cases as $k = 0, 1, 2, 3$ or 4. Note that if $p^a - p^{a-1} = 2^m$, then $a = 1$ and $p = 2^m = 1$. Also note that $p^a - p^{a-1}$ is always even. In every case, $3 \mid p_1^{a_1} - p_1^{a_1-1}$. Every other factor in the formula for $\phi(n)$ is of the form 2^{k-1} or $p - 1$ where p is a Mersenne prime. If $k = 4$, then $p_1^{a_1} - p_1^{a_1-1} = 3$ which is impossible. If $k = 3$, then $\phi(n) = \phi(8 \cdot m) = \phi(8)\phi(m) = 4\phi(m)$, so $\phi(m) = 6$. By part (d) $m = 7$ or 9 so $n = 56$ or 72. If $k = 2$, then $\phi(n) = \phi(4 \cdot m) = \phi(4)\phi(m) = 2\phi(m)$, so $\phi(m) = 12$. Since $p_1^{a_1} - p_1^{a_1-1} \neq 3, p_1^{a_1} - p_1^{a_1-1} = 6$ or 12 so $p_1^{a_1} = 13$ and $n = 52$ or $p_1^{a_1} = 9$ and $p_2^{a_2} - p_2^{a_2-1} = 2$ which is impossible. If $k = 1$, then $\phi(n) = \phi(2m) = \phi(m)$ so the case $k = 0$ is covered here also. We have $p_1^{a_1} - p_1^{a_1-1} = 24$ or $(p_1^{a_1} - p_1^{a_1-1})(p_2^{a_1} - p_2^{a_2-1}) = 24$. In the first case we have $p_1 = 3$ since $3 \mid 24$ and this leads to $p_2 = 13$ so $n = 39$ or 78. In the second case, either $p_1 = 3$ and we have $(3-1)(p_2^{a_2} - p_2^{a_2-1}) = 24$ so $p_2 = 13$ as in the last case, or $p_1 = 5$ and $(5-1)(p_2^{a_2} - p_2^{a_2-1}) = 24$, so $p_2^{a_2} = 9$ or 7 which leads to $n = 45$ and 35 respectively if $k = 0$ and 90 and 70 if $k = 1$. Then the totality of all solutions is 35, 39, 45, 52, 56, 70, 72, 78, 84, and 90.

7.1.8. If $\phi(n) = 14$, then $7 \mid p_1^{a_1} - p_1^{a_1-1}$ for some odd prime p_1. Since the only factors of 14 are 2 and 7, either $p_1 = 7$ and $a_1 > 1$ and hence $p_1 - 1 = 6 \mid 14$ which is false, or $7 \mid p_1 - 1$, but $p_1 - 1$ is even, so $p_1 - 1 = 14$ or $p_1 = 15$ which is not prime. Therefore there are no solutions.

7.1.9. Studying Table E.2 on page 609 and 610, we discover that the nth term of this sequence is given by $\phi(2n)$.

7.1.10. Studying Table E.2 on page 609 and 610, we discover that the nth term is the number of solutions to $\phi(k) = n$.

7.1.11. Let $n = 3^k m$, where $(3, m) = 1$. If $k = 0$, then $\phi(3n) = 2\phi(n) \neq 3\phi(n)$. On the other hand, if $k \geq 1$, then $\phi(3n) = \phi(3^{k+1}m) = (3^{k+1} - 3^k)\phi(m) = 3(3^k - 3^{k-1})\phi(m) = 3\phi(3^k m) = 3\phi(n)$. Therefore, $\phi(3n) = 3\phi(n)$ if and only if $3 \mid n$.

7.1.12. If $n = 2^k p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ then $\phi(n) = 2^{k-1} p_1^{a_1-1}(p_1 - 1) \cdots p_r^{a_r-1}(p_r - 1)$. Since p_i is odd, $p_i - 1$ is even. So $\phi(n)$ is divisible by 4 if n satisfies any of the following: (1) $n = 2^k$ with $k \geq 3$; (2) n has an odd prime divisor of the form $4k + 1$; (3) n is divisible by 4 (i.e. $k = 2$) and n has an odd prime divisor; (4) n has 2 odd prime divisors.

7.1.13. If $n = 2^k p_1^{a_1} \cdots p_r^{a_r}$ then $\phi(n) = n(p_1 - 1)/p_1 \cdots (p_r - 1)/p_r$. If $\phi(n) = n/2$, we have $(p_1 - 1)/p_1 \cdots (p_r - 1)/p_r = 1/2$. Let p_r be the largest prime dividing n, then p_r divides none of $p_1 - 1, p_2 - 1, \cdots, p_r - 1$, so it must appear in the denominator of $(p_1 - 1)/p_1 \cdots (p_r - 1)/p_r$ in lowest terms. But $1/2$ is in lowest terms, therefore n has no odd prime divisors. Since $\phi(2^k) = 2^{k-1} = 2^k/2$, for $k = 1, 2, \cdots$ we have $n = 2, 2^2, \cdots$ as the only solutions.

7.1.14. If $n = p_1^{a_1} \cdots p_r^{a_r}$ and $\phi(n) \mid n$ we have $k\phi(n) = kn(p_1 - 1)/p_1 \cdots (p_r - 1)/p_r = n$ so that $k = p_1/(p_1 - 1) \cdots p_r/(p_r - 1)$ is an integer. The numerator can have at most one factor of 2, so the denominator can have at most one factor of the form $p_i - 1$ where p_i is an odd prime. Thus either $n = 2^{a_1}$ and $\phi(n) = (n/2) \mid n$ or $n = 2^{a_1} p^{a_2}$ and $\phi(n) = n((2 - 1)/2))((p - 1)/p)$ and $l = (2/(2 - 1)) \cdot (p/(p - 1) = 2p/(p - 1)$. So $p - 1 = 2$ or $p = 3$ and we have $n = 2^{a_1} 3^{a_2}$. So the solutions are $n = 1, 2^{a_1}, 2^{a_1} 3^{a_2}$ with $a_1, a_2 \geq 1$.

7.1.15. If n is odd, then $(2, n) = 1$ and $\phi(2n) = \phi(2)\phi(n) = 1 \cdot \phi(n) = \phi(n)$. If n is even, say $n = 2^s t$ with t odd. Then $\phi(2n) = \phi(2^{s+1}t) = \phi(2^{s+1})\phi(t) = 2^s\phi(t) = 2(2^{s-1}\phi(t)) = 2(\phi(2^s)\phi(t)) = 2(\phi(2^s t)) = 2\phi(n)$.

7.1.16. Suppose that the prime factorization of n is $n = \prod_{j=1}^{k} p_j^{a_j}$. Then since ϕ is multiplicative, $\phi(n) = \prod_{j=1}^{k} \phi(p_j^{a_j})$. Note that $\phi(p_j^{a_j}) = p_j^{a_j-1}(p_j - 1)$. If p_j is odd, then $2 \mid \phi(p_j)$. Hence $2^k \mid \phi(n)$ if n has k distinct odd prime divisors.

7.1.17. If $\phi(n)$ is a power of 2 then every factor $p_i^{a_i} - p_i^{a_i-1} = p_i^{a_i-1}(p_i - 1)$ must be a power of 2. Then either $p_i = 2$ or $a_i = 1$ and $p_i - 1 = 2^{k_i}$ and so p_i is a Mersenne prime. Therefore $\phi(n)$ is a power of 2 if and only if $n = 2^k p_1 p_2 \cdots p_r$ where each p_i is a Mersenne prime.

7.1.18. Since n is odd, we have $(4, n) = 1$. Then since ϕ is multiplicative, we compute $\phi(4n) = \phi(4)\phi(n) = 2\phi(n)$.

7.1.19. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the factorization for n. If $n = 2\phi(n)$ then $p_1^{a_1} \cdots p_r^{a_r} = 2\prod_{j=1}^{r} p_j^{a_j-1}(p_j - 1)$. Cancelling the powers of all p_j's yields $p_1 \cdots p_r = 2\prod_{j=1}^{r}(p_j - 1)$. If any p_j is an odd prime, then the factor $(p_j - 1)$ is even and must divide the product on the left-hand side. But there can be at most one factor of 2 on the left-hand side and it is accounted for by the factor of 2 in front of the product on the right hand side. Therefore, no odd primes appear in the product. That is, $n = 2^j$ for some j.

7.1.20. First, if $p \nmid n$, then $(p, n) = 1$. Since ϕ is multiplicative, we have $\phi(pn) = \phi(p)\phi(n) = (p - 1)\phi(n)$. Conversely, if $p|n$, say $n = p^a m$, where $(p, m) = 1$, then we compute $\phi(pn) = \phi(p^{a+1}m) = \phi(p^{a+1})\phi(m) = (p^{a+1} - p^a)\phi(m) = (p - 1)p^a\phi(m)$. On the other hand $(p-1)\phi(n) = (p-1)\phi(p^a m) = (p-1)\phi(p^a)\phi(m) = (p-1)(p-1)p^{a-1}\phi(m)$. If we form the ratio of these two expressions, we get $\phi(pn)/(p-1)\phi(n) = p/(p-1)$. Since this last expression can not be equal to 1, we know that $\phi(pn) \neq (p - 1)\phi(n)$.

7.1.21. Since $(m, n) = p$, p divides one of the terms, say n exactly once, so $n = kp$ with $(m, k) = 1 = (n, k)$. Then $\phi(n) = \phi(kp) = \phi(k)\phi(p) = \phi(k)(p - 1)$, and $\phi(mp) = p\phi(m)$ by the formula in Example 7.7. Then, $\phi(mn) = \phi(mkp) = \phi(mp)\phi(k) = (p\phi(m))(\phi(n)/(p - 1))$.

7.1.22. Suppose that the prime factorization of m is $m = \prod_{i=1}^{r} p_i^{a_i}$. Then $\phi(m) = \prod_{i=1}^{r} \phi(p_i^{a_i})$. Since $m^k = \prod_{i=1}^{r} p_i^{ka_i}$, $\phi(m^k) = \prod_{i=1}^{r} \phi(p_i^{ka_i})$. Note that $\phi(p_i^{ka_i}) = p_i^{ka_i-1}(p_i - 1) = p_i^{(k-1)a_i} p_i^{a_i-1}(p_i - 1) = p_i^{(k-1)a_1}\phi(p_i^{a_i})$. Hence $\phi(m^k) = \prod_{i=1}^{r} p_i^{(k-1)a_i}\phi(p_i^{a_i}) = \prod_{i=1}^{r} p_i^{(k-1)a_i} \prod_{i=1}^{r} \phi(p_i^{a_i}) = m^{k-1}\phi(m)$.

7.1.23. Let $p_1, \cdots, p_r$ be those primes dividing a but not b. Let $q_1, \cdots, q_s$ be those primes dividing b but not a. Let $r_1, \cdots r_t$ be those primes dividing a and b. Let $P = \prod(1 - \frac{1}{p_i}), Q = \prod(1 - \frac{1}{q_i})$ and $R = \prod(1 - \frac{1}{r_i})$. Then we have $\phi(ab) = abPQR = \frac{aPRbQR}{R} = \frac{\phi(a)\phi(b)}{R}$. But $\phi((a,b)) = (a,b)R$ so $R = \frac{\phi((a,b))}{(a,b)}$ and we have $\phi(ab) = \frac{\phi(a)\phi(b)}{R} = \frac{(a,b)\phi(a)\phi(b)}{\phi((a,b))}$ as desired.

7.1.24. If n is prime, then $\phi(n) = n-1 \geq 10^k$. If n is not prime, then by Exercise 18, $n \geq 10^k + \sqrt{n} \geq 10^k + 10^{k/2}$. So in each case, we seek the smallest prime between 10^k and $10^k + 10^{k/2}$: **a.** 101 **b.** 1013 **c.** 10007 **d.** 100003

7.1.25. From the formula for the ϕ function, we see that if $p|n$, then $p - 1|k$. Since k has only finitely many divisors, there are only finitely many possibilities for prime divisors of n. Further, if p is prime and $p^a|n$, then $p^{a-1}|k$. Hence, $a \leq \log_p(k) + 1$. Therefore, each of the finitely many primes which might divide n may appear to only finitely many exponents. Therefore, there are only finitely many possibilities for n.

7.1.26. If n is odd, then $\phi(2n) = \phi(n) = k$, giving two solutions, so n must be even. If $n = 2m$ with m odd, then $\phi(m) = \phi(2m) = \phi(n) = k$, also giving two solutions, so $4 \mid n$. Say $n = 2^t m$ with m odd and $t \geq 2$. Then $k = \phi(n) = 2^{t-1}\phi(m)$. If $3 \nmid m$, then $\phi(2^{t-1}3m) = 2^{t-2}(3-1)\phi(m) = 2^{t-1}\phi(m) = k$, giving two solutions, so $3 \mid m$. If $m = 3s$ with $(3, s) = 1$, then $k = \phi(n) = 2^{t-1}(3-1)\phi(s) = 2^t\phi(s)$. But $\phi(2n/3) = \phi(2^{t+1}s) = 2^t\phi(s) = k$, again giving two solutions. Therefore $3 \mid s$ and so $4 \cdot 9 = 36 \mid n$.

7.1.27. From the formula for the ϕ function, we see that if $p|n$, then $p - 1|k$. Since k has only finitely many divisors, there are only finitely many possibilities for prime divisors of n. Further, if p is prime and $p^a|n$, then $p^{a-1}|k$. Hence, $a \leq \log_p(k) + 1$. Therefore, each of the finitely many primes which might divide n may appear to only finitely many exponents. Therefore, there are only finitely many possibilities for n.

7.1.28. We further assume that $2^a p + 1$ is not prime for $a = 1, 2, \ldots, r$, and that p is not a Fermat prime. Suppose that $\phi(n) = \prod_{i=0}^{k}(p_i^{a_i} - p_i^{a_i - 1}) = 2^r p$. Then only one odd non-Fermat prime can appear to a power greater than 1 in the prime power factorization of n. There are two cases. First, $p^2 \parallel n$, so that $\phi(n) = p(p-1)2^a = 2^r p$. But this forces $p - 1 = 2^{r-a}$, so that p is a Fermat prime, which contradicts the hypotheses. Second, there is a prime $q \parallel n$ such that $p \mid q - 1$ so that $\phi(n) = m(q-1) = 2^r p$. So there is an integer a with $1 \leq a \leq r$ such that $2^a p + 1 = q$ is prime. This also contradicts the hypotheses.

7.1.29. As suggested, we take $k = 2 \cdot 3^{6j+1}$ with $j \geq 1$, and suppose that $\phi(n) = k$. From the formula for $\phi(n)$ we see that $\phi(n)$ has a factor of $(p-1)$, which is even for every odd prime that divides n. Since there is only one factor of 2 in k, there is at most one odd prime divisor of n. Further, since $2 \parallel k$, we know that $4 \not| n$. Since k is not a power of 2, we know that an odd prime p must divide n. So n is of the form p^a or $2p^a$. Recall that $\phi(p^a) = \phi(2p^a)$. It remains to discover the value of p. If $a = 1$, then $\phi(p^a) = p - 1 = 2 \cdot 3^{6j+1}$. But then, $p = 2 \cdot 3^{6j+1} + 1 \equiv 6 \cdot (3^6)^j + 1 \equiv (-1)(1)^j + 1 \equiv 0 \pmod 7$. Hence $p = 7$. But $\phi(7) = 6 = 2 \cdot 3^{6j+1}$ implies that $j = 0$, contrary to hypothesis, so this is not a solution. Therefore $a > 1$ and we have $\phi(p^a) = (p-1)p^{a-1} = 2 \cdot 3^{6j+1}$, from which we conclude that $p = 3$ and $a = 6j + 2$. Therefore the only solutions are $n = p^{6j+2}$ and $n = 2p^{6j+2}$.

7.1.30. Let p be a prime. If $a > 1$, then $\phi(p^a) = p^{a-1}(p-1) \geq p^{a-1} \geq p^{a/2} = \sqrt{p^a}$. If p is odd, then $\phi(p) = p - 1 > \sqrt{p}$ and $\phi(2p^a) = p^{a-1}(p-1) \geq p^{a-1}2 \geq \sqrt{2p^a}$. Finally, if $p > 4$, then $p^2 + 1 > 4p$, so $(p-1)^2 = p^2 - 2p + 1 > 2p$, and we have $\phi(2p) = p - 1 \geq \sqrt{2p}$. Now suppose $n = 2^{a_0}p_1^{a_1} \cdots p_r^{a_r}$. If $a_0 \neq 1$, then by multiplicativity of the ϕ-function and the square root function, we have $\phi(n) = \prod_{i=0}^{r}\phi(p_i^{a_i}) \geq \prod_{i=0}^{r}\sqrt{p_i^{a_i}} = \sqrt{n}$. If $a_0 = 1$ and, by rearrangement if necessary, $p_1^{a_1}$ has either $a_1 > 1$ or $p_1 > 4$, then again by multiplicativity, we have $\phi(n) = \phi(2p_1^{a_1})\prod_{i=2}^{r}\phi(p_i^{a_i}) \geq \sqrt{2p_1^{a_1}}\prod_{2=0}^{r}\sqrt{p_i^{a_i}} = \sqrt{n}$. The remaining cases are for n exactly divisible by 2, not divisible by a prime greater than 4 and not divisible by a prime to a power greater than 1. This leaves only $n = 2$ and $n = 6$, which are the only exceptions to the proposition.

7.1.31. If $n = p^r m$, then $\phi(p^r m) = (p^r - p^{r-1})\phi(m) \mid (p^r m - 1)$, hence $p \mid 1$ or $r = 1$. So n is square-free. If $n = pq$, then $\phi(pq) = (p-1)(q-1) \mid (pq - 1)$. Then $(p-1) \mid (pq-1) - (p-1)q = q - 1$. Similarly $(q-1) \mid (p-1)$, a contradiction.

7.1.32. Suppose that $n = p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$. Using the formula for $\phi(n)$ in terms of its prime factorization, we find that $n/\phi(n) = \frac{p_1p_2\cdots p_m}{(p_1-1)(p_2-1)\cdots(pm-1)}$. We see that $\phi(n) \mid n$ if and only if $(p_1-1)(p_2-1)\cdots(p_m-1) \mid p_1p_2\cdots p_m$. The right hand side contains at most one factor of 2, so that there is at most one even factor on the left hand side. Hence there can be at most one odd prime in the factorization.

We see that $\phi(1) = 1 \mid 1$. If $n = 2^a$ where a is positive integer then $\phi(n) + 2^{a-1}$ so that $n \mid \phi(n)$. Otherwise, $n = 2^a p^b$ where a and b are positive integers and p is an odd prime. Then $\phi(n) = 2^{a-1}p^{b-1}(p-1)$. We first show that $p \neq 3$. If $p > 3$ then $(p-1)$does not divide $2p$. Suppose that it does, then $(p-1)d = 2p$ where d is an odd positive integer so that $d = \frac{2p}{(p-1)} = \frac{2}{1-1/p} < \frac{2}{2/3} = 3$, which is a contradiction. Now suppose that $p = 3$. Then if $a \geq 1$, $\phi(n) = 2^a 3^{b-1}$ so that $n = 3\phi(n)$ and $n \mid \phi(n)$. If $a = 0$ then $\phi(n) = 2 \cdot 3^{b-1}$ so that n is odd but $\phi(n)$ is even. Hence $\phi(n)$ does not divide n. In summary, $\phi(n)$ divides n if and only if $n = 1$, 2^a, or $2^a 3^b$ where a and b are positive integers.

7.1.33. Let $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$. Let P_i be the property that an integer is divisible by p_i. Let S be the set $\{1, 2, \ldots, n-1\}$. To compute $\phi(n)$ we need to correct the elements of S with more of the properties $P_1, P_2, \cdots, P_k$. Let $n(P_{i_1}, P_{i_2}, \cdots, P_{i_m})$ be the number of elements of S with all of properties $P_{i_1}, P_{i_2}, \cdots, P_{i_m}$. Then $n(p_{i_1}, \cdots P_{i_m}) = \frac{n}{p_{i_1}}p_{i_2}\cdots p_{i_m}$. By Exercise 18 of Section 1.4, we have $\phi(n) = n - (\frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_k}) + (\frac{n}{p_1p_2} + \cdots + \frac{n}{p_{k-1}p_k}) + \cdots + (-1)^k(\frac{n}{p_1\cdots P_k}) = n(1 - \sum_{p_i|n}\frac{1}{p_i} + \sum_{p_{i_1}p_{i_2}|n}\frac{1}{p_{i_1}p_{i_2}} - \sum_{p_{i_1}p_{i_2}p_{i_3}}\frac{1}{p_{i_1}p_{i_2}p_{i_3}} + \cdots + (-1)^k\frac{n}{p_1}\cdots p_k)$. On the other hand, notice that each term in the expansion of $(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k})$ is obtained by choosing either 1 or $-\frac{1}{p_i}$ from each factor and multiply the choice together. This gives each term the form $\frac{(-1)^m}{p_{i_1}p_{i_2}\cdots p_{i_m}}$. Note that each term can occur in only one way. Thus $n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k}) = n(1 - \sum_{p_i|n}\frac{1}{p_i} + \sum_{p_{i_1}p_{i_2}}\frac{1}{p_{i_1}p_{i_2}} - \cdots (-1)^k\frac{n}{p_1\cdots p_k}) = \phi(n)$.

7.1.34. If n is prime, $\phi(n) = n - 1 > n - \sqrt{n}$. If n is not prime say $n = ab$ with $a, b > 1$. Then $b \geq \sqrt{n}$ say. By Exercise 23, $\phi(ab) = \frac{(a,b)\phi(a)\phi(b)}{\phi((a,b))} < \phi(a)\phi(b) \leq (a-1)(b-1) = ab - a - b + 1 \leq n - b + 1 \leq n - \sqrt{n} + 1$. So $\phi(a,b) < n - \sqrt{n+1}$ but since both sides are integers we have $\phi(a,b) \leq n - \sqrt{n}$.

7.1.35. Note that $1 \leq \phi(m) \leq m - 1$ for $m > 1$. Hence if $n \geq 2$, $n > n_1 > n_2 > \cdots \geq 1$ where $n_i = \phi(n)$ and $n_i = \phi(n_{i-1})$ for $i > 1$. Since $n_i, i = 1, 2, 3, \ldots$ is a decreasing sequence of positive integers, there must be a positive integer r such that $n_r = 1$.

7.1.36. We have $f(p^k) = \frac{\phi(p^k)}{p^k} = \frac{(p^k - p^{k-1})}{p^k} = \frac{(p-1)}{p} = \frac{\phi(p)}{p}$. Hence $f(n) = \frac{\phi(n)}{n}$ is strongly multiplicative.

7.1.37. Note that the definition of $f * g$ can also be expressed as $(f * g)(n) = \sum_{a \cdot b = n} f(a)g(b)$. Then the fact that $f * g = g * f$ is evident.

7.1.38. Using the form of the definition in the solution to Exercise 37 above, we have $((f * g) * h)(n)$

$$\begin{aligned} &= \sum_{ab=n}(f*g)(a)h(b) \\ &= \sum_{ab=n}\sum_{cd=a} f(c)g(d)h(b) \\ &= \sum_{cdb=n} f(c)g(d)h(b) \end{aligned}$$

Similarly, $(f * (g * h))(n) = \sum_{cdb=n} f(c)g(d)h(b)$ and we're done.

7.1.39. a. If either $m > 1$ or $n > 1$ then $mn > 1$ and one of $\iota(m)$ or $\iota(n)$ is equal to zero. Then $\iota(mn) = 0 = \iota(m)\iota(n)$. Otherwise, $m = n = 1$ and we have $\iota(mn) = 1 = 1 \cdot 1 = \iota(m)\iota(n)$. Therefore $\iota(n)$ is multiplicative.

b. $(\iota * f)(n) = \sum_{d|n}\iota(d)f(\frac{n}{d}) = \iota(1)f(\frac{n}{1}) = f(n)$ since $\iota(d) = 0$ except when $d = 1$.$(f * \iota)(n) = (\iota * f)(n) = f(n)$ by Exercise 37.

7.1.40. We need, first of all, for $(f*f^{-1})(1)=\iota(1)=1$ which reduces to $f(1)f^{-1}(1)=1$. Therefore if $f(1)=0$, there is no solution and hence no inverse. If $f(1)\neq 0$, we define $f^{-1}(1)=\frac{1}{f(1)}$, the unique solution. Now assume that $f^{-1}(k)$ has been uniquely determined for all $k<n$. We solve the equation $(f*f^{-1})(n)=\iota(n)$, or $\sum_{d|n} f(\frac{n}{d})f^{-1}(d)=0$. Rewriting gives

$$f(1)f^{-1}(n)+\sum_{\substack{d|n\\ d<n}} f(\frac{n}{d})f^{-1}(d)=0.$$

If $d<n$, then $\frac{n}{d}<n$ and every quantity in the equation is uniquely determined except for $f^{-1}(n)$. Since $f(1)\neq 0$ we can solve uniquely for $f^{-1}(n)$ and by induction we're done.

7.1.41. Let $h=f*g$ and let $(m,n)=1$. Then $h(mn)=\sum_{d|mn} f(d)g(\frac{mn}{d})$. Since $(m,n)=1$, each divisor d of mn can be expressed in exactly one way as $d=ab$ where $a\mid m$ and $b\mid n$. Then $(a,b)=1$ and $(\frac{m}{a},\frac{n}{b})=1$. Then there is a one-to-one correspondence between the divisors d of mn and the pairs of products ab where $a\mid m$ and $b\mid n$. Then

$$\begin{aligned} h(mn)&=\sum_{\substack{a|m\\ b|n}} f(ab)g(\frac{mn}{ab})=\sum_{\substack{a|m\\ b|n}} f(a)f(b)g(\frac{m}{a})g(\frac{n}{b})\\ &=\sum_{a|m} f(a)g(\frac{m}{a})\sum_{b|n} f(b)g(\frac{n}{b})=h(m)h(n)\end{aligned}$$

as desired.

7.1.42. By Exercise 38, Dirichlet product is associative. We compute $F*h=(f*g)*h=f*(g*h)=f*\iota$ by the definition in Exercise 40. Then, by Exercise 39, $f*\iota=f$, which proves the theorem.

7.1.43. a. Since $12=2^2 3$, we have $\lambda(12)=(-1)^{2+1}=-1$.

b. Since $20=2^2 5$, we have $\lambda(20)=(-1)^{2+1}=-1$.

c. Since $210=2\cdot 3\cdot 5\cdot 7$, we have $\lambda(210)=(-1)^{1+1+1+1}=1$.

d. Since $1000=2^3 5^3$, we have $\lambda(1000)=(-1)^{3+3}=1$.

e. Since $1001=7\cdot 11\cdot 13$, we have $\lambda(1001)=(-1)^{1+1+1}=-1$.

f. Since $10!=2^8 3^4 5^2 7$, we have $\lambda(10!)=(-1)^{8+4+2+1}=-1$.

g. Since $20!=2^{18}3^8 5^4 7^2 11\cdot 13\cdot 17\cdot 10$, we have $\lambda(20!)=(-1)^{18+8+4+2+1+1+1+1}=1$.

7.1.44. We see that $\lambda(n)=(-1)^s$ where S is the sum of the powers in the prime factorization of n (with $S=0$ for $n=1$). Suppose that m and n are positive integers with prime factorizations $m=p_1^{a_1}\cdots p_s^{a_s}$ and $n=q_1^{b_1}\cdots q_t^{b_t}$. Then$\lambda(m)=(-1)^s$ and $\lambda(n)=(-1)^T$ where $S=\sum a_i$ and $T=\sum b_i$. But $\lambda(mn)=(-1)^{S+T}=(-1)^S(-1)^T$ since the prime powers in the factorization of mn are formed by multiplying the prime powers in the factorizations of m and n. Hence $\lambda(mn)=\lambda(m)\lambda(n)$.

7.1.45. Let $f(n)=\sum_{d|n}\lambda(d)$. Then f is the Dirichlet product of λ and the constant function $h(n)=1$. Since h and λ are both multiplicative, so is f, by Exercise 41. Now, $f(p^t)=\lambda(1)+\lambda(p)+\lambda(p^2)+\cdots+\lambda(p^t)=1-1+1-\cdots+(-1)^t=0$ if t is odd and $=1$ if t is even. Then $f(p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r})=\prod f(p_i^{a_i})=0$ if any a_i is odd and $=1$ if all a_i are even and hence n is a square.

7.1.46. Suppose that f and g are multiplicative functions. Then $f(mn)=f(m)f(n)$ and $g(mn)=g(m)g(n)$ whenever $(m,n)=1$. It follows that $(fg)(mn)=f(mn)g(mn)=f(m)f(n)g(m)g(n)=f(m)g(m)f(n)g(n)=(fg)(m)(fg)(n)$ whenever $(m,n)=1$. We conclude that fg is completely multiplicative.

7.1.47. If f and g are completely multiplicative and m and n are positive integers we have $(fg)(mn) = f(mn)g(mn) = f(m)f(n)g(m)g(n) = f(m)g(m)f(n)g(n) = (fg)(m)(fg)(n)$, so fg is also completely multiplicative.

7.1.48. If p is prime and f is completely multiplicative then $f(p^a) = f(p)f(p)\cdots f(p), a-$times, $= f(p)^a$. Since f is multiplicative we have $f(n) = f(p_1^{a_1}\cdots p_m^{a_m}) = f(p_1^{a_1})f(p_2^{a_2})\cdots f(p_m^{a_m}) = f(p_1)^{a_1}\cdots f(p_m)^{a_m}$.

7.1.49. We have $f(mn) = \log mn = \log m + \log n = f(m) + f(n)$. Hence $f(n) = \log n$ is completely additive.

7.1.50. a. Since 1 has no prime factors, $\omega(1) = 0$.

b. Since 2 is prime, $\omega(2) = 1$.

c. Since $20 = 2^2 5$, we have $\omega(20) = 2$.

d. Since $84 = 2^2 3 \cdot 7$, we have $\omega(84) = 3$.

e. Since $128 = 2^7$, we have $\omega(128) = 1$.

7.1.51. a. Since $12 = 2^2 3$, we have $\omega(12) = 2$.

b. Since $30 = 2 \cdot 2 \cdot 5$, we have $\omega(30) = 3$.

c. Since $32 = 2^5$, we have $\omega(32) = 1$.

d. Since the primes that divide $10!$ are exactly those primes less than or equal to 10, namely $2, 3, 5$ and 7, we have $\omega(10!) = 4$.

e. Since the primes that divide $20!$ are exactly those primes less than or equal to 20, namely 2, 3, 5, 7, 11, 13, 17 and 19, we have $\omega(20!) = 8$.

f. Since the primes that divide $50!$ are exactly those primes less than or equal to 50, and $\pi(50) = 15$, we have $\omega(50!) = 15$.

7.1.52. Suppose that $(m, n) = 1$. Then m and n have no common prime factors. Let the prime power factorizations of m and n be $m = p_1^{a_1}\cdots p_s^{a_s}$ and $n = q_1^{b_1}\cdots q_s^{b_s}$, so that $\omega(m) = s$ and $\omega(n) = t$. Then since the primes p_i and q_j are distinct, the prime power factorization of mn is $mn = p_1^{a_1}\cdots p_s^{a_s} q_1^{b_1}\cdots q_t^{b_t}$, so that $\omega(mn) = s + t$. Hence $\omega(mn) = \omega(m) + \omega(n)$, which shows that ω is additive. To see that ω is not completely additive, note that $\omega(4) = \omega(2 \cdot 2) = 1$ but $\omega(2) + \omega(2) = 1 + 1 = 2$.

7.1.53. Let $(m, n) = 1$, then by the additivity of f we have $f(mn) = f(m) + f(n)$. Then $g(mn) = 2^{f(mn)} = 2^{f(m)+f(n)} = 2^{f(m)}2^{f(n)} = g(m)g(n)$, so g is multiplicative.

7.1.54. Let $f(n) = n^k$. Then if n and m are any two positive integers, we have, by the ordinary rules of exponents, $f(mn) = (mn)^k = m^k n^k = f(m)f(n)$. Therefore f is completely multiplicative.

7.2. The Sum and Number of Divisors

7.2.1. a. Since $35 = 5 \cdot 7$ and σ is multiplicative, we see that $\sigma(35) = (1 + 5)(1 + 7) = 6 \cdot 8 = 48$.

b. Since $196 = 2^2 7^2$ and σ is multiplicative, we see that $\sigma(196) = (1+2+2^2)(1+7+7^2) = 7 \cdot 57 = 399$.

c. Since $1000 = 2^3 5^3$ and σ is multiplicative, we see that $\sigma(1000) = (1 + 2 + 2^2 + 2^3) \cdot (1 + 5 + 5^2 + 5^3) = 15 \cdot 156 = 2340$.

d. By Lemma 7.1 we have $\sigma(2^{100}) = \frac{2^{101}-1}{2-1} = 2^{101} - 1$.

e. Since $\sigma(n)$ is a multiplicative function, we have $\sigma(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) = (1+2)(1+3)(1+5) \cdot (1+7)(1+11) = 6912$.

f. Since σ is multiplicative, it follows that $\sigma(2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11) = (1+2+2^2+2^3+2^4+2^5) \cdot (1+3+3^2+3^3+3^4) \cdot (1+5+5^2+5^3) \cdot (1+7+7^2) \cdot (1+11) = 63 \cdot 121 \cdot 156 \cdot 57 \cdot 12 = 813404592$.

g. The prime factorization of 10! is 10! = $2^8 3^4 5^2 7$. By Theorem 6.8, we conclude that $\sigma(10) = \frac{2^9-1}{2-1} \cdot \frac{3^5-1}{3-1} \cdot \frac{5^2-1}{5-1} \cdot \frac{7^2-1}{7-1} = 511 \cdot 242 \cdot 6 \cdot 8 = 5935776$.

h. The prime factorization of 20! is 20! = $2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. By Theorem 6.8 it follows that $\sigma(20) = \frac{2^{19}-1}{2-1} \cdot \frac{3^9-1}{3-1} \cdot \frac{5^5-1}{5-1} \cdot \frac{7^3-1}{7-1} \cdot \frac{11^2-1}{11-1} \cdot \frac{13^2-1}{13-1} \cdot \frac{17^2-1}{17-1} \cdot \frac{19^2-1}{19-1} \cdot = 9841 \cdot 781 \cdot 57 \cdot 12 \cdot 14 \cdot 18 \cdot 20 = 26495791882560$.

7.2.2. a. Since the prime factorization of 36 is $36 = 2^2 3^2$ and τ is multiplicative it follows that 36 has $\tau(36) = (2+1)(2+1) = 3 \cdot 3 = 9$ positive integer divisors.

b. Since the prime factorization of 99 is $99 = 3^2 \cdot 11$ and τ is multiplicative, it follows that 99 has $\tau(99) = (2+1)(1+1) = 3 \cdot 2 = 6$ positive integer divisors.

c. Since the prime factorization of 144 is $144 = 2^4 \cdot 3^2$ and τ is multiplicative, it follows that 144 has $\tau(144) = (4+1)(2+1) = 15$ positive integer divisors.

d. Since τ is multiplicative, it follows that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ has $\tau(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19) = (1+1)^8 = 2^8 = 256$ positive integer divisors.

e. By Theorem 7.9 we find that $2 \cdot 3^2 \cdot 5^3 \cdot 7^4 \cdot 11^5 \cdot 13^4 \cdot 17^5 \cdot 19^5$ has $\tau(2 \cdot 3^2 \cdot 5^3 \cdot 7^4 \cdot 11^5 \cdot 13^4 \cdot 17^5 \cdot 19^5) = (1+1)(2+1)(3+1)(4+1)(5+1)(4+1)(5+1)(5+1) = 129600$ divisors.

f. Since the prime factorization of 20! is 20! = $2^{18} 3^8 5^4 7^2 11 \cdot 13 \cdot 17 \cdot 19$, and τ is multiplicative, by Theorem 7.9 it follows that 20! has $\tau(20) = (18+1)(8+1)(4+1)(2+1)(1+1)(1+1)(1+1)(1+1) = 19 \cdot 9 \cdot 5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 41040$ divisors.

7.2.3. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$. We need to find when $\tau(n)$ is odd. By Theorem 7.9 $\tau(n) = (a_1+1)(a_2+1) \cdots (a_s+1)$, so each factor $a_i + 1$ must be odd, hence each a_i must be even. Therefore n is a perfect square.

7.2.4. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$. We need to find when $\sigma(n)$ is odd. By Lemma 7.1 and the multiplicativity of σ we have $\sigma(n) = (1 + p_1 + \cdots + p_1^{a_1})(1 + p_2 + \cdots + p_2^{a_2}) \cdots (1 + p_s + \cdots + p_s^{a_s})$. So we need each factor $(1 + p_i + \cdots + p_i^{a_i})$ to be odd. Each factor has $a_i + 1$ terms. If p_i is odd, then each term $1, p_i, \cdots, p_i^{a_i}$ is odd, so their sum will be odd if and only if there is an odd number of terms, that is a_i must be even. If $p_i = 2$, then $1 + 2 + 2^2 + \cdots + 2^{a_i}$ is always odd. Therefore $\phi(n)$ is odd if $n = 2^k t$ with t odd and t is a perfect square.

7.2.5. a. For each part of this exercise let the prime factorization of n be $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. Then since σ is multiplicative, we have $\sigma(n) = \prod_{i=1}^{r} (1 + p_i + \cdots + p_i^{a_i})$.

Suppose that $\sigma(n) = 12$. Each factor in the formula for $\sigma(n)$ must divide 12. The only ways to get factors, other than 1, of 12 for sums of this type are $(1+2) = 3, (1+3) = 4, (1+5) = 6, (1+11) = 12$. Hence the only values of n for which $\sigma(n) = 12$ are $n = 2 \cdot 3 = 6$ and $n = 11$.

b. Suppose that $\sigma(n) = 18$. Each factor in the formula for $\sigma(n)$ must divide 18 and the product of these factors must be 18. The only ways to get factors, other that 1, of 18 for sums of this type are $(1+2) = 3$, $(1+5) = 6$, and$(1+17) = 18$. It follows that the only solutions of $\sigma(n) = 18$ are $n = 2 \cdot 5 = 10$ and $n = 17$.

c. Suppose that $\sigma(n) = 24$. Each factor in the formula for $\sigma(n)$ must divide 24 and the product of these factors must be 24. The only ways to get factors, other than 1, of 24 for sums of this type are $(1+2) = 3$, $(1+3) = 4$, $(1+5) = 6$, $(1+7) = 8$, $(1+11) = 12$, and $(1+23) = 24$. It follows that the only solutions of $\sigma(n) = 24$ are $n = 2 \cdot 7 = 14$,$n = 3 \cdot 5 = 15$, and $n = 23$.

d. Suppose that $\sigma(n) = 48$. Each factor in the formula for $\sigma(n)$ must divide 48 and the product of these factors must be 48. The only ways to get factors, other than 1, of 48 for sums of this type are $(1+2) = 3$, $(1+3) = 4$, $(1+5) = 6, (1+7) = 8$, $(1+11) = 12$, $(1+23) = 24$, and $(1+47) = 48$. If follows that the only solutions of $\sigma(n) + 48$ are $n = 3 \cdot 11 = 33$, $n = 5 \cdot 7 = 35$, and $n = 47$.

e. Suppose that $\sigma(n) = 52$. Each factor in the formula for $\sigma(n)$ must divide 52 and the product of these factors must be 52. The only ways to get factors, other than 1, of 52 for sums of this type are $(1+3) = 4$ and $(1+3+9) = 13$. Since only one factor for each prime can be included, there are no solutions of $\sigma(n) = 52$.

f. Suppose that $\sigma(n) = 84$. Each factor in the formula for $\sigma(n)$ must divide 84 and the product of these factors must be 84. The only ways to get factors, other than 1, of 84 for sums of this type are $(1+2) = 3$, $(1+3) = 4$, $(1+5) = 6$, $(1+2+4) = 7$, $(1+11) = 12$, $(1+13) = 14$, and $(1+83) = 84$. It follows that the only solutions of $\sigma(n) = 84$ are $n = 5 \cdot 13 = 65$, $n = 4 \cdot 11 = 44$, and $n = 83$.

7.2.6. a. Suppose that $n = \prod_{j=1}^{r} p_j^{a_j}$ is the prime factorization of n. By Theorem 7.9 we know that $\tau(n) = \prod_{j=1}^{r}(1+a_j)$.

For $\tau(1) = 1$ it is necessary that $n = 1$. Hence $n = 1$ is the smallest positive integer such that $\tau(n) = 1$.

b. For $\tau(n) = 2$ we must have $n = p$ where p is a prime. Consequently the smallest n for which $\tau(n) = 2$ is $p = 2$.

c. For $\tau(n) = 3$ we must have $n = p^2$ where p is prime. Hence $2^2 = 4$ is the smallest n for which $\tau(n) = 4$.

d. For $\tau(n) = 6$, by the formula for $\tau(n)$ we see that n must have the form $n = pq^2$ or $n = p^5$ where p and q are prime. The smallest n of the first kind is $3 \cdot 2^2 = 12$ and the smallest n of the second kind is $2^5 = 32$. Hence the smallest n such that $\tau(n) = 6$ is $n = 12$.

e. For $\tau(n) = 14$, by the formula for $\tau(n)$, we see that $n = pq^6$ or $n = p^{13}$ where p and q are primes. The smallest such integer is $n = 3 \cdot 2^6 = 192$.

f. For $\tau(n) = 100$, by the formula for $\tau(n)$ we see that $n = pq^{49}$, $n = p^3q^{24}$, $n = p^4q^{19}$, $n = p^9q^9$, $n = pqr^{24}$, $n = p^3r^4q^4$ or $n = pqr^4s^4$ where p, q, r, and s are primes. We can easily see that the smallest such integer is of the final form listed, with $n = 2^4 \cdot 3^4 \cdot 5 \cdot 7 = 45360$.

7.2.7. Note that $\tau(p^{k-1}) = k$ whenever p is prime and k is a positive integer $k > 1$. Hence the equation $\tau(n) = k$ has infinitely many solutions.

7.2.8. The positive integers with exactly two prime divisors are the primes.

7.2.9. The only positive integers with exactly three prime divisors are those integers of the form p^2 where p is prime. We see this using the formula given in Theorem 7.9. We have $\tau(p_1^{a_1} \cdots p_t^{a_t}) = (a_1+1)(a_2+1)\cdots(a_t+1)$. Since the terms on the right-hand side are all at least 2, this product can equal 3 if and only if there is precisely one term on the right-hand side that is equal to 3.

7.2.10. We need $\tau(n) = 4$. If $n = p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s}$ then $\tau(n) = (a_1+1)(a_2+1)\cdots(a_{s+1}) = 4$, so there are two possibilities. Either $(a_1+1) = 4$ or $(a_1+1) = (a_2+1) = 2$. In the first case $a_1 = 3$ and $n = p^3$. In the second case $a_1 = a_2 = 1$ and $n = p_1p_2$.

7.2.11. We first suppose that n is not a perfect square. Then the divisors of n come in pairs with product n, that is, when d is a divisor, so is n/d and conversely. Since there are $\tau(n)/2$ such pairs, the product of all divisors is $n^{\tau(n)/2}$. Now suppose that n is a perfect square. Then there are $\frac{\tau(n)-1}{2}$ pairs with product n and the extra divisor $\sqrt{n}$. Hence the product of all the divisors of n is $n^{(\tau(n)-1)/2} \cdot n^{1/2} = n^{\tau(n)/2}$.

7.2.12. If $n > k$ then $\sigma(n) \geq n > k$. So a solution must be a positive integer less than or equal to k. Since there are only finitely many of these, we're done.

7.2.13. a. The nth term is given by $\sigma(2n)$.

b. The nth term is given by $\sigma(n) - \tau(n)$.

c. The nth term of this sequence is the least positive integer m with $\tau(m) = n$.

d. The nth term is the number of solutions k to the equation $\sigma(k) = n$.

7.2.14. a. The nth term is given by $\sigma(n) + \tau(n)$.

b. The nth term is given by $\sigma(2n-1)$.

c. The nth term is the nth smallest solution to $\tau(n) = 4$.

d. The nth term is given by $\tau(2n-1)$.

7.2.15. If we list the values of $\tau(n)$ for $n = 2, 3, 4, \ldots$, in order, we can identify highly composite integers by noting the first occurrence of a value which is larger than all previous values. From Table 2 in Appendix E we see that $\tau(2) = 2$ is the first occurrence of 2, and is larger than all previous values. This is first exceeded when we find $\tau(4) = 3$. This is exceeded when we find $\tau(6) = 4$. This is exceeded when we find $\tau(12) = 6$. This is exceeded when we find $\tau(24) = 8$. This is exceeded when we find $\tau(36) = 9$. So the first six highly composite numbers are $2, 4, 6, 12, 24$ and 36.

7.2.16. Since n is highly composite, $\tau(m) > \tau(n) > \tau(j)$ for $j = 1, 2, \ldots, n-1$. Let k be the smallest integer greater than n for which $\tau(k) \geq \tau(m)$. If no integer between m and n works, then $k = m$, so the existence of k is assured. Then $\tau(k) > \tau(j)$ for $j = 1, 2, \ldots, k-1$, and hence k is highly composite. Since, for instance $\tau(2^a) = a+1$ is arbitrarily large, such an m always exists, and hence, for each highly composite number we can find a larger one. By induction, there are infinitely many highly composite numbers.

7.2.17. Let a be the largest highly composite integer less than or equal to n. Note that $2a$ is less than or equal to $2n$ and has more divisors than a and hence $\tau(2a) > \tau(a)$. By Exercise 16, there must be a highly composite integer b with $a < b \leq 2a$. If $b \leq n$, this contradicts the choice of a. Therefore $n < b \leq 2n$. It follows that there must be a highly composite integer k with $2^m < k \leq 2^{m+1}$ for every nonnegative integer m. Therefore, there are at least m highly composite integers less than or equal to 2^m. Thus the mth highly composite integer is less than or equal to 2^m.

7.2.18. Let the prime power factorization of n be $2^{a_1}3^{a_2}\cdots p_k^{a_k}$, and suppose there is some pair $i < j$ such that $a_i < a_j$. Form a new integer $m = 2^{a_1}\cdots p_i^{a_j}\cdots p_j^{a_i}\cdots p_k^{a_k}$. Then $\tau(m) = \tau(n)$ since both integers have exactly the same set of exponents in their prime power factorization. But $n/m = (p_i^{a_i}p_j^{a_j})/(p_i^{a_j}p_j^{a_i}) = p_j^{a_j-a_i}/p_i^{a_j-a_i} > 1$, since $p_j > p_i$. Therefore $m < n$ with $\tau(m) = \tau(n)$ and so n is not highly composite, a contradiction. Therefore the sequence $a_1, a_2, \ldots, a_k$ is strictly decreasing.

7.2.19. If $n = 2^a3^b$ is highly composite, then by Exercise 18 we have $a \geq b$. Since $2^a3^b > 2^{a-1}b3^{b-1}5$, we must have $\tau(2^a3^b) > \tau(2^{a-1}b3^{b-1}5)$, that is $(a+1)(b+1) > 2ab$. Rearranging the inequality yields $(a-1)(b-1) < 2$. Hence, either $a = b = 2$ or $b < 2$. In the first case we have $n = 36$, which is highly composite. If $b = 1$, assume $a > 3$. We have $n = 2^a3 > 2^{a-1}3 \cdot 5$. Then $\tau(2^a3) = 2(a+1) > 4(a-2) = \tau(2^{a-1}3 \cdot 5)$. This reduces to $a < 5$, so we need only check $a = 1, 2, 3$ and 4, which correspond to the numbers $6, 12, 24$

and 48, all of which are highly composite. Finally, if $b = 0$ assume $a > 2$. Then $n = 2^a > 2^{a-2}3$, and so $tau(2^a) = a + 1 > 2(a - 1) = \tau(2^{a-2}3$, which reduces to $a < 5$. So we need only check the cases $a = 0, 1, 2, 3$ and 4, which correspond to the numbers $1, 2, 4, 8$ and 16. Of these, only $1, 2$, and 4 are highly composite. The complete list of the highly composite numbers of the form 2^a3^b is $1, 2, 4, 6, 12, 24, 36$ and 48.

7.2.20. We have $\sigma_3(4) = \sum_{d|4} d^3 = 1^3 + 2^3 + 4^3 = 1 + 8 + 64 = 73$. Similarly, we have $\sigma_3(6) = \sum d \mid 6d^3 = 1^3 + 2^3 + 3^3 + 6^3 = 1 + 8 + 27 + 216 = 252$. Likewise, we have $\sigma_3(12) = \sum_{d|12} d^3 = 1^3 = 2^3 + 3^3 + 4^3 + 6^3 + 12^3 = 1 + 8 + 27 + 64 + 216 + 1728 = 2044$.

7.2.21. We find that $\sigma_k(p) = \sum_{d|p} d^k = 1^k + p^k = 1 + p^k$.

7.2.22. We find that $\sigma_k(p^a) = \sum_{d|p^a} d^k = 1^k + p^k + (p^2)^k + \cdots + (p^a)^k = 1^k + p^k + p^{2k} + \cdots + p^{ak} = \frac{(p^{ak+1}-1)}{(p-1)}$.

7.2.23. Suppose that a and b are positive integers with $(a, b) = 1$. Then $\sum_{d|ab} d^k = \sum_{d_1|a,d_2|b} (d_1d_2)^k = \sum_{d_2|a} d_1^k \sum_{d_2|a} d_2^k = \sigma_k(a)\sigma_k(b)$.

7.2.24. Suppose that $n = \prod_{j=1}^r p_j^{a_j}$. Then since σ_k is multiplicative, $\sigma_k(n) = \prod_{j=1}^r \sigma_k(p_j^{a_j}) = \prod_{j=1}^r \frac{(p_j^{a_j k+1}-1)}{(p-1)}$.

7.2.25. Let $n = p^{a_1} \cdots p_r^{a_r}$. Then $\phi(n) = (p_1^{a_1} - p_1^{a_1-1}) \cdots (p_r^{a_r} - p_r^{a_r-1}) = \sum T_j$. Where $\sum T_j$ is this product is expanded. Each term T_j is of the form $T_j = (-1)^k p_1^{b_1} \cdots p_r^{b_r}$ where $b_i = a_i$ or $a_i - 1$. Note that each one of these terms is a divisor of n, and note that one of the terms is $p_1^{a_1} \cdots p_r^{a_r} = n$. Now since $\sigma(n)$ is the sum of the divisors of n, each of the terms T_j above appears in the sum $\sigma(n) = \sum_{d|n} d$, without the $(-1)^k$. Note that n also appears in this sum. Then we have $\sigma(n) + \phi(n) = \sum_{d|n} d + \sum T_j = 2n + \sum_{\substack{d|n \\ d<n}} d + \sum_{T_j \neq n} T_j$. Now if T_j is negative, then the $\mid T_j \mid$ appearing in the first sum will cancel it. But if T_j is positive we get two terms T_j in the last sum. Then we have $\sigma(n) + \phi(n) = 2n + \sum_{d|n, d<n, d\neq T_j} d + 2\sum_{T_j>0, t_j \neq n} T_j$. Since both of these last sums are nonnegative, we need them both to be zero in order to have a solution. In particular the expansion of $\phi(n) = (p_1^{a_1} - p_1^{a_1-1}) \cdots (p_r^{a_r} - p_r^{a_r-1})$ can have no positive terms other than n, and therefore we must have $\phi(n) = (p_1^{a_1} - p_1^{a_1-1})$. Now the term $d = 1$ appears in the first sum unless $T_j = -1$ for some j. Therefore $\phi(n) = p_1 - 1$ and so n is prime.

7.2.26. First we find an explicit formula for $f(n) = \prod_{d|n} d$. Notice that

$$n^{\tau(n)} = \prod_{d|n} n = \prod_{d|n} d(\frac{n}{d}) = \prod_{d|n} d \prod_{d|n} \frac{n}{d}.$$

Now as d runs through the divisors of n, so does $\frac{n}{d}$, so the last two products are the same. Then $n^{\tau(n)} = \left(\prod_{d|n} d\right)^2$ and therefore $f(n) = n^{\frac{\tau(n)}{2}}$. Now if $f(n) = f(m)$, with $n < m$, we have $n^{\frac{\tau(n)}{2}} = m^{\frac{\tau(m)}{2}}$ which implies there exists an integer a such that $a^s = n$ and $a^t = m$ for some nonnegative integers s and t. Since $n < m$, we have $s < t$ so any divisor of n is also a divisor of m, but not vice versa. Therefore $f(n) < f(m)$ a contradiction.

7.2.27. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ and let x and y be integers such that $[x, y] = n$. then $x \mid n$ and $y \mid n$ so we have $x = p_1^{b_1} p_2^{b_2} \cdots p_r^{a_r}$ and $y = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}$, where b_i and $c_i = 0, 1, 2, \ldots, a_i$. Since $[x, y] = n$, we must have $\max\{b_i, c_i\} = a_i$ for each i. Then one of b_i and c_i must be equal to a_i and the other can range over $0, 1, \ldots, a_i$. Therefore we have $2a_i + 1$ ways to choose the pair (b_i, c_i) for each i. Then in total, we can choose the exponents $b_1, b_2, \ldots b_r, c_1, \ldots, c_r$ in $(2a_1 + 1)(2a_2 + 1) \cdots (2a_r + 1) = \tau(n^2)$ ways.

7.2.28. If p is an odd prime, then $p^a > (1 + 2)^a = 1 + 2a + \ldots, +2^a > a + 1$ for integers $a \geq 1$. Also $2^a = (1 + 1)^a = 1 + a + \binom{a}{2} + \ldots + 1 > a + 1$ for $a \geq 2$. Therefore, for any prime $p, \tau(p^a) = (a + 1) < p^a$ unless $p^a = 2^1$. Then $\tau(n) = \prod \tau(p_i^{a_i}) < \prod p_i^{a_i} = n$ if any $p_i^{a_i}$ is different from 2. Thus $n > n_1 > n_2 > \cdots$ until, for some $r, n_r = 2$, and then $n_{r+1} = \tau(n_r) = \tau(2) = 2$ etc.

7.2.29. Suppose that n is composite. Then $n = ab$ where a and b are integers with $1 < a \leq b < n$. It follows that either $a \geq \sqrt{n}$ or $b \geq \sqrt{n}$. Consequently $\sigma(n) \geq 1 + a + b + n > 1 + \sqrt{n} + n > n + \sqrt{n}$. Conversely, suppose that n is prime. Then $\sigma(n) = n + 1$ so that $\sigma(n) \leq n + \sqrt{n}$. Hence $\sigma(n) > n + \sqrt{n}$ implies that n is composite.

7.2.30. If d is a divisor of n, then there is an integer k such that $n = dk$. It follows that $2^d - 1$ is a divisor of $2^n - 1$ since $2^n - 1 = (2^d - 1)(2^{d(k-1)} + 2^{d(k-2)} + \cdots + 2^d + 1)$. Hence $2^n - 1$ has at least as many divisors as n does, or in other words, $\tau(2^n - 1) \geq \tau(n)$.

7.2.31. For $n = 1$, the statement is true. Suppose that $\sum_{j=1}^{n-1} \tau(j) = 2\sum_{j=1}^{[\sqrt{n-1}]} \left[\frac{n-1}{j}\right] - [\sqrt{n-1}]^2$. For the induction step, it suffices to show that $\tau(n) = 2\sum_{j=1}^{[\sqrt{n-1}]} \left(\left[\frac{n}{j}\right] - \left[\frac{n-1}{j}\right]\right) = 2\sum_{\substack{j \leq [\sqrt{n-1}] \\ j|n}} 1$, which is true by the definition of $\tau(n)$, since there is one factor less than $\sqrt{n}$ for every factor greater than $\sqrt{n}$. Note that if n is a perfect square, we must add the term $2\sqrt{n} - (2\sqrt{n} - 1) = 1$ to the last two sums. For $n = 100$, we have $\sum_{j=1}^{100} \tau(j) = 2\sum_{j=1}^{10} \left[\frac{n}{j}\right] - 100 = 482$.

7.2.32. Let $f(n) = \frac{\sigma(n)}{n}$. Then f is multiplicative. $f(p^s) = \frac{\sigma(p^s)}{p^s} = \frac{p^s + p^{s-1} + \cdots + p + 1}{p^s} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^s}$. Then it is clear that $f(p^s) < f(p^t)$ whenever $s < t$. Suppose $n = p_1^{a_1} \cdots p_r^{a_r}$ and $d = p_1^{b_1} \cdots p_1^{b_r}$ is a divisor of n so that $b_i \leq a_i$ for $i = 1, 2, \ldots, r$. Then $f(n) = \prod f(p_i^{a_i}) \geq \prod f(p_i^{b_i}) = f(d)$. Since $a \mid ab$ this proves the first inequality. Now if $d \mid ab$, then d can be written as $d = xy$ where $x \mid a$ and $y \mid b$ in at least one way. Then every d in $\sum_{d|ab} d$ appears in the double sum $\sum_{x|a} \sum_{y|b} xy = \sum_{x|a} x \sum_{y|b} y$ at least once, which is to say $\sigma(ab) \leq \sigma(a)\sigma(b)$ which proves the second inequality.

7.2.33. We use the identity $\sum_{j=0}^{\min\{a,b\}} (p^{a+b-j} + p^{a+b-j-1} + \cdots + p^j) = (p^a + p^{a-1} + \cdots + 1)(p^b + p^{b-1} + \cdots + 1)$. If $a = \prod p^{a_i}$ and $b = \prod p^{b_i}$, then $\sigma(a)\sigma(b) = \prod (p^{a_i} + p^{a_i - 1} + \cdots + 1)(p^{b_i} + p^{b_i - 1} + \cdots + 1) = \prod \sum_{j=0}^{\min\{a_i, b_i\}} (p_i^{a_i + b_i - j} + p_i^{a_i + b_i - j - 1} + \cdots + p_i^j)$

7.2.34. By Theorem 7.8 we know that $\sum_{d|n} \tau(n)$ is multiplicative since $\tau(n)$ is multiplicative. By Exercise 46 in Section 7.1 it follows that $\left(\sum_{d|n} \tau(d)\right)^2$ is multiplicative and $\tau(n)^3$ is multiplicative. Moreover, by Theorem 7.8 we see that $\sum_{d|n} \tau(d)^3$ is multiplicative. Hence both sides of this identity are multiplicative, so to verify that the identity holds for all positive integers n it suffices to prove it holds for powers of primes. So suppose that $n = p^k$ where p is prime and k is a positive integer. Then $\left(\sum_{d|p^k} \tau(d)\right)^2 = \left(\sum_{j=0}^{k} \tau(p^j)\right)^2 = \left(\sum_{j=0}^{k} (j = 1)\right)^2 = \left(\sum_{j=1}^{k+1} k\right)^2 = \left(\frac{(k+1)(k+2)}{2}\right)^2$, using the formula for the sum of the first k positive integers given in Exercise 6 of Section 1.2. On the other hand, $\sum_{d|p^k} \tau(d)^3 = \sum_{j=0}^{K} \tau(p^j)^3 = \sum_{j=0}^{k} (j+1)^3 = \sum_{j=1}^{k+1} j^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$, using the formula for the sum of the cubes of the first k positive integers given in Exercise 8 of Section 1.2. It follows that both sides agree when n is a power of a prime. Since both sides are multiplicative,Theorem 6.1 shows that they agree for all positive integers n.

7.2.35. From Exercises 52 and 53 in Section 7.1 we know that the arithmetic function $f(n) = 2^{\omega(n)}$ is multiplicative. Further, since the Dirichlet product $h(n) = \sum_{d|n} 2^{\omega(d)} = f * g(n)$, where $g(n) = 1$ is also multiplicative, we know that $h(n)$ is also multiplicative. See Exercise 41 in Section 7.1. Since $\tau(n)$ and n^2 are multiplicative, so is $\tau(n^2)$. Therefore, it sufficient to prove the identity for n equal to a prime power, p^a. We have $\tau(p^{2a}) = (2a+1)$. On the other hand we have $\sum_{d|p^a} 2^{\omega(d)} = \sum_{i=0}^{a} 2^{\omega(p^i)} = 1 + \sum_{i=1}^{a} 2^1 = 2a + 1$, which completes the proof.

7.2.36. By Exercises 41 and 46 of Section 7.1, and Theorem 7.8, we know that both sides of the identity represent multiplicative functions. Therefore it suffices to prove the identity for prime powers. Suppose $n = p^a$. Then we have $\sum_{d|p^a} p^a \sigma(d)/d = \sum_{i=0}^{a} p^a \sigma(p^i)/p^i = \sum_{i=0}^{a} p^{a-i} \sigma(p^i) = \sum_{i=0}^{a} p^{a-i}(p^i + p^{i-1} + \cdots p + 1) = \sum_{i=0}^{a} \sum_{j=0}^{i} p^{a-j} = \sum_{j=0}^{a} \sum_{i=j}^{a} p^{a-j} = \sum_{j=0}^{a} (a - j + 1)p^{a-j} = \sum_{k=0}^{a} (k+1)p^k = \sum_{k=0}^{a} p^k \tau(p^k) =$

$\sum_{d|p^a} d\tau(d)$, which is the right hand side.

7.2.37. Let **M** be the matrix. Let **D** be the matrix with entries $\phi(1), \phi(2), \ldots, \phi(n)$ on the diagonal and zeros elsewhere. Let **A** be the matrix of 0's and 1's defined by the rule: If i divides j then the (i, j) entry is 1, otherwise, it is 0. Then **A** has all 0's below the main diagonal, and 1's on the main diagonal, therefore $\det(\mathbf{A}) = 1$. Check that $\mathbf{M} = \mathbf{ADA}^T$. Then $\det(\mathbf{M}) = 1 \cdot \det(\mathbf{D}) \cdot 1 = \phi(1)\phi(2)\cdots\phi(n)$.

7.2.38. We have $n \equiv 23 \pmod{24}$ so $n \equiv 2 \pmod 3$, and n is odd. If every prime dividing n was $\equiv 1 \pmod 3$ then n would be $\equiv 1 \pmod 3$ so n has a prime divisor $p \equiv 2 \pmod 3$, say $p^a \| n$. If a were even, then $p^a \equiv 1 \pmod 3$ and then so is n, so there exists a prime divisor with a odd. Then $\sigma(p^a) \equiv p^a + p^{a-1} + \cdots + p + 1 \equiv 2 + 1 + 2 + 1 + \cdots + 2 + 1 \equiv 3 + 3 + \cdots + 3 \equiv 0 \pmod 3$, so $3 \mid \sigma(n)$. Similarly $n \equiv 7 \pmod 8$ so either $p^a \| n$ with $p \equiv 7 \pmod 8$ and a odd or $q^b \| n$ and $r^c \| n$ with $q \equiv 3 \pmod 8$ and $r \equiv 5 \pmod 8$ and b and c odd. In the first case $\sigma(p^a) = p^a + p^{a-1} + \cdots + p + 1 \equiv 7 + 1 + 7 + 1 + \cdots + 7 + 1 \equiv 0 \pmod 8$ and so $8 \mid \sigma(n)$. In the second case $\sigma(q^b) \equiv 3 + 1 + 3 + 1 + \cdots + 3 + 1 \equiv 0 \pmod 4$ so $4 \mid \sigma(n)$ and $\sigma(r^c) \equiv 5 + 1 + 5 + 1 + \cdots + 5 + 1 \equiv 0 \pmod 2$ so $2 \mid \frac{\sigma(n)}{4}$, hence $8 \mid \sigma(n)$. Since $3 \mid \sigma(n)$ we have $24 \mid \sigma(n)$.

7.2.39. Suppose there are infinitely many pairs of twin primes. Let p and $p + 2$ be a pair of twin primes. Then $\sigma(p) = p + 1$ and $\phi(p + 2) = p + 2 - 1 = p + 1$. So each pair of twin primes is a solution to the equation. Next Suppose there are infinitely many primes of the form $2^q - 1$ with q prime. Then $\phi(2^{q+1}) = 2^q$ and $\sigma(2^q - 1) = 2^q - 1 + 1 = 2^q$. So once again we have infinitely many solutions.

7.2.40. By Theorem 7.8, the function $F(n) = \sum_{d|n} \phi(d)$ is multiplicative. Therefore, to prove Theorem 7.7, it suffices to show only that the identity holds for n a prime power. Suppose $n = p^a$. Then $F(n) = \sum_{d|p^a} \phi(d) = \phi(1) + \phi(p) + \phi(p^2) + \cdots + \phi(p^a) = 1 + (p - 1) + (p^2 - p) + (p^3 - p^2) + \cdots + (p^a - p^{a-1})$. This is a telescoping sum, and every term cancels expect p^a, and so $F(p^a) = p^a$ as desired.

7.3. Perfect Numbers and Mersenne Primes

7.3.1. From the table on page 264, the first six Mersenne primes are given by $2^p - 1$ with p = 2,3,5,7,13, and 17. Then Theorem 6.9 gives the first six even perfect numbers as $2^1(2^2 - 1) = 6$;$2^2(2^3 - 1) = 28$; $2^4(2^5 - 1) = 496$; $2^6(2^7 - 1) = 8,128$; $2^{12}(2^{13} - 1) = 33,550,336$; and $2^{16}(2^{17} - 1) = 8589869056$.

7.3.2. $2^{19} - 1$ is prime, so the seventh even perfect number is 137438691328. $2^{23} - 1$ and $2^{29} - 1$ are composite, but $2^{31} - 1$ is prime, so the eighth even perfect number is 2305843008139952128.

7.3.3. a. By the difference of cubes factorization we have $2^15 - 1 = (2^5)^3 - 1 = (2^5 - 1)(2^10 + 2^5 + 1)$, so $2^5 - 1 = 31$ is a factor.

b. Since $7 \mid 91, 127 = 2^7 - 1 \mid 2^91 - 1$.

c. Since $7 \mid 1001, 127 = 2^7 - 1 \mid 2^1001 - 1$.

7.3.4. a. Since $3 \mid 111, 7 = 2^3 - 1 \mid 2^111 - 1$.

b. Since $17 \mid 289, 131071 = 2^17 - 1 \mid 2^289 - 1$.

c. Since $11 \mid 46189, 2047 = 2^11 - 1 \mid 2^46189 - 1$.

7.3.5. We have $\sigma(12) = 28, \sigma(18) = 39, \sigma(20) = 42, \sigma(24) = 60, \sigma(30) = 72$ and $\sigma(36) = 91$.

7.3.6. If $n = p^a q^b$, then $\frac{\sigma(n)}{n} = \frac{p^{a+1}-1}{(p-1)pa} \cdot \frac{q^{b+1}-1}{(q-1)q^b} < \frac{p}{p-1} \cdot \frac{q}{q-1} \leq \frac{3}{2} \cdot \frac{5}{4} < 2$ so n has at least 3 distinct prime factors. If $p < q$ are primes, then check that $\frac{\sigma(p^a)}{p^a} > \frac{\sigma(q^a)}{q^a}$, so we may take the 3 prime factors to be 3,5, and 7. Try the possibilities in order: $3 \cdot 5 \cdot 7, 3^2 \cdot 5 \cdot 7, 3^3 \cdot 5 \cdot 7, 3^2 5^2 \cdot 7, 3 \cdot 5 \cdot 7 \cdot 11$, etc. and find that $\sigma(3^3 \cdot 5 \cdot 7) = \sigma(945) = 1920$ is the smallest example.

7.3.7. Suppose that $n = p^k$ where p is prime and k is a positive integer. Then $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$. Note that $2p^k - 1 < p^{k+1}$ since $p \geq 2$. It follows that $p^{k+1} - 1 < 2(p^{k+1} - p^k) = 2p^k(p-1)$, so that $\frac{(p^{k+1}-1)}{p-1} < 2p^k = 2n$. It follows that $n = p^k$ is deficient.

7.3.8. Let m and n be integers and write $mn = \prod p_i^{a_i}$ and $n = \prod p_i^{b_i}$ where the p_i are distinct primes and $a_i \geq b_i$. Then $\frac{\sigma(mn)}{mn} = \prod \frac{p_i - p_i^{-a_i}}{p_i - 1} > \prod \frac{p_i - p_i^{b_i}}{p_i - 1} = \frac{\sigma(n)}{n}$. So if mn is deficient, then $\frac{\sigma(n)}{n} < \frac{\sigma(mn)}{mn} < 2$, so n is also deficient.

7.3.9. Suppose that n is abundant or perfect. Then $\sigma(n) \geq 2n$. Suppose that $n \mid m$. Then $m = nk$ for some integer k. The divisors of m include the integers kd and $d \mid n$. Hence $\sigma(m) \geq \sum_{d|n}(k+1)d = (k+1)\sum_{d|n} d = (k+1)\sigma(n) \geq (k+1)2n > 2kn = 2m$. Hence m is abundant.

7.3.10. $\frac{\sigma(2^{m-1})\sigma(2^m-1)}{n} = \frac{(2^m-1)\sigma(2^m-1)}{2^{m-1}2^{(m}-1)} = \frac{\sigma(2^m-1)}{2^{m-1}} > 2$, since $\sigma(n) \geq n+1$ with equality if and only if n is prime.

7.3.11. If p is any prime, then $\sigma(p) = p + 1 < 2p$, so p is deficient. Since there are infinitely many primes, we must have infinitely many deficient numbers.

7.3.12. The solution to Exercise 10 provides an infinite set of even abundant numbers, since $2^m - 1$ is composite whenever m is composite.

7.3.13. See Exercises 6 and 9. For a positive integer a let $n = 3^a 5 \cdot 7$ and compute $\sigma(n) = \sigma(3^a 5 \cdot 7) = (3^{a+1} - 1)/(3-1)(5+1)(7+1) = (3^{a+1} - 1)24 = 3^{a+1}24 - 24 = 2 \cdot 3^a(36) - 24 = 2 \cdot 3^a(35) + 2 \cdot 3^a - 24 = 2n + 2 \cdot 3^a - 24$, which will be greater than $2n$ whenever $a \geq 3$. This demonstrates infinitely many odd abundant integers.

7.3.14. We have $\sigma(p^a q^b) = \frac{(p^{a+1}-1)(q^{b+1}-1)}{(p-1)(q-1)} < \frac{(p^{a+1})(q^{b+1})}{(p-1)(q-1)} = \frac{p}{p-1}\frac{q}{q-1}p^a q^b < 2p^a q^b$. Therefore $n = p^a q^b$ is deficient.

7.3.15. a. The prime factorizations of 220 and 284 are $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$. Hence $\sigma(220) = \sigma(2^2)\sigma(5)\sigma(11) = 7 \cdot 6 \cdot 12 = 504$ and $\sigma(284) = \sigma(2^2)\sigma(71) = 7 \cdot 72 = 504$. Since $\sigma(220) = \sigma(284) = 220 + 284 = 504$, it follows that 220 and 284 form an amicable pair.

b. The prime factorizations of 1184 and 1210 are $1184 = 2^5 \cdot 37$ and $1210 = 2 \cdot 5 \cdot 11^2$. Hence $\sigma(1184) = \sigma(2^5)\sigma(37) = 63 \cdot 38 = 2394$ and $\sigma(1210) = \sigma(2)\sigma(5)\sigma(11^2) = 3 \cdot 6 \cdot 133 = 2394$. Since $\sigma(1184) = \sigma(1210) = 1184 + 1210 = 2394$, 1184 and 1210 form an amicable pair.

c. The prime factorizations of 79750 and 88730 are $79750 = 2 \cdot 5^3 \cdot 11 \cdot 29$ and $88730 = 2 \cdot 5 \cdot 19 \cdot 467$. Hence $\sigma(79750) + \sigma(2)\sigma(5^3)\sigma(11)\sigma(29) = 3 \cdot 156 \cdot 12 \cdot 30 = 168480$ and similarly $\sigma(88730) = \sigma(2)\sigma(5)\sigma(19)\sigma(467) = 3 \cdot \cdot 6 \cdot 20 \cdot 468 = 168480$. Since $\sigma(79750) = \sigma(88730) = 79750 + 88730 = 168480$ it follows that 79750 and 88730 form an amicable pair.

7.3.16. a. $\sigma\left(2^n(3 \cdot 2^{n-1} - 1)(3 \cdot 2^n - 1)\right) = (2^{n+1}-1)3 \cdot 2^{n-1} \cdot 3 \cdot 2^n = (2^{n+1}-1) \cdot 3^2 \cdot 2^{2n-1}$ and $\sigma\left(2^n(3^2 \cdot 2^{2n-1} - 1)\right) = (2^{n+1} - 1) \cdot 3^2 \cdot 2^{2n-1}$. Also $2^n(3 \cdot 2^{n-1} - 1)(3 \cdot 2^n - 1) + 2^n(3^2 \cdot 2^{2n-1} - 1) = 2^n(3^2 2^{2n-1} - 3 \cdot 2^{n-1} - 3 \cdot 2^n + 1 + 3^2 2^{2n-1} - 1) = 2^n(3^2 2^{2n-1} - 3 \cdot 2^{n-1}(1+2) + 3^2 2^{2n-1}) = 2^n 3^2(2^{2n-1} - 2^{n-1} + 2^{2n-1}) = 2^n 3^2(2^{2n} - 2^{n-1}) = 2^{2n-1}3^2(2^{n+1} - 1)$.

b. We find the following amicable pairs, $(220, 284), (17296, 18416), (9363584, 9437056)$, corresponding to $n = 2, 4$, and 7 in the formulae in part (a).

7.3.17. Since $120 = 2^3 \cdot 3 \cdot 5$ and σ is multiplicative, we have $\sigma(120) = \sigma(2^3 \cdot 3 \cdot 5) = \sigma(2^3)\sigma(3)\sigma(5) = 15 \cdot 4 \cdot 6 = 360$. Since $\sigma(120) = 360 = 3 \cdot 120$, it follows that 120 is 3-perfect.

7.3.18. $\sigma(2^5 3^2 5 \cdot 7) = \frac{2^6-1}{2-1} \cdot \frac{3^3-1}{3-1} \cdot (5+1)(7+1) = 120960 = 4 \cdot 30240$.

7.3.19. $\sigma(2^7 3^4 5 \cdot 7 \cdot 11^2 \cdot 17 \cdot 19) = \frac{2^8-1}{2-1} \cdot \frac{3^5-1}{3-1}(5+1)(7+1)\frac{11^3-1}{11-1}(17+1)(19+1) = 255 \cdot 121 \cdot 6 \cdot 8 \cdot 133 \cdot 18 \cdot 20 = 5 \cdot 14182439040$.

7.3.20. Suppose that n is 3-perfect and 3 does not divide n. Then $\sigma(3n) = \sigma(3)\sigma(n) = 4 \cdot 3n = 12n = 4 \cdot 3n$. Hence $3n$ is 4-perfect.

7.3.21. Suppose that n is 3-perfect and 3 does not divide n. Then $\sigma(3n) = \sigma(3)\sigma(n) = 4 \cdot 3n = 12n = 4 \cdot 3n$. Hence $3n$ is 4-perfect.

7.3.22. For example, $\sigma(2^4 3^3 5^2 7) = \sigma(75600) = 307520 > 4 \cdot 75600$.

7.3.23. For example, $\sigma(2^6 3^4 5^2 7^2 11 \cdot 13) = \sigma(908107200) = 4561786152 > 5 \cdot 908107200$.

7.3.24. The argument in the solution to Exercise 6 shows that a multiple of ak-abundant number is k-abundant. So it suffices that there are k-abundant numbers for arbitrarily large k. Let $n_r = \prod_{i=1}^{r} p_i$ be the product of the first r primes. Then $\lim_{n\to\infty} \frac{\sigma(n_r)}{n_r} = \lim_{n\to\infty} \prod_{i=1}^{r}(1 + \frac{1}{p_i}) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

7.3.25. We have $\sigma(\sigma(16)) = \sigma(31) = 32 = 2 \cdot 16$. Hence 16 is superperfect.

7.3.26. We compute $\sigma\left(\sigma(2^q)\right) = \sigma(2^{q+1} - 1) = (2^{q+1} - 1) + 1 = 2^{q+1} = 2 \cdot 2^q$. Therefore n is superperfect.

7.3.27. Certainly if r and s are integers, then $\sigma(rs) \geq rs + r + s + 1$. Suppose $n = 2^q t$ is superperfect with t odd and $t > 1$. Then $2n = 2^{q+1}t = \sigma\left(\sigma(2^q t)\right) = \sigma\left(\left(2^{q+1} - 1\right)\sigma(t)\right) \geq (2^{q+1} - 1)\sigma(t) + (2^{q+1} - 1) + \sigma(t) + 1 > 2^{q+1}\sigma(t) \geq 2^{q+t}(t + 1)$. Then $t > t + 1$, a contradiction. Therefore we must have $n = 2^q$, in which case we have $2n = 2^{q+1} = \sigma\left(\sigma(2^q)\right) = \sigma\left(2^{q+1} - 1\right) = \sigma(2n - 1)$. Therefore $2n - 1 = 2^{q+1} - 1$ is prime.

7.3.28. Suppose $\sigma\left(\sigma(p^2)\right) = 2p^2$, then $\sigma(p^2 + p + 1) = 2p^2$. If $p^2 + p + 1$ has three distinct prime factors, then for one of them we have $\sigma(q^a) = 2$, which is impossible. Therefore $p^2 + p + 1 = q^a r^b$ with q and r primes. Then $\sigma(q^a r^b) = 2p^2$, so $q^a + \cdots + q + 1 = p$ and $r^b + \cdots + r + 1 = 2p$. Then $p \equiv 1 \pmod{q}$ and $p^2 = p = 1 = q^a r^b \equiv 0 \pmod{q}$, so $3 \equiv 0 \pmod{q}$ so $q = 3$. But $p^2 + p + 1 \equiv 3 \pmod{9}$ since $p \equiv 1 \pmod{3}$, therefore $a = 1$ and $\sigma(q^a) = \sigma(3) = 4$, a contradiction.

7.3.29. a. By Theorem 7.12 any divisor of $M_7 = 127$ must be of the form $14k + 1$ where k is a positive integer. There are no primes of this form less than $\sqrt{127}$ so $M_7 = 127$ is prime.

b. By Theorem 7.12 any divisor of $M_{11} = 2047$ must be of the form $22k + 1$ where k is a positive integer. The only prime of this form less than $\sqrt{2047}$ is 23. Since 23 does not divide 2047, it follows that $M_{11} = 2047$ is prime.

c. By Theorem 7.12 any divisor of $M_{17} = 131071$ must be of the form $34k + 1$ where k is a positive integer. The primes of this form less than $\sqrt{131071} < 363$ are 103, 137, 239, and 307, but none of these divide 131071. Hence $M_{17} = 131071$ is prime.

d. By Theorem 7.12 any divisor of $M_{29} = 536870911$ must be of the form $58k + 1$ where k is a positive integer. We first note neither that 59 nor 107 divides 536870911. However, $233 = 58 \cdot 4 = 1$ does divide 536870911 since $536870911 = 233 \cdot 2304167$. Hence M_{29} is not prime.

7.3.30. a. Note that $M_3 = 2^3 - 1 = 7$. We have $r_1 = 4$, and $r_2 \equiv 4^2 - 2 = 14 \equiv 0 \pmod{7}$. Since $r_2 = 14 \equiv 0 \pmod{7}$ if follows that $M_3 = 7$ is prime.

b. Note that $M_7 = 2^7 - 1 = 127$. we have $r_1 = 4$, $r_2 \equiv 4^2 - 2 = 14 \pmod{127}$, $r_3 \equiv 14^2 - 2 = 194 \equiv 67 \pmod{127}$, $r_4 \equiv 67^2 - 2 = 4487 \equiv 42 \pmod{127}$, $r_5 \equiv 42^2 - 2 = 1762 \equiv 111 \pmod{127}$,and $r_6 \equiv 111^2 - 2 = 12319 = 0 \pmod{127}$. Since $r_{7-1} = r_6 \equiv 0 \pmod{M_7}$, it follows that $M_7 = 127$ is prime.

c. $M_{11} = 2^{11} - 1 = 2047$. Then $r_1 = 4, r_2 = 16 - 2 = 14, r_3 = 196 - 2 = 194, r_4 \equiv 194^2 - 2 \equiv 788 \pmod{2047}, r_s \equiv 788^2 - 2 \equiv 701 \pmod{2047}, r_6 \equiv 701^2 - 2 \equiv 119 \pmod{2047}, r_7 \equiv 119^2 \equiv$

$1877 \pmod{2047}, r_8 \equiv 1877^2 - 2 \equiv 240 \pmod{2047}, r_9 \equiv 240^2 - 2 \equiv 282 \pmod{2047}, r_{10} \equiv 282^2 - 2 \equiv 1736 \not\equiv 0 \pmod{2047}$, therefore 2047 is not prime.

d. $M_{13} = 8191$. $r_1 = 4, r_2 = 14, r_3 = 194, r_4 = 4870, r_5 = 3953, r_6 = 5970, r_7 = 1857, r_8 = 36, r_9 = 1294, r_{10} = 3470, r_{11} = 128, r_{12} = 0$,so M_{13} is prime.

7.3.31. $M_n(M_n+2) = (2^n-1)(2^n+1) = 2^{2n}-1$. If $2n+1$ is prime then $\phi(2n+1) = 2n$ and $2^{2n} \equiv 1 \pmod{2n+1}$. Then $(2n+1) \mid 2^{2n} - 1 = M_n(M_n+2)$. Therefore $(2n+1) \mid M_n$ or $(2n+1) \mid (M_n+2)$.

7.3.32. a. Suppose that n is an odd perfect number with $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$. Then $\sigma(n) = 2n = \prod_{j=1}^{m}\sigma(p_j^{k_j})$. Since n is odd it follows that $2n \equiv 2 \pmod 4$ and consequently $\sigma(n) = 2n$ is divisible by 2 but not by 4. Hence $\sigma(n) \equiv 2 \pmod 4$). It follows that exactly one of the terms $\sigma(p_j^{k_j})$ is even and not divisible by 4 and all other terms are odd. We now relabel the terms so that $\sigma(p_1^{k_1})$ is the even term.

Suppose that $p_i \equiv -1 \pmod 4$. Note that $\sigma(p_i^{k_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{k_i} \equiv 1 + (-1) + 1 + \cdots + (-1)^{k_i} = 0 \pmod 4$ if k_i is odd and $1 \pmod 4$ if k_i is even. It follows that p_1 is not congruent to -1 modulo 4 and that all primes congruent to -1 modulo 4 occur to an even power in the prime factorization of n.

Now suppose that $p_i \equiv 1 \pmod 4$. We have $\sigma(p_i^{k_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{k_i} \equiv 1 + 1^1 + 1^2 + \cdots + 1^{k_i} = k_i = 1 \pmod 4$. We know that $p_1 \equiv 1 \pmod 4$ and that $\sigma(p_1^{k_i}) \equiv 2 \pmod 4$. It follows that $k_i \equiv 1 \pmod 4$. For the primes p_i in the factorization other than p_i that are congruence to 1 modulo 4 it follows that $k_i \equiv$ or $2 \pmod 4$ since $\sigma(k_i) = k_i + 1$ is odd. Summarizing we see that the prime factorization of n consists of a prime congruent to 1 modulo 4 to a power which is congruent to 1 modulo 4 and a product of even powers of other odd primes. It follows that $n = p^a m^2$ where p is prime and $p \equiv a \equiv 1 \pmod 4$.

b. We see from part a) that $n = p^a m^2$ where p is prime and $p \equiv a \equiv 1 \pmod 4$. It follows that $p^a \equiv 1^1 \equiv 1 \pmod 4$ and since m is odd, $m^2 \equiv 1 \pmod 4$. Hence $n = 1 \pmod 4$.

7.3.33. Since m is odd, $m^2 \equiv 1 \pmod 8$, so $n = p^a m^2 \equiv p^a \pmod 8$. By Exercise 32 (a), $a \equiv 1 \pmod 4$, so $p^a \equiv p^{4k}p \equiv p \pmod 8$,since p^{4k} is an odd square. Therefore $n = p \pmod 8$.

7.3.34. Let $n = 3^a5^b7^c\prod p_i^{a_i}$. Since 3 and 7 $\not\equiv 1 \pmod 4$, by Exercise 32 (a), we must have $a, c \geq 2$. Then $2 < \frac{13\cdot6\cdot57}{9\cdot5\cdot49} = \frac{\sigma(3^2\cdot5\cdot7^2)}{3^2\cdot5\cdot7^2} \leq \frac{\sigma(n)}{n} = 2$, a contradiction.

7.3.35. First suppose that $n = p^a$ where p is prime and a is a positive integer. Then $\sigma(n) = \frac{p^{a+1}-1}{p-1} < \frac{p^{a+1}}{p-1} = \frac{np}{p-1} = \frac{n}{1-\frac{1}{p}} \leq \frac{n}{\frac{2}{3}} < \frac{3n}{2}$ so that $\sigma(n) \neq 2n$ and n is not perfect. Next suppose that $n = p^aq^b$ where a and b are primes and a and b are positive integers. Then $\sigma(n) = \frac{p^{a+1}-1}{p-1}\cdot\frac{q^{b+1}-1}{q-1} < \frac{p^{a+1}q^{b+1}}{(p-1)(q-1)} = \frac{npq}{(p-1)(q-1)} = \frac{n}{(1-\frac{1}{p})(1-\frac{1}{q})} \leq \frac{n}{(\frac{2}{3})(\frac{4}{5})} = \frac{15n}{8} < 2n$. Hence $\sigma(n) \neq 2n$ and n is not perfect.

7.3.36. Suppose $n = p^aq^br^c$. Then $\frac{\sigma(n)}{n} < \frac{\sigma(pqr)}{pqr} = \frac{(p+1)(q+1)(r+1)}{pqr}$. If (p, q, r) is not $(3, 5, 7)$ or$(3, 5, 11)$, then the last expression is < 2 and so $\frac{\sigma(n)}{n} < 2$ and n is not perfect. Exercise 28 eliminates $(3, 5, 7)$. Exercise 32 (a) gives that if $n = 3^a5^b11^c$, then $a, c, \geq 2$. So $\frac{\sigma(n)}{n} < \frac{\sigma(9\cdot5\cdot121)}{9\cdot5\cdot121} < 2$.

7.3.37. By Exercise 11 of Section 7.2 it follows that the product of all positive divisors of an integer n is $n^{\frac{\tau(n)}{2}}$. If the product of all divisors of n other than n is n^2 then $n^{\frac{\tau(n)}{2-1}} = n^2$ so that $\frac{\tau(n)}{2} = 3$. This implies that $\tau(n) = 6$. The integers with $\tau(n) = 6$ are those of the form p^5 and p^2q where p and q are primes.

7.3.38. a. Suppose that $n = n_1$ is perfect. Then $n_2 = \sigma(n) - n = 2n - n = n$, so that $n_j = \sigma(n_{j-1}) - n_{j-1} = \sigma(n) - n = n$ for all $j \geq 1$. It follows that $n = n_1 = n_2 = n_3 = \cdots$.

b. Suppose that m and n are an amicable pair. Then $\sigma(m) = \sigma(n) = m + n$. If $n_1 = m$ then $n_2 = \sigma(n_1) - n_1 = (m + n) - m = n$. We see that $n_3 = \sigma(n_2) - n_2 = \sigma(m) - m = (m + n) - n = m$. We

see that the terms n_j are periodic, with $n1 = m$, $n_2 = n$, $n_3 = m$, $n_4 = n$, and so on.

c. Let $n_1 = 12496 = 2^4 \cdot 11 \cdot 71$. Then $n_2 = \sigma(n_1) - n_1 = 26784 - 12496 = 14288$, since $\sigma(12496) = \sigma(2^4)\sigma(11)\sigma(71) = 31 \cdot 12 \cdot 72 = 26784$.

Iterating, we find that $n_3 = \sigma(n_2) - n_2 = 29760 - 14288 = 15472$, since $14288 = 2^4 \cdot 19 \cdot 47$ and $\sigma(14288) = \sigma(2^4)\sigma(19)\sigma(47) = 31 \cdot 20 \cdot 48 = 29760$.

Continuing, we see that $n_4 = \sigma(n_3) - n_3 = 30008 - 15472 = 14536$ since $15472 = 2^4 \cdot 967$ and $\sigma(15472) = \sigma(2^4\sigma(967) = 31 \cdot 968 = 30008$.

Carrying the computation to the next stage, we see that $n_5 = \sigma(n_4) - n_4 = 28800 - 14536 = 14264$. Since $14536 = 2^3 \cdot 23 \cdot 79$ and $\sigma(14536) = \sigma(2^3)\sigma(23)\sigma(79) = 15 \cdot 24 \cdot 80 = 28800$.

The next iteration shows that $n_6 = \sigma(n_5) - n_5 = 26760 - 14264 = 12496$. Since $14264 = 2^3 \cdot 1783$ and $\sigma(14264) = \sigma(2^3)\sigma(1784) = 15 \cdot 1784 = 26760$.

Since $n_6 = n_1$, it follows that the sequence $n_1, n_2, n_3, n_4 \cdots$ is periodic with period equal to five, with $n_j = n_{j-5}$ for $j = 6, 7, 8, \cdots$.

7.3.39. Suppose $M_n = 2^n - 1 = a^k$, with n and k integers greater than 1. Then a must be odd. If $k = 2j$, then $2^n - 1 = (a^j)^2$. Since $n > 1$ and the square of an odd integer is congruent to 1 modulo 4, reduction of the last equation modulo 4 yields the contradiction $-1 \equiv 1 \pmod 4$, therefore k must be odd. Then $2^n = a^k + 1 = (a+1)(a^{k-1} - a^{k-2} + \cdots + 1)$. So $a + 1 = 2^m$ for some integer m. Then $2^n - 1 = (2^m - 1)^k$. Now $n > mk$ so reduction modulo 2^{2m} gives $-1 \equiv -k2^m - 1 \pmod{2^{2m}}$ or, since k is odd, $2^m \equiv 0 \pmod{2^{2m}}$, a contradiction.

7.4. Möbius Inversion

7.4.1. a. Since $12 = 2^2 3$ is not squarefree, $\mu(12) = 0$.

b. Since $15 = 3 \cdot 5$ is the product of two primes, $\mu(15) = (-1)^2 = 1$.

c. Since $30 = 2 \cdot 3 \cdot 5$ is the product of three primes, $\mu(30) = (-1)^3 = -1$.

d. Since $50 = 2 \cdot 5^2$ is not squarefree, $\mu(50) = 0$.

e. Since $1001 = 7 \cdot 11 \cdot 13$ is the product of three primes, $\mu(1001) = (-1)^3 = -1$.

f. Since $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ is the product of six primes, $\mu(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) = (-1)^6 = 1$.

g. Since $4|10!$, we know $10!$ is not squarefree, so we have $\mu(10!) = 0$.

7.4.2. a. Since $33 = 3 \cdot 11$, the product of two primes, $\mu(33) = (-1)^2 = 1$.

b. Since $105 = 3 \cdot 5 \cdot 7$, we have $\mu(105) = (-1)^3 = -1$.

c. Since $110 = 2 \cdot 5 \cdot 11$, we have $\mu(110) = (-1)^3 = -1$.

d. Since $2^2 \mid 740$, we have $\mu(740) = 0$.

e. Since $3^2 \mid 999$, we have $\mu(999) = 0$.

f. Since $3 \cdot 7 \cdot 13 \cdot 19 \cdot 23$ is the product of five distinct primes, $\mu(3 \cdot 7 \cdot 13 \cdot 19 \cdot 23) = (-1)^5 = -1$.

g. Let $n = 10!/(5!)^2$. The highest power of 2 dividing the numerator is 2^8. The highest power of 2 dividing $5!$ is 2^3, so the highest power of 2 dividing the denominator is 2^6. Therefore $2^2 \mid n$, and so $\mu(n) = 0$.

7.4.3. Since 4 divides $100, 104$, and 108, the value of μ for each of these is 0. Since $101, 103, 107$, and 109 are prime, μ for each of these values is -1. Then $\mu(102) = \mu(2 \cdot 3 \cdot 17) = (-1)^3 = -1$, $\mu(105) = \mu(3 \cdot 5 \cdot 7) =$

$(-1)^3 = -1$, $\mu(106) = \mu(2 \cdot 53) = (-1)^2 = 1$, and $\mu(110) = \mu(2 \cdot 5 \cdot 11) = (-1)^3 = -1$.

7.4.4. Since 4 divides $1000, 1004$, and 1008, the value of μ at these numbers is 0. From Exercise 1(e), $\mu(1001) = -1$. Since $1002 = 2 \cdot 3 \cdot 167$, we have $\mu(1002) = -1$. Since $1003 = 17 \cdot 59$, we have $\mu(1003) = 1$. Since $1005 = 3 \cdot 5 \cdot 67$, we have $\mu(1005) = -1$. Since $1006 = 2 \cdot 503$, we have $\mu(1006) = 1$. Since $1007 = 19 \cdot 53$, we have $\mu(1007) = 1$. Since 1009 is prime, we have $\mu(1009) = -1$. Since $1010 = 2 \cdot 5 \cdot 101$, we have $\mu(1010) = -1$.

7.4.5. Such n must be the product of an even number of distinct primes. The only product of zero primes is 1. The products of 2 primes which are less than or equal to 100, are $2 \cdot 3 = 6$, $2 \cdot 5 = 10$, $2 \cdot 7 = 14$, $2 \cdot 11 = 22$, $2 \cdot 13 = 26$, $2 \cdot 17 = 34$, $2 \cdot 19 = 38$, $2 \cdot 23 = 46$, $2 \cdot 29 = 58$, $2 \cdot 31 = 62$, $2 \cdot 37 = 74$, $2 \cdot 41 = 82$, $2 \cdot 43 = 86$, $2 \cdot 47 = 94$, $3 \cdot 5 = 15$, $3 \cdot 7 = 21$, $3 \cdot 11 = 33$, $3 \cdot 13 = 39$, $3 \cdot 17 = 51$, $3 \cdot 19 = 57$, $3 \cdot 23 = 69$, $3 \cdot 29 = 87$, $3 \cdot 31 = 93$, $5 \cdot 7 = 35$, $5 \cdot 11 = 55$, $5 \cdot 13 = 65$, $5 \cdot 17 = 85$, $5 \cdot 19 = 95$, $7 \cdot 11 = 77$, and $7 \cdot 13 = 91$. Since the product of the four smallest primes is $210 > 100$, the above list is exhaustive.

7.4.6. The product of the 4 smallest primes is 210, so we need only find those integers with exactly one or three distinct prime factors. The primes between 100 and 200 are given in the table in the appendix. The integers in this range which are products of three primes are: 102, 105, 110, 114, 130, 138, 154, 165, 170, 174, 182, 186, 190, and 195.

7.4.7. Starting with the values $\mu(1) = \mu(6) = \mu(10) = 1$, $\mu(2) = \mu(3) = \mu(5) = \mu(7) = -1$, and $\mu(4) = \mu(8) = \mu(9) = 0$, we compute $M(1) = 1$, $M(2) = 1 + (-1) = 0$, $M(3) = 0 + (-1) = -1$, $M(4) = -1 + 0 = -1$, $M(5) = -1 + (-1) = -2$, $M(6) = -2 + 1 = -1$, $M(7) = -1 + -1 = -2$, $M(8) = -2 + 0 = -2$, $M(9) = -2 + 0 = -2$, and $M(10) = -2 + 1 = -1$,

7.4.8. If p is an odd prime between 1 and 50, then both $\mu(p) = -1$ and $\mu(2p) = 1$ are in the sum, and add to zero, so we need not consider these numbers in the sum. We also need not consider numbers which are not squarefree. There are 10 primes between 50 and 100 which contribute, collectively -10 to the sum. This leaves only $1, 2, 15, 21, 30, 33, 35, 39, 42, 51, 55, 57, 65, 66, 69, 70, 77, 78, 85, 87, 91, 93$, and 95. But 16 of these are the product of two primes, so they contribute 16 to the sum. This leaves $1, 2, 30, 42, 66, 70$, and 78. The last 5 of these are the products of 3 primes, and so contribute -5 to the sum. Then we have $M(100) = \mu(1) + \mu(2) - 10 + 16 - 5 = 1 - 1 + 1 = 1$.

7.4.9. Since $\mu(n)$ is 0 for nonsquarefree n, 1 for n a product of an even number of distinct primes and -1 for n a product of a odd number of distinct primes, the sum $M(n) = \sum_{i=1}^{n} \mu(i)$ is unaffected by the nonsquarefree numbers, but counts 1 for every even product and -1 for every odd product. Thus $M(n)$ counts how many more even products than odd products there are.

7.4.10. Since $n, n+1, n+2, n+3$ form a complete residue system modulo 4, one of them is divisible by 4, and so not squarefree. Therefore one of the factors in $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3)$ is 0, making the product 0.

7.4.11. For any nonnegative integer k, the numbers $n = 36k + 8$ and $n + 1 = 36k + 9$ are consecutive and divisible by $4 = 2^2$ and $9 = 3^2$ respectively. Therefore $\mu(36k + 8) + \mu(36k + 9) = 0 + 0 = 0$.

7.4.12. If n is a solution to the system of congruences $n \equiv -1 \pmod 4, n \equiv 0 \pmod 9, n \equiv 1 \pmod{25}$, then $4 \mid (n + 1)$, $9 \mid n$ and $25 \mid (n - 1)$, and none of $n - 1, n$, or $n + 1$ is squarefree, making each term of the sum 0. Since the Chinese remainder theorem ensures infinitely many solutions, all $n \equiv 351 \pmod{900}$, there are infinitely many such n.

7.4.13. Since every multiple of 4 is nonsquarefree, we can have at most 3 consecutive integers for which μ takes on nonzero values.

7.4.14. We can have $\mu(n) = 0$ for arbitrarily long strings of integers. To see this, let k be a positive integer and p_i be the ith prime. By the Chinese remainder theorem, there is a solution n to the system of congruences $n \equiv 0 \pmod{p_1^2}, n \equiv -1 \pmod{p_2^2}, n \equiv -2 \pmod{p_3^2}, \ldots, n \equiv -k + 1 \pmod{p_k^2}$. Then for every $i = 0, 1, \ldots, k - 1$, we have $p_{i+1}^2 \mid n + i$, and so none of the numbers $n, n+1, n+2, \ldots, n+k-1$ is squarefree.

Hence μ takes on the value 0 for all of them.

7.4.15. Let $h(n) = n$ be the identity function. Then from Theorem 7.7 we have $h(n) = n = \sum_{d|n} \phi(n)$. Then by the Möbius inversion formula, we have $\phi(n) = \sum_{d|n} \mu(d)h(n/d) = \sum_{d|n} \mu(d)(n/d) = n\sum_{d|n} \mu(d)/d$, as desired.

7.4.16. a. Let $F(n) = n$ for all n. Then we have $F(n) = \sum_{d|n} \phi(n/d)$. By the Möbius inversion formula we have $\phi(n) = \sum_{d|n} \mu(d)F(n/d)$. The divisors of p^t are $1, p, p^2, \ldots, p^t$ of which only 1 and p are square-free so we have $\phi(p^t) = \sum_{j=0}^{t} \mu(p^j)F(p^t/p^j) = \mu(1)F(p^t) + \mu(p)F(p^{t-1}) = p^t - p^{t-1}$.

b. Since F as defined in part (a) is multiplicative, and since μ is multiplicative, we have $\phi = \mu * F$ is also multiplicative by Exercise 41 in Section 7.1.

7.4.17. Since μ and f are multiplicative, then so is their product μf, by Exercise 46 of Section 7.1. Further, the summatory function $\sum_{d|n} \mu(d)f(d)$ is also multiplicative by Theorem 7.17. Therefore it suffices to prove the proposition for n a prime power. We compute $\sum_{d|p^a} \mu(d)f(d) = \mu(p^a)f(p^a) + \mu(p^{a-1})f(p^{a-1}) + \cdots + \mu(p)f(p) + \mu(1)f(d)$. But for exponents greater than 1, $\mu(p^j) = 0$, so the above sum equals $\mu(p)f(p) + \mu(1)f(1) = -f(p) + 1$, as desired.

7.4.18. Let $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ be the prime power factorization for n. Using $f(n) = n$ in Exercise 17, we have $\sum_{d|n} d\mu(d) = \prod_{i=1}^{k}(1 - p_i)$.

7.4.19. Here we let $1/n$ play the role of $f(n)$ in the identity in Exercise 17. This gives $\sum_{d|n} \mu(d)/d = \prod_{j=1}^{k}(1 - 1/p_j)$. We might note that this resembles the formula for $\phi(n)$, indeed, it equals $\phi(n)/n$. Compare Exercise 15.

7.4.20. Let $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ be the prime power factorization for n. Using $f(n) = \tau(n)$ in Exercise 17, we have $\sum_{d|n} \mu(d)\tau(d) = \prod_{i=1}^{k}(1 - \tau(p_i)) = \prod_{i=1}^{k}(1 - 2) = (-1)^k$.

7.4.21. Here we let σ play the role of f in the identity. Then the sum equals $\prod_{j=1}^{k}(1 - \sigma(p_j)) \prod_{j=1}^{k}(1 - (p_j + 1)) = \prod_{j=1}^{k} p_i$.

7.4.22. If n is prime, then $\prod_{d|n} \mu(d) = \mu(1)\mu(n) = 1(-1) = -1$. If $s^2 \mid n$ for some $s > 1$, then $\mu(s^2) = 0$ appears in the product, making the whole product 0. Finally, if $n = p_1p_2\cdots p_k$, then $\prod_{d|n} \mu(d) = 1 \cdot \prod_{p_i} \mu(p_i) \cdot \prod_{p_i,p_j} \mu(p_ip_j) \cdots \mu(p_1p_2\cdots p_k)$. The first of these products contributes k (-1)s to the whole product. The second product contributes $\binom{k}{2}$ (-1)s to the product, and in general, the ith product contributes $\binom{k}{i}$ $(-1)^i$s to the product. Therefore, we need only count the number of (-1)s in the product, namely, $\binom{k}{1} + \binom{k}{3} + \binom{k}{5} + \cdots$. By Exercise 6 of Appendix B, this last sum is 2^{k-1}, which is even. (If $k = 1$, then n is prime.) Since the product consists of an even number of (-1)s, it must equal 1.

7.4.23. Since both sides of the equation are known to be multiplicative, (see Exercise 35 in Section 7.2) it suffices to prove the identity for $n = p^a$, a prime power. On one hand we have $\sum_{d|p^a} \mu^2(d) = \mu^2(p) + \mu^2(1) = 1 + 1 = 2$. On the other hand, we have $\omega(p^a) = 1$, so the right side is $2^1 = 2$, which equals the left side.

7.4.24. Exercises 52 and 53 in Section 7.1 show that $g(n) = 2^{\omega(n)}$ is multiplicative. Then by the Möbius inversion formula, we have $\mu^2(n) = \sum_{d|n} \mu(d)g(n/d)) = \sum_{d|n} \mu(d)2^{\omega(n/d)}$, as desired.

7.4.25. Let λ play the role of f in the identity of Exercise 17. Then the left side equals $\prod_{j=1}^{k}(1 - \lambda(p_j)) = \prod_{j=1}^{k}(1 - (-1)) = 2^k$. But $\omega(n) = k$ by definition, so we're done.

7.4.26. Since $\lambda(n)$ and $2^{\omega(n)}$ are multiplicative, so is their Dirichlet product, which is the sum in question. Therefore it suffices to prove the identity for $n = p^a$, where p is prime. Then we have $\sum_{d|n} \lambda(n/d)2^{\omega(d)} = \sum_{i=0}^{a} \lambda(p^{a-i})2^{\omega(p^i)} = \lambda(p^a)2^{\omega(1)} + \sum_{i=1}^{a}(-1)^{a-i}2^1 = (-1)^a(1) + 2\sum_{i=1}^{a}(-1)^{a-i}$ which equals $(-1) +$

$2(1) = 1$ if a is odd, and $1 + 2(0) = 1$ if a is even. This completes the proof.

7.4.27. We compute $\mu * \nu(n) = \sum_{d|n} \mu(d)\nu(n/d) = \sum_{d|n} \mu(d) = \iota(n)$, by Theorem 7.15.

7.4.28. Suppose f and g are multiplicative functions and $f = \sum_{d|n} g(d) = g \star \nu$. From Exercise 42 in Section 7.1 we can Dirichlet multiply on both sides of the equation by μ, which is the inverse for ν, and get $f \star \mu = g$, that is, $g(n) = \sum_{d|n} \mu(d) f(n/d)$, which is the Möbius inversion formula.

7.4.29. Since $\nu(n)$ is identically 1, we have $F(n) = \sum_{d|n} f(d) = \sum_{d|n} f(d)\nu(n/d) = f * \nu(n)$. If we Dirichlet multiply both sides by μ, we have $F * \mu = f * \nu * \mu = f * \iota = f$. as desired.

7.4.30. Let $n = p_1^{a_1} \cdots p_r^{a_r}$. Since $\Lambda(d) = 0$ unless d is a prime power, we have $\sum_{d|n} \Lambda(d) = \sum_{i=1}^{r} \sum_{d|p_i^{a_i}} \Lambda(d) = \sum_{i=1}^{r} \sum_{j=1}^{a_i} \Lambda(p_i^j) = \sum_{i=1}^{r} \sum_{j=1}^{a_i} \log(p_i) = \sum_{i=1}^{r} a_i \log(p_i) = \sum_{i=1}^{r} \log(p_i^{a_i}) = \log n$.

7.4.31. From the Möbius inversion formula, we have $\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d) = \sum_{d|n} \mu(d)(\log n - \log d) = \sum_{d|n} \mu(d) \log(n) - \sum_{d|n} \mu(d) \log(d) = \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log(d) = \log n \nu(n) - \sum_{d|n} \mu(d) \log(d) = -\sum_{d|n} \mu(d) \log(d)$, since $\nu(n) = 0$ if n is not 1, and $\log n = 0$ if $n = 1$.

CHAPTER 8

Cryptology

8.1. Character Ciphers

8.1.1. We translate ATTACK AT DAWN into the corresponding numbers. We obtain 0 19 19 0 2 10 0 19 3 0 22 13. When encrypting this message using the Caesar cipher we obtain the numbers 3 22 22 3 5 13 3 22 6 3 25 16. Translating this into letters give DWWDF NDWGD ZQ.

8.1.2. Using Table 8.2 we find L in the ciphertext corresponds to I in plaintext. Continuing in this manner we have ICAME ISAWI CONQU ERED.

8.1.3. We first translate the message SURRENDER IMMEDIATELY into the corresponding numbers. We obtain 18 20 17 17 4 13 3 4 17 8 12 12 4 3 8 0 19 4 11 24. We encipher each of these numbers using the transformation $C \equiv 11P + 18 \pmod{26}$. This gives 8 4 23 23 10 5 25 10 23 2 20 20 10 25 2 18 19 10 9 22. Translating back to letters gives IEXXK FZKXC UUKZC STKJW.

8.1.4. First we convert each letter of the plaintext to its corresponding number, to get $19, 7, 4, 17, 8, 6, 7, 19, 2, 7, 14, 8, 2, 4$. Then we apply the affine transformation $C \equiv 15P + 14 \pmod{26}$ to each number. For instance $15 \cdot 19 + 14 \equiv 13 \pmod{26}$. Continuing in this fashion, we get $13, 15, 22, 9, 4, 0, 15, 13, 18, 15, 16, 4, 18, 22$ which are the numberical equivalents to NPWJE APNSP QSW.

8.1.5. Since 5 is an inverse for 21 modulo 26, we have $P \equiv 5(C - 5) \equiv 5C + 1 \pmod{26}$ as the deciphering transformation. Converting the ciphertext to numbers gives us $24, 11, 5, 16, 23, 15, 2, 17, 8, 19$. We apply the deciphering transform to each number, for instance, $5 \cdot 24 + 1 \equiv 17 \pmod{26}$. Continuing in this fashion gives us $17, 4, 0, 3, 12, 24, 11, 8, 15, 18$ which are the numerical equivalents of READM YLIPS.

8.1.6. Since $3 \cdot 9 = 27 \equiv 1 \pmod{26}, 9$ is an inverse for 3. Then we have $p \equiv 9(C - 24) \equiv 9C - 9(24) \equiv 9C - 9(-2) \equiv 9C + 18 \equiv 9C - 8 \pmod{26}$. Converting the ciphertext to numbers gives: 17 19 14 11 10 19 14 8 10. Applying the transformation $P \equiv 9C - 8 \pmod{26}$ to each number gives: 15 7 14 13 4 7 14 12 4 and converting back to letters: PHONE HOME.

8.1.7. Since E is the most common letter suppose that E is sent to Q. Since E corresponds to 4 and Q corresponds to 16, we have $4 + k \equiv \pmod{16}$. Hence $k = 12$.

8.1.8. In the ciphertext V occurs 7 times, which is more than any other letter. We guess that V corresponds to E since E is the most common letter in English text. This implies that $21 \equiv 4 + k \pmod{26}$, or that $k = 17$. If this is correct, deciphering is carried out using the relationship $P \equiv C - 17 \equiv C + 9 \pmod{26}$. Attempting to decipher, we obtain THEVA LUEOF THEKE YISSE VENTE EN. Hence, the plaintext message was "The value of the key is seventeen."

8.1.9. By counting letter frequencies, we find that M is the most common letter, occuring 8 times. We guess that M stands for E, which would be a shift of 8. We subtract 8 from each letter and get ANIDE AISLI KEACH ILDNO NEISB ETTER THANY OUROW NFROM CHINE SEFOR TUNEC OOKIE

8.1.10. Since E and T are the most common letters, and E=4, T=19, Q=16, and X=23, we suspect that $23 \equiv a4 + b \pmod{26}$, and $16 \equiv a19 + b \pmod{26}$, which has solution $a = 3, b = 11$.

8.1.11. Since E=4 and T=19 are the most common letters in plaintext and W=22 and B=1, we have $22 \equiv a4 + b$ and $1 \equiv a19 + b \pmod{26}$. By Theorem 3.14, the solution to the system is $a \equiv 9, b \equiv 12 \pmod{26}$.

8.1.12. We find that J occurs 11 times followed by F at 7 times and O at 5 times. Our first guess is $J \to E$ and $F \to T$. Hence we solve $9 \equiv a4 + b, 5 \equiv a19 + b \pmod{26}$. This yields $a = 24$, but $(24, 26) = 2 \neq 1$ so we try $J \to E$ and $O \to T$. We need to solve $9 \equiv a4 + b$ and $14 \equiv a19 + b \pmod{26}$. Therefore P$\equiv 3C + 3 \pmod{26}$ and we have WEUSE FREQU ENCIE SOFLE TTERS TODEC RYPTS ECRET MESSA GES.

8.1.13. We count the frequencies of letters in the ciphertext and discover that A, B, T, and N appear most often, namely 6 times each. Let $P = D(C) \equiv cC + d \pmod{26}$. Then $D(A) = d$ is one of A, E, N, or S. From which we deduce that $d = 0, 4, 13$ or 18. Also $D(B) = c + d$, must be another one of these numbers. Since A, B, T, N is not a simple shift of A, E, N, S, we see that c is not 0. Assuming that d is also not zero, the possible pairs for (c, d) are $(9, 4), (14, 4), (5, 13), (17, 13), (12, 18)$, and $(21, 18)$. We try various of these and discover that $P \equiv 5C + 13$ is the deciphering transformation. Applying this to the ciphertext gives us THISM ESSAG EWASE NCIPH EREDU SINGA NAFFI NETRA NSFOR MATIO N

8.1.14. A frequency count shows that J occurs 12 times, F occurs 7 times and O occurs 5 times. Our first guess is that J is the ciphertext for E and F is the ciphertext for T. But when we solve for a we get an even number, which is not relatively prime to 26, so we guess that O is the ciphertext for T instead. Then we solve $9 \equiv a4 + b, 14 \equiv a19 + b \pmod{26}$ and get $a = 9, b = 25$. Then we solve $C \equiv 9P + 25 \pmod{26}$ for P and get $P \equiv 3C + 3 \pmod{26}$. This decrypts the message as follows: WEUSE FREQU ENCIE SOFLE TTERS TODEC RYPTS ECRET MESSA GES.

8.1.15. We have $C \equiv 17(5P + 13) + 3 \equiv 85P + 224 \equiv 7P + 16 \pmod{26}$.

8.1.16. We have $C \equiv c(aP + b) + d \equiv acP + bc + d \pmod{26}$.

8.2. Block and Stream Ciphers

8.2.1. We first translate the letters of the message DO NOT OPEN THIS ENVELOPE into the corresponding numbers, grouping letters into blocks of six. This gives 3 14 13 14 19 14 15 4 13 19 7 8 18 4 13 21 4 11 14 15 4. The letters of the word SECRET, which is the key, translate to 184217419. For each block $p_1p_2p_3p_4p_5p_6$we find $c_i \equiv p_i + k_i \pmod{26}$ where $0 \leq c_i \leq 25$ where $k_i = 18$,$k_2 = 4$, $k_3 = 2$, $k_4 = 17$, $k_5 = 4$, and $k_6 = 19$. This gives 21 18 15 5 23 7 7 8 15 10 11 1 10 8 15 12 8 4 6 19 6. Translating this back to letters gives VSPFXH HIPLKB KIPMIE GTG.

8.2.2. We have $p_i \equiv c_i - k_i \pmod{26}$, so we subtract the numerical equivalents of the letters SECRET from the numerical equivalents of the letters of the ciphertext respectively. SECRET = 18 4 2 17 4 19. The cyphertext is 22 1 17 2 18 11 0 25 6 9 12 6 10 12 5 21. Subtracting the key gives 4 23 15 11 14 18 8 21 4 18 8 13 18 8 3 4, which has letter equivalents EXPLOS IVESIN SIDE.

8.2.3. The numerical equivalents for the key TWAIN are 19 22 0 8 13. The numerical equivalents for ANENGLISHMAN are 0 13 4 13 6 11 8 18 7 12 0 13. Adding the key numbers to the corresponding first five plaintext numbers yields 19 9 4 21 19, $\pmod{26}$ which stand for TJEVT. Adding the key numbers to the corresponding next five plaintext numbers yields 4 4 18 15 25, $\pmod{26}$ which stand for EESPZ. Adding the numbers for TA to the last two letters yields TJ. Continuing in this fashion we find the cipher text to be TJEVT EESPZ TJIAN IARAB GSHWQ HASBU BJGAO XYACF XPHML AWVMO XANLB GABMS HNEIA TIEZV VWNQF TLEZF HJWPB WKEAG AENOF UACIH LATPR RDADR GKTJR XJDWA XXENB KA

8.2.4. The numerical equivalents for the key TWAIN are 19 22 0 8 13. The numerical equivalents for PACWH EZUAR are 15 0 2 22 7 4 25 20 0 17. We substract successively each number of the key from the cipher text numbers and get 22 4 2 14 20 11 3 20 18 4, $\pmod{26}$ which stands for WECOU LDUSE. Continuing in this fashion we discover the plaintext to be WECOU LDUSE UPTWO ETERN ITIES INLEA RNING ALLTH ATIST OBELE ARNED ABOUT OUROW NWORL DANDT HETHO USAND SOFNA TIONS THATH AVEAR ISENA NDFLO URISH EDAND VANIS HEDFR OMITM

ATHEM ATICS ALONE WOULD OCCUP YMEEI GHTMI LLION YEARS.

8.2.5. Let n be the key length, and suppose $k_1, k_2, \ldots, k_n$ are the numerical equivalents of the letters of the keyword. If $p_i = p_j$ are two plaintext characters separated by a multiple of the key length, when we separate the plaintext into blocks of length n, p_i and p_j will be in the same position in their respective blocks, say the mth position. So when we encrypt them, we get $c_i \equiv p_i + k_m \equiv p_j + k_m \equiv c_j \pmod{26}$.

8.2.6. We see that the initial UCY is repeated 9 letters later. Likewise with the BS in the second row. The HF in the fourth block is repeated 21 letters later, so we guess that the period is $(9, 21) = 3$. A frequency count of letters in the 1st, 4th, 7th, etc. positions gives us 6 U's, 5 F's and 4 B's and T's. Since $T - E = 15$ and $U - F = 15$, we are quite sure that $F \to E$ and $U \to T$, that is, $l_1 =$ B. The other two cases are not so clear. A frequency count of letters in the 2nd, 5th, 8th, etc positions gives us 6 C's, 6 H's and 4 V's. None of these have a difference of 15, so we guess that one of E or T doesn't appear among C, H, and V. So we have 6 reasonable guesses to try. $C \to E, H \to E, V \to E, C \to T, H \to T$, and $V \to T$. Each of these 6 cases determines a different second letter, namely E, Z, T, T, O, and A respectively. It is unlikely that Z or T would follow the first letter, which we believe to be T, so we easily discard three of the cases. A frequency count of letters in the 3rd, 6th, 9th, etc positions gives us 6 B's, 5 O's and 4 K's. We try, in order, $B \to E$, which implies $l_3 =$X, with each of the three remaining cases for l_2, and discover that $H \to T$, which corresponds to $l_2 = O$, is the correct choice. Therefore the key is BOX and the plaintext is TOBEO RNOTT OBETH ATIST HEQUE STION WHETH ERTIS NOBLE RINTH EMIND TOSUF FERTH ESLIN GSAND ARROW SOFOU TRAGE OUSFO RTUNE.

8.2.7. Searching the ciphertext, we find two occurrences of KMK which are 42 positions apart, and two occurrences of PWQW which are 39 positions apart, so we guess that the period is $(42, 39) = 3$. The index of coincidence for the letters in positions 1, 4, 7,... is 0.064, the index of coincidence for the letters in positions 2, 5, 8,... is 0.072, and the index of coincidence for the letters in positions 3, 6, 9,... is 0.068. Since these indexes are all about 0.065, we are sure that the period is 3. Counting frequencies of the letters in positions 1, 4, 7,..., we find 9 R's and 7 C's, and since $R - C = 15$, we suspect that $R \to T$ and $C \to E$, which implies that $l_1 = Y$. Counting frequencies for the letters in positions 2, 5, 8,..., we find 11 E's, 6 M's, 5 W's and 5 I's. We seek a difference among these letters which is the same as a difference among the commonly occurring letters E, T, N, R, I, and O. We find that $M - I = 4$ and $E - I = 4$, so we suspect that $M \to I$ and $I \to E$, which implies that $l_2 = E$. Counting frequencies for the letters in positions 3, 6, 9,... we find 8 S's, 7 W's and 6 K's. We try successively assuming that each of these has plaintext E, and discover that $W \to E$ yields a sensible message with key YES. The plaintext is MISTA KESAR EAPAR TOFBE INGHU MANAP PRECI ATEYO URMIS TAKES FORWH ATTHE YAREP RECIO USLIF ELESS ONSTH ATCAN ONLYB ELEAR NEDTH EHARD WAYUN LESSI TISAF ATALM ISTAK EWHIC HATLE ASTOT HERSC ANLEA RNFRO M.

8.2.8. Searching the ciphertext, we find two occurrences of GP which are 15 positions apart. We also find two occurrences of WV which are 42 positions apart, so we guess that the period is $(15, 42) = 3$. We compute the indexes of coincidence for each of the three groups of letters and get 0.057, 0.053, and 0.095, respectively, which shows that the period is likely 3. Counting frequencies of the letters in positions 1, 4, 7,..., we find 7 W's, and 4 each of A, J and P. Comparing the differences of these letters with the differences of E, T, N, R, I and O (the most common plaintext letters) we see that $P - J = 6$ and $T - N = 6$, so we guess that $P \to T$ and $J \to N$, which implies $l_1 = W$. Counting frequencies of the letters in positions 2, 5, 8,..., we find 7 X's and 4 I's. Since $X - I = 15$ and $T - E = 15$ we guess that $X \to T$ which implies that $l_2 = E$. Counting frequencies of letters in positions 3, 6, 9,..., we find 9 F's and 8 P's. Since $P - F = 10$ and $O - E = 10$, we guess that $P \to O$, which implies $l_3 = B$. Using the key WEB, we find the plaintext to be WEHAV EHEAR DTHAT AMILL IONMO NKEYS ATAMI LLION KEYBO ARDSC OULDP RODUC ETHEC OMPLE TEWOR KSOFS HAKES PEARE NOWTH ANKST OTHEI NTERN ETWEK NOWTH ATISN OTTRU E.

8.2.9. Searching the ciphertext, we find two occurrences of UPRW, which are 16 positions apart. We also find two occurrences of UQ, which are 12 positions apart, so we guess that the period is $(16, 12) = 4$. We compute the indexes of coincidence for each of the four letter groups and get 0.059, 0.055, 0.058, and 0.043, which are all significantly greater than 0.038, so we believe the keyword has 4 letters. Counting

frequencies of the letters in positions 1, 5, 9,..., we find 6 F's and 5 U's, and since $U - F = 15$ and $T - E = 15$, we guess that $U \to T$ and so $l_1 = B$. Counting frequencies of the letters in positions 2, 6, 10,..., we find 6 Q's, 5 P's and 4 W's. Since $W - Q = O - I = 6$, we guess that $W \to O$, so that $l_2 = I$. Counting frequencies of letters in positions 3, 7, 11,..., we find 6 E's and 5 V's and R's. Since $V - E = 17 = E - N$, we guess that $V \to E$, and so $l_3 = R$. Counting frequencies of letters in positions 4, 8, 12,..., we find 5 D's, 4 K's and 3 each of B, H, J, O, U, and W. (Since the index of coincidence for this group was relatively small, we are not surprised at the more random-seeming distribution of letters.) After several attempts we guess that $W \to T$ and so $l_4 = D$. Using the keyword BIRD, we find the plaintext to be IONCE HADAS PARRO WALIG HTUPO NMYSH OULDE RFORA MOMEN TWHIL EIWAS HOEIN GINAV ILLAG EGARD ENAND IFELT THATI WASMO REDIS TINGU ISHED BYTHA TCIRC UMSTA NCETH ATISH OULDH AVEBE ENBYA NYEPA ULETI COULD HAVEW ORN.

8.2.10. Searching the ciphertext, we find two occurrences of YLP which are 72 positions apart. We also find two occurrences of RR which are 20 positions apart, so we guess that the period is $(72, 20) = 4$. We compute the indexes of coincidence for each of the four letter groups and get 0.060, 0.060, 0.053, and 0.066, which are all significantly greater than 0.038, so we believe the keyword has 4 letters. Counting frequencies of the letters in positions 1, 5, 9,..., we find 7 P's, and 6 E's and Z's. Since $Z - P = 10 = O - E$, we guess that $Z \to O$, and so $l_1 = L$. Counting frequencies of the letters in positions 2, 6, 10,..., we find 8 M's, and 7 Q's and W's. Since $W - M = 10 = O - E$, we guess that $W \to O$, and so $l_2 = I$. Counting frequencies of the letters in positions 3, 7, 11,..., we find 6 each of J, W and Y. Since these match the numerical pattern of E, R, and T, we guess that $J \to E$ and so $l_3 = F$. Counting frequencies of the letters in positions 4, 8, 12, ..., we find 11 I's and no other letter approaching that frequency. We guess that $I \to E$, so that $l_4 = E$. Using the keyword LIFE, we find the plaintext to be EVERY DAYYO UMAYM AKEPR OGRES SEVER YSTEP MAYBE FRUIT FULYE TTHER EWILL STRET CHOUT BEFOR EYOUA NEVER LENGT HENIN GEVER ASCEN DINGE VERIM PROVI NGPAT HY-OUK NOWYO UWILL NEVER GETTO THEEN DOFTH EJOUR NEYBU TTHIS SOFAR FROMD IS-COU RAGIN GONLY ADDST OTHEJ OYAND GLORY OFTHE CLIMB.

8.2.11. Searching the ciphertext, we find two occurrences of ZEELN which are 40 positions apart. We also find two occurrences of SUMHR which are 45 positions apart, so we guess that the period is $(40, 45) = 5$. The indexes of coincidence for the five letter groups are 0.073, 0.062, 0.062, 0.059, and 0.089, which are all significantly greater than 0.038, so we are confirmed in our guess that the keyword has length 5. Counting frequencies of the letters in positions 1, 6, 11,..., we find 5 W's, S's and M's, and 4 G's and L's. Since $W - L = E - T \equiv 15 \pmod{26}$, we guess that $W \to E$, and so $l_1 = S$. Counting frequencies of the letters in positions 2, 7, 12,..., we find 6 A's, 5 T's and 4 E's. Since these letters should be frequent in the plaintext, we suspect that $A \to A$ and that $l_2 = A$. Counting frequencies of the letters in positions 3, 8, 13,..., we find 5 K's and 5 Z's, and since $Z - K = 15 = T - E$, we guess that $Z \to T$ and so $l_3 = G$. Counting frequencies of the letters in positions 4, 9, 14,..., we find 5 T's, 4 S's and H's, and 3 E's and N's. Since these letters should be frequent in the plaintext, we suspect that $A \to A$ and that $l_4 = A$. Counting frequencies of the letters in positions 5, 10, 15,..., we find 7 R's, 6 U's and 5 G's. Since $R - G = E - T = 15$, we guess that $R \to E$ and so $l_5 = N$. Using the keyword SAGAN we discover the plaintext to be BUTTH EFACT THATS OMEGE NIUSE SWERE LAUGH EDATD OESNO TIMPL YTHAT ALLWH OAREL AUGHE DATAR EGENI USEST HEYLA UGHED ATCOL UMBUS THEYL AUGHE DATFU LTONT HEYLA UGHED ATTHE WRIGH TBROT HERSB UTTHE YALSO LAUGH EDATB OZOTH ECLOW N.

8.2.12. Counting frequencies of the letters in positions 1, 4, 7,..., we find 7 N's, 6 L's and 5 Y's. Since $N - Y = 15$ and $T - E = 15$, we guess that $N \to T$ and $Y \to E$, which would make the first letter of the keyword equal to $N - T = 13 - 19 \equiv 20 \pmod{26} = U$. Counting frequencies of the letters in positions 2, 5, 8,..., we find 13 W's and no other frequencies nearly so high. We guess that $W \to E$ and so the second letter of the keyword would be S. Counting frequencies of the letters in positions 3, 6, 9,..., we find 10 E's and 8 T's, which is typical for plaintext. We guess that the third letter of the keyword is A which has numerical equivalent 0.

8.2.13. We first translate BEWARE OF THE MESSENGER into numerical equivalents. This gives 01 04 22 00 17 04 14 05 19 07 04 12 04 18 18 04 13 06 04 17. We now encipher each block. We have $3 \cdot 1 + 10 \cdot 4 \equiv$

$17 \pmod{26}$, $9 \cdot 1 + 7 \cdot 4 \equiv 11 \pmod{26}$;$3 \cdot 22 + 10 \cdot 0 \equiv 14 \pmod{26}$,$9 \cdot 22 + 7 \cdot 0 \equiv 16 \pmod{26}$;$3 \cdot 17 + 10 \cdot 4 \equiv 13 \pmod{26}$, $9 \cdot 17 + 7 \cdot 4 \equiv 25 \pmod{26}$;$3 \cdot 14 + 10 \cdot 5 \equiv 14 \pmod{26}$, $9 \cdot 14 + 7 \cdot 5 \equiv 5 \pmod{26}$; $3 \cdot 19 + 10 \cdot 7 \equiv 23 \pmod{26}$, $9 \cdot 19 + 7 \cdot 7 \equiv 12 \pmod{26}$; $3 \cdot 4 + 10 \cdot 12 \equiv 2 \pmod{26}$, $9 \cdot 4 + 7 \cdot 12 \equiv 16 \pmod{26}$; $3 \cdot 4 + 10 \cdot 18 \equiv 10 \pmod{26}$, $9 \cdot 4 + 7 \cdot 18 \equiv 6 \pmod{26}$; $3 \cdot 18 + 10 \cdot 4 \equiv 16 \pmod{26}$, $9 \cdot 18 + 7 \cdot 4 \equiv 8 \pmod{26}$; $3 \cdot 13 + 10 \cdot 6 \equiv 21 \pmod{26}$, $9 \cdot 13 + 7 \cdot 6 \equiv 3 \pmod{26}$; $3 \cdot 4 + 10 \cdot 17 \equiv 0 \pmod{26}$,$9 \cdot 4 + 7 \cdot 17 \equiv 25 \pmod{26}$. This gives the enciphered values 17 11 14 16 13 25 14 5 23 12 2 16 10 6 16 8 21 3 0 25. Translating back to letters gives R L O Q N Z O F X M C Q K G Q I V D A Z.

8.2.14. We break the plaintext into blocks of 2 and convert to numerical equivalents to get 3 14 13 14 19 18 7 14 14 19 19 7 4 12 4 18 18 4 13 6 4 17. Applying the transformation to each block gives us 20 7 22 11 2 21 0 19 23 17 7 4 10 14 12 2 24 20 2 1 3 17 which converted to letters is UH WL CV AT XR HE KO MC YU CB DR.

8.2.15. The matrix $\begin{pmatrix} 13 & 4 \\ 9 & 1 \end{pmatrix}$ has inverse $\begin{pmatrix} 9 & 16 \\ 23 & 13 \end{pmatrix}$. The numerical values of R D are 17 and 3. Then $\begin{pmatrix} 9 & 16 \\ 23 & 13 \end{pmatrix}\begin{pmatrix} 17 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} 19 \\ 14 \end{pmatrix} \pmod{26}$, and 19 14 are the numerical values for TO. Continuing in this fashion we have TO SL EE PP UR CH AN CE TO DR EA MX

8.2.16. The matrix $\begin{pmatrix} 23 & 3 \\ 10 & 25 \end{pmatrix}$ has inverse $\begin{pmatrix} 1 & 3 \\ 10 & 3 \end{pmatrix}$. The numerical values of U W are 20 and 22. Then $\begin{pmatrix} 1 & 3 \\ 10 & 3 \end{pmatrix}\begin{pmatrix} 20 \\ 22 \end{pmatrix} \equiv \begin{pmatrix} 8 \\ 6 \end{pmatrix} \pmod{26}$, and 8 6 are the numerical values for IG. Similarly we have DM→ NO, NK→ RE, QB→ TH, and EK→ IS. So the plaintext is IGNORE THIS.

8.2.17. RH NI TH and HE correspond to 17 7 13 8 19 7 and 7 4 respectively, so we have $\begin{pmatrix} 17 & 13 \\ 7 & 8 \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 19 & 7 \\ 7 & 4 \end{pmatrix} \pmod{26}$. Since $\begin{pmatrix} 4 & 19 \\ 19 & 19 \end{pmatrix}$ is an inverse for $\begin{pmatrix} 19 & 7 \\ 7 & 4 \end{pmatrix}$ we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 17 & 13 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 4 & 19 \\ 19 & 19 \end{pmatrix} \equiv \begin{pmatrix} 3 & 24 \\ 24 & 25 \end{pmatrix} \pmod{26}$.

8.2.18. a. If the pair P_1P_2 remained unchanged, we have $P_1 \equiv 4P_1 + 5P_2$ and $P_2 \equiv 3P_1 + P_2 \pmod{26}$. Then we have $3P_1 \equiv 0 \pmod{26}$ hence $P_1 \equiv 0$ from the second congruence. Then the first congruence gives $5P_2 \equiv 0$ or $P_2 \equiv 0 \pmod{26}$. Since 0 0 corresponds to the block AA, this is the only unchanged pair.

b. As above, we need to solve the congruences $P_1 \equiv 7P_1 + 17P_2$ and $P_2 \equiv P_1 + 6P_2 \pmod{26}$, or $6P_1 + 17P_2 \equiv 0$ and $P_1 + 5P_2 \equiv 0 \pmod{26}$. Subracting six times the second from the first gives $-13P_2 \equiv 0 \pmod{26}$ which has solutions $P_1 \equiv 0, 2, 4, \ldots, 24$. Then $P_1 \equiv -5P_2 \equiv 21P_2 \pmod{26}$ and since 21 has a unique inverse modulo 26, we have a unique P_2 for each P_1, so we have 13 solutions.

c. As above, we solve the congruences $P_1 \equiv 3P_1 + 5P_2$ and $P_2 \equiv 6P_1 + 3P_2 \pmod{26}$, or $2P_1 + 5P_2 \equiv 0$ and $6P_1 + 2P_2 \equiv 0 \pmod{26}$. If we take 3 times the first from the second we have $-13P_2 \equiv 0$, so $P_2 \equiv 0, 2, 4, \ldots, 24$ are all solutions. The second congruence implies $6P_1 \equiv -2P_2 \pmod{26}$ which reduces to $3P_1 \equiv -P_2 \pmod{13}$. Three has an inverse mod 13, so given P_2, we can solve for P_1 modulo 13. This gives two solutions modulo 26, namely P_1 and $P_1 + 13$. Therefore we have 26 solutions.

8.2.19. We have $\mathbf{C} \equiv \mathbf{AP} \pmod{26}$. Multiplying both sides on the left by $\mathbf{A}$ gives $\mathbf{AC} \equiv \mathbf{A^2P} \equiv \mathbf{IP} \equiv \mathbf{P} \pmod{26}$. The congruence $\mathbf{A^2} \equiv \mathbf{I} \pmod{26}$ follows since $\mathbf{A}$ is involutory. It follows that $\mathbf{A}$ is also a deciphering matrix.

8.2.20. The numerical equivalents of LME, WRI and ZYC are 11 12 4 22 17 8 and 25 24 2. The numerical equivalents of THE, AND and THA are 19 7 4 0 13 3 and 19 7 0. Then $\mathbf{C} = \begin{pmatrix} 11 & 22 & 25 \\ 12 & 17 & 24 \\ 4 & 8 & 2 \end{pmatrix}$ and $\mathbf{P}$=

$\begin{pmatrix} 19 & 0 & 19 \\ 7 & 13 & 7 \\ 4 & 3 & 0 \end{pmatrix}$. Now note that $\det \mathbf{P} \equiv 0 \pmod{26}$ so **P** doesn't have an inverse. We can still find a suitable **A** however. Our matrix conguence is

$$\begin{pmatrix} 11 & 22 & 25 \\ 12 & 17 & 24 \\ 4 & 8 & 2 \end{pmatrix} \equiv \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 19 & 0 & 19 \\ 7 & 13 & 7 \\ 4 & 3 & 0 \end{pmatrix}.$$

If we perform the multiplication on the right and equate corresponding entries we get the follow system of 9 congruences in 9 unknowns, modulo 26:

$$\begin{aligned} 19a + 7b + 4c &\equiv 11 \\ 13b + 3c &\equiv 22 \\ 19a + 7b &\equiv 25 \\ 19d + 7e + 4f &\equiv 12 \\ 13e + 3f &\equiv 17 \\ 19d + 7e &\equiv 24 \\ 19g + 7h + 4i &\equiv 4 \\ 13h + 3i &\equiv 8 \\ 19g + 7h &\equiv 2 \end{aligned}$$

Notice that this is really three 3×3 systems. We can solve the first three congruences for a, b, and c. Subtracting the third from the first gives $4c \equiv -14 \pmod{26}$. Hence $2c \equiv -7 \pmod{13}$ and so $c \equiv 3 \pmod{13}$. Therefore $c \equiv 3$ or $16 \pmod{26}$. We can choose either of these and we'll take $c \equiv 3 \pmod{26}$. Then the second congruence yields $13b + 3(3) \equiv 22 \pmod{26}$ or $13b \equiv 13 \pmod{26}$ hence b can be any odd number. Take $b \equiv 1 \pmod{26}$, then the third congruence becomes $19a + 7(1) \equiv 25 \pmod{26}$ which forces $a \equiv 16 \pmod{26}$. Similarly, we solve the other systems and get, among the several possibilities, $\mathbf{A} = \begin{pmatrix} 16 & 1 & 3 \\ 6 & 2 & 23 \\ 22 & 0 & 20 \end{pmatrix}$.

8.2.21. We have $C \equiv C_1C_2P \equiv \begin{pmatrix} 5 & 1 \\ 25 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} P \equiv \begin{pmatrix} 11 & 32 \\ 54 & 143 \end{pmatrix} P \equiv \begin{pmatrix} 11 & 6 \\ 2 & 13 \end{pmatrix} P$. Hence the product cipher is given by $C \equiv AP \pmod{26}$ where $A = \begin{pmatrix} 11 & 6 \\ 2 & 13 \end{pmatrix}$.

8.2.22. Let **A** be the enciphering matrix for the first Hill cipher and **B** be the enciphering matrix for the second Hill cipher. Then if **P** is a plaintext vector we have $\mathbf{C}_1 \equiv \mathbf{AP} \pmod{26}$ for the first encryption. Then the second encryption is $\mathbf{C}_2 \equiv \mathbf{BC}_1 \equiv \mathbf{BAP} \pmod{26}$. So the final result is the same if we just use the matrix **BA** as our encryption matrix.

8.2.23. If the plaintext is grouped into blocks of size m, we may take $\frac{[m,n]}{m}$ of these blocks to form a superblock of size $[m, n]$. If **A** is the $m \times m$ enciphering matrix, form the $[m, n] \times [m, n]$ matrix **B** with $\frac{[m,n]}{m}$ copies of **A** on the diagonal and zeros elsewhere: $\mathbf{B} = \begin{pmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & \mathbf{A} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \mathbf{A} \end{pmatrix}$ Then **B** will encipher $\frac{[m,n]}{m}$ blocks of size m at once. Similarly, if **C** is the $n \times n$ enciphering matrix, form the corresponding $[m, n] \times [m, n]$ matrix **D**. Then by Exercise 8, **BD** is an $[m, n] \times [m, n]$ enciphering matrix which does everything at once.

8.2.24. As described in Exercise 23, we form two 6×6 matrices and multiply them:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 10 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 & 2 \end{pmatrix}.$$

8.2.25. Multiplication of $(0 \cdots 010 \cdots 0) \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$ with the 1 in the ith place yields the 1×1 matrix (P_i). So if the jth row of a matrix $\mathbf{A}$ is $(0 \cdots 010 \cdots 0)$ then $\mathbf{A} \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$ gives $C_j = P_i$. So if every row of $\mathbf{A}$ has its 1 in a different column, then each C_j is equal to a different P_i. Hence $\mathbf{A}$ is a "permutation" matrix.

8.2.26. We break the plaintext into blocks of two and convert to numerical equivalents to get 7 0 21 4 0 13 8 2 4 3 0 24. Applying the transformation to each pair gives us 3 16 1 2 8 6 10 19 0 2 4 23 which are the numerical equivalents for DQ BC IG KT AC EX.

8.2.27. The matrix $\begin{pmatrix} 3 & 2 \\ 7 & 11 \end{pmatrix}$ has inverse $\begin{pmatrix} 17 & 4 \\ 1 & 7 \end{pmatrix}$ modulo 26. We compute $\mathbf{P} \equiv \begin{pmatrix} 17 & 4 \\ 1 & 7 \end{pmatrix} \left(C - \begin{pmatrix} 8 \\ 19 \end{pmatrix} \right) \equiv \begin{pmatrix} 17 & 4 \\ 1 & 7 \end{pmatrix} \mathbf{C} + \begin{pmatrix} 22 \\ 15 \end{pmatrix} \pmod{26}$.

8.2.28. We need to solve the congruence for $\mathbf{P}$.Since $\mathbf{C}$ and $\mathbf{B}$ are $n \times 1$ matrices we can subtract $\mathbf{B}$ from both sides and get $\mathbf{C} - \mathbf{B} \equiv \mathbf{AP} \pmod{26}$. Since (det$\mathbf{A}$, 26) = 1, $\mathbf{A}$ has an inverse $\overline{\mathbf{A}}$ modulo 26. We multiply both sides by $\overline{\mathbf{A}}$ and get $\overline{\mathbf{A}}\,(\mathbf{C} - \mathbf{B}) \equiv \mathbf{P} \pmod{26}$, or $\overline{\mathbf{A}}\mathbf{C} - \overline{\mathbf{A}}\mathbf{B} \equiv \mathbf{P} \pmod{26}$ as the deciphering transformation.

8.2.29. The matrix $\begin{pmatrix} 5 & 2 \\ 11 & 15 \end{pmatrix}$ has inverse $\begin{pmatrix} 15 & 24 \\ 15 & 5 \end{pmatrix}$ modulo 26. We compute $\mathbf{P} \equiv \begin{pmatrix} 15 & 24 \\ 15 & 5 \end{pmatrix} \left(C - \begin{pmatrix} 14 \\ 3 \end{pmatrix} \right) \equiv \begin{pmatrix} 17 & 4 \\ 1 & 7 \end{pmatrix} \mathbf{C} + \begin{pmatrix} 4 \\ 9 \end{pmatrix} \pmod{26}$. Applying this deciphering transformation to the numeric equivalents of the ciphertext and converting back to letters gives TOXIC WASTE as the plaintext.

8.2.30. First make a frequency count of digraphs in the ciphertext. Since there are 6 variables to determine, guesses about 3 digraphs will be needed. We would first guess that the most common digraph has plaintext TH, the next most common has plaintext HE, etc. Then we could solve the 6 corresponding congruences in 6 variables.

8.2.31. Make a frequency count of the trigraphs and use a published English language count of frequencies of trigraphs. Then proceed as in problem 18. There are 12 variables to determine, so 4 guesses are needed.

8.2.32. Yes. Let the first transformation be $\mathbf{C}_1 \equiv \mathbf{A}_1\mathbf{P}+\mathbf{B}_1 \pmod{26}$ and the second be $\mathbf{C}_2 \equiv \mathbf{A}_2\mathbf{P}+\mathbf{B}_2 \pmod{26}$. Then composition of these transformations is $\mathbf{C} \equiv \mathbf{A}_2(\mathbf{A}_1\mathbf{P}+\mathbf{B}_1)+\mathbf{B}_2 \equiv \mathbf{A}_2\mathbf{A}_1\mathbf{P}+ \mathbf{A}_2\mathbf{B}_1+\mathbf{B}_2 \pmod{26}$,

which is an affine transformation.

8.2.33. Let **A** be an $m \times m$ matrix, **B** be an $m \times 1$ matrix, **D** be an $n \times n$ matrix, and **E** be an $n \times 1$ matrix. Form $mn \times mn$ matrices **X** and **Y**by placing n copies of **A** along the diagonal of **X** and m copies of **D** along the diagonal of **Y**. Form $mn \times 1$ matrices **Z** and **W** by stringing n copies of **B** together and m copies of **E**, respectively. Then the product tranformation is given by $\mathbf{C} = \mathbf{YXP} + \mathbf{YZ} + \mathbf{W}$ which is an affine transformation based on a block size of mn.

8.2.34. To encrypt the string, we need to add corresponding bits of the string to the keystream to produce the string $21\ 1121\ 1012$ and then reduce modulo 2 to get $01\ 1101\ 1010$ as the ciphertext.

8.2.35. To decrypt the string, we need to add corresponding bits of the string to the keystream to produce the string $21\ 1121\ 1012$ and then reduce modulo 2 to get $01\ 1101\ 1010$ as the plaintext.

8.2.36. Converting MIDDLETOWN and Z to numerical equivalents gives us $12, 8, 3, 3, 11, 4, 19, 14, 22, 13$ for the plaintext and 25 for the seed value. We add 25 to 12 and reduce modulo 26 to get the first ciphertext as 11. Then we add 12 to 8 and reduce modulo 26 to get 20. Then we add 8 to 3 and reduce to get 11 and so on. This generates the ciphertext $11, 20, 11, 6, 14, 15, 23, 7, 10, 9$, which are the numerical equivalents of LULGOPXHKJ.

8.2.37. We first convert the ciphertext to its numerical equivalents: $25\ 21\ 17\ 16\ 7\ 3\ 20\ 9\ 8\ 12$. The seed is I which has numerical value 8. We subtract 8 from 25 to get 17 which stands for R. Then we subtract 17 from 21 to get 4 which stands for E. Then we subtract 4 from 17 to get 13 which stands for N. Then we subtract 13 from 16 to get 3 which stands for D. Then we subtract 3 from 7 to get 4 which stands for E. Then we subtract 4 from 3 to get $-1 \equiv 25 \pmod{26}$ which stands for Z. Then we subtract 25 from 20 to get $-5 \equiv 21 \pmod{26}$ which stands for V. Then we subtract 21 from 9 to get $-12 \equiv 14 \pmod{26}$ which stands for O. Then we subtract 14 from 8 to get $-6 \equiv 20 \pmod{26}$ which stands for U. Then we subtract 20 from 12 to get $-8 \equiv 18 \pmod{26}$ which stands for S. So the plaintext is RENDE ZVOUS.

8.2.38. Suppose the plaintext is $p_1p_2 \ldots p_k$, where each p_i is a binary digit. Also let $c_1c_2 \ldots c_k$ be the resulting ciphertext. Then for each $i = 1, 2, \ldots, k$, if $p_i = c_i$ then the ith digit of the keystream is 0 and if $p_i \neq c_i$ then the ith digit of the keystream is 1. Thus the entire keystream can be determined.

8.2.39. Let $p_1p_2 \cdots p_m$ and $q_1q_2 \cdots q_m$ be two different plaintext bit streams. Let $k_1, k_2, \ldots, k_m$ be the keystream by which the plaintexts are encrypted. Then note that for any $i = 1, 2, \ldots, m$, $E_{k_i}(p_i) + E_{k_i}(q_i) = k_i + p_i + k_i + q_i = 2k_i + p_i + q_i \equiv p_i + q_i \pmod{2}$. Therefore, by adding corresponding bits of the ciphertext streams, we get the sums of the corresponding bits of the plaintext streams. This can lead to the discovery of portions of the keystream. For instance if $p_i + q_i$ is known to be 2, then it is known that both p_i and q_i are 1. Then if $E_{p_i} = 1$ we know that $k_i = 0$, but if $E_{p_i} = 0$ then $k_i = 1$. Likewise, if $p_i + q_i$ is known to be 0, then it is known that both p_i and q_i are 0. Then if $E_{p_i} = 0$ we know that $k_i = 0$, but if $E_{p_i} = 1$ then $k_i = 1$. If significant portions of the keystream are discovered in this way, then decoded parts of each message will aid in deducing further pieces of the keystream, perhaps resulting in complete cryptanalysis.

8.3. Exponentiation Ciphers

8.3.1. Since $25 < p < 2525, m = 1$ The numerical equivalents for GOOD MORNING, in blocks of $2 = 2m$ digits, are 06 14 14 03 12 14 17 13 08 12 06. Raising each of these 2-digit numbers to the 3rd power and reducing modulo 101 gives: 14 17 17 27 11 17 65 76 07 76 14.

8.3.2. Converting the plaintext to numerical equivalents and grouping in 4-digit blocks gives $1822\ 0404\ 1903\ 1704\ 0012$. Raising each of these blocks to the 7th power and reducing modulo 2621 yields $0394\ 1679\ 1804\ 0755\ 0117$ for the ciphertext.

8.3.3. We find that 17 is an inverse of 5 modulo $28 = \phi(29)$. We raise each block to the 17th power and reduce modulo 29 to get 01 04 00 12 12 04 20 15, which are the numerical equivalents of BEAM ME UP.

8.3.4. An inverse for 13 modulo 2590 is 797. For each block of ciphertext **C**, we find the corresponding block of plaintext by the formula $\mathbf{P} \equiv \mathbf{C}^{797} \pmod{2591}$. This gives us 0314 1314 1917 0400 0319 0708 1823, which is the numerical equivalent of DO NO TR EA DT HI SX.

8.3.5. We encipher messages using the transformation $c \equiv P^{11} \pmod{31}$. The deciphering exponent is the inverse of 11 modulo 30 since $\phi(31) = 30$. But 11 is its own inverse modulo 30 since $11 \cdot 11 \equiv 121 \equiv 1 \pmod{30}$. It follows that 11 is both the enciphering and deciphering exponent.

8.3.6. We have $24 \equiv 20^e \pmod{29}$ and we know that $(e, 28) = 1$ so e must be odd and not 7 or 21. We try 3: $20^3 \equiv 25 \pmod{29}$. We try 5: $20^5 \equiv 24 \pmod{29}$ so $e = 5$. Now $d = 17$ is an inverse for 5 modulo 28, so we raise each cipher block to the 17th power and reduce modulo 29: $(04)^{17} \equiv 06, (19)^{17} \equiv 14, 11^{17} \equiv 03, 24^{17} \equiv 20, 09^{17} \equiv 04$, and $15^{17} \equiv 18 \pmod{29}$. And 06 14 14 03 06 20 04 15 15 stands for GOOD GUESS.

8.4. Public Key Cryptography

8.4.1. Suppose that $n = pq = 14647$ and $\phi(n) = 14400$. Since $\phi(n) = (p-1)(q-1) = pq - (p+q) + 1$, we have $14400 = 14647 - (p+q) + 1$ we have $p + q = 248$. Also, we have $p - q = \sqrt{(p+q)^2 - 4n} = \sqrt{248^2 - 4 \cdot 14647} = \sqrt{2916} = 54$. When we add $p + q = 248$ and $p - q = 54$ we see that $2_p = 302$. Hence $p = 151$ and $q = 97$.

8.4.2. Since $\phi(n) = pq - p - q + 1 = 4386607 - p - q + 1 = 4382136$, hence $p + 1 = 4472$. Also $\phi(n) = (p-1)(q-1) = 4382136 = 8 \cdot 9 \cdot 11 \cdot 11 \cdot 503$, so $503 \mid p - 1$, say. Now $p < 4472$ so $\frac{p-1}{503} < \frac{4472}{503} < 9$. So the possibilities for p are: $503 \cdot 2 + 1, 503 \cdot 3 + 1, 503 \cdot 4 + 1, 503 \cdot 6 + 1$, and $503 \cdot 8 + 1$. Of these only $503 \cdot 6 + 1 = 3019$ is prime. Then $q = 4472 - 3019 = 1453$.

8.4.3. Since a block of ciphertext p is less than n, we must have $(p, n) = p$ or q. Therefore the cryptanalyst has a factor of n.

8.4.4. The probability a message P is not relatively prime to $n = pq$ is the probability that a randomly selected integer between 0 and $n - 1$, inclusive, is divisible by p or by q. The probability such an integer is divisible by p is $1/p$ since there are q integers in the range of pq integers divisible by p, that it is divisible by q is $1/q$ since there are p integers in the range divisible by q, and that it is divisible by both p and q is $1/pq$ since among the integers in the range only 0 is divisible by both p and q. Hence the probability that $(P, n) > 1$ is $1/p + 1/q - 1/pq$. When p and q are both greater than 10^{100} this probability is less than $1/10^{100} + 1/10^{100} - 1/(10^{100})^2 = 2/10^{100} - 1/10^{200} < 1/10^{99}$.

8.4.5. We first translate the letters of BEST WISHES into their numerical equivalents. We group together numbers into blocks of four digits since $n = 2669$. This gives 01041819220818070418. We use the transformation $C \equiv P^3 \pmod{2669}$ to encipher the message. We have $104^3 \equiv 1215 \pmod{2669}$, $1819^3 \equiv 1224 \pmod{2669}$, $2208^3 \equiv 1471 \pmod{2669}$, $1807^3 \equiv 23 \pmod{2669}$, $418^3 \equiv 116 \pmod{2669}$. Hence the ciphertext is 1215 1224 1471 0023 0116.

8.4.6. We group the letters into block of two and convert to the numerical equivalents to get 1108 0504 0818 0003 1704 0012. We raise each block to the 7th power and reduce modulo 2627 to get the ciphertext 1019 0014 1066 2187 1349 2155.

8.4.7. Since $2747 = 41 \cdot 67$, we have $\phi(2747) = 40 \cdot 66 = 2640$. An inverse for 13 modulo 2640 is 2437, so we raise each ciphertext block to the 2437 power modulo 2747. For instance, $2206^{2437} \equiv 0617 \pmod{2747}$. The entire plain text is 0617 0404 1908 1306 1823, which corresponds to the message GR EE TI NG SX.

8.4.8. Since $2881 = 43 \cdot 67, \phi(2881) = 42 \cdot 66 = 2772$. Since $5 \cdot 1109 \equiv 1 \pmod{2772}, 1109$ is an inverse for 5 modulo 2772. Therefore we perform the transformation $P \equiv C^{1109} \pmod{2881}$ to each 4-digit block

of ciphertext. For instance $0504^{1109} \equiv 0400 \pmod{2881}$. Similarly we find 1902, 0714, 0214, 1100, 1904, 0200, and 1004 as the other blocks of plaintext. The letters for these are EA TC HO CO LA TE CA KE.

8.4.9. We convert the plaintext into numerical equivalents and group into blocks of 4 (appending an X) to get 1804 1111 1314 2223. Applying the enciphering algorithm to the first block yields $C \equiv 1804 \cdot 1809 \equiv 2145 \pmod{2573}$. We encrypt the other blocks the same way. The ciphertext is 0872 2263 1537 2392.

8.4.10. We convert the plaintext into numerical equivalents and group into blocks of 4 (appending an X) to get 1104 0021 0419 1422 1323. Applying the enciphering algorithm to the first block yields $C \equiv 1104 \cdot 1115 \equiv 2145 \pmod{3901}$. We encrypt the other blocks the same way. The ciphertext is 2145 0672 0724 1404 1630.

8.4.11. No. It is as if the encryption key were (e_1e_2, n), and it is no more difficult (or easy) to discover the inverse of $e = e_1e_2$ than it would be to discover the inverse of either of the factors modulo $\phi(n)$.

8.4.12. Suppose $n = pq$. If $(P, n) > 1$, we must have $(P, n) = p$ or q, assuming that the numerical plaintext block $P < n$. Without loss of generality, suppose $(P, n) = p$. The corresponding ciphertext is $C \equiv P^e \pmod{n}$. Then $C \equiv 0 \pmod{p}$, and so $p|C$. Since it is computationally feasible to compute greatest common divisors, we can find one factor of n, by computing $(C, n) = p$. Then $q = n/p$ and we have factored n. Having factored n, we can easily compute $\phi(n)$ and then $d \equiv e^{-1} \pmod{n}$, which is the private key.

8.4.13. Suppose P is a plaintext message and the two encrypting exponents are e_1 and e_2. Let $a = (e_1, e_2)$. Then there exist integers x and y such that $e_1x + e_2y = 1$. Let $C_1 \equiv P^{e_1} \pmod{n}$ and $C_2 \equiv P^{e_2} \pmod{n}$ be the two cipher texts. Since C_1, C_2, e_1, and e_2 are known to the decipherer, and since x and y are relatively easy to compute, then it is also easy to compute $C_1^x C_2^y \equiv P^{e_1x} P^{e_2y} \equiv P^{e_1x+e_2y} \equiv P^a \pmod{n}$. If $a = 1$, then P has been recovered. If a is fairly small, then it may not be too difficult to compute ath roots of P^a and thereby recover P.

8.4.14. Assuming the three moduli are pairwise relatively prime, we can use the Chinese remainder theorem to solve the system of congruences to give us a least nonnegative integer $x \equiv P^3 \pmod{n_1n_2n_3}$. Since, by construction $P < n_i$ for $i = 1, 2, 3$, we have $P^3 < n_1n_2n_3$. By the uniqueness guaranteed by the Chinese remainder theorem, x must be a perfect cube, whose cube root is, therefore, easy to compute. This will be the plaintext P.

8.4.15. Encryption works the same as for the two prime case. For decryption, we must compute an inverse d for e modulo $\phi(n) = (p-1)(q-1)(r-1)$ where $n = pqr$ the product of three primes. Then we proceed as in the two prime case.

8.4.16. Suppose $n_1 = p_1q_1$ and $n_2 = p_2q_2$. Then if $(n_1, n_2) > 1$ we must have $(n_1, n_2) = p_1$ or q_1, since $n_1 \neq n_2$. Without loss of generality, suppose $(n_1, n_2) = p_1$. Since it is computationally feasible to compute greatest common divisors, we have a factor of n_1, and can easily compute $q_1 = n/p_1$, to have the complete prime factorization for n_1. Further, since $p_1 \mid n_2$, we have that $p_1 = p_2$ or q_2, and we also have the complete factorization for n_2, and the system is broken.

8.5. Knapsack Ciphers

8.5.1. a. We have $3 < 5$, $3 + 5 = 8 < 9$, $3 + 5 + 9 = 17 < 19$, and $3 + 5 + 9 + 19 = 36 < 40$. Hence the sequence is super-increasing.

b. We have $2 < 6$, $2 + 6 = 8 < 10$, but $2 + 6 + 10 = 18 > 15$. Hence the sequence is not super-increasing.

c. We have $3 < 7$, $3 + 7 = 10 < 17$, $3 + 7 + 17 = 27 < 30$, and $3 + 7 + 17 + 30 = 57 < 59$. Hence the sequence is super-increasing.

d. We have $11 < 21$, $11 + 21 = 32 < 41$, $11 + 21 + 41 = 73 < 81$, but $11 + 21 + 41 + 81 = 154 > 151$. Hence the sequence is not super -increasing.

8.5.2. Suppose that $a_1, a_2, \cdots, a_n$ is super-increasing. We will prove that $a_j \geq 2^{j-1}$ using the second principle of mathematical induction. For $j = 1$ we have $a_1 \geq 1$ since a_1 is a positive integer. Hence $a_1 \geq 2^{1-1} = 2^0 = 1$. Now assume that $a_j > 2^{j-1}$ for every positive integer j with $1 \leq j < k$. Since $a_i + a_2 + \cdots + a_{k-1} \leq a_k$ the inductive hypothesis shows that $a_k \geq 1 + 2 + \cdots + 2^{k-2} = 2^{k-1} - 1$. Hence $a_k \geq 2^k$. This completes the proof.

8.5.3. Proceed by induction. Certainly $a_1 < 2a_1 < a_2$. Suppose $\sum_{j=1}^{n-1} a_j < a_n$. Then $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n-1} a_j + a_n < a_n + a_n = 2a_n < a_{n+1}$. This completes the induction step.

8.5.4. If the largest integer in the sum is 16 we can only obtain 18 by taking $16 + 2$; every other sum with 16 in it is greater than 18. If 13 is the largest integer in the sum we can include 2 and then also include 3 to find that $13 + 3 + 2 = 18$. This is the only way to obtain a sum of 18 with 13 included. Suppose the largest integer in the sum is 11. If we also include 2 we find that $11 + 5 + 2 = 18$. If we include 3 we find that $11 + 4 + 3 = 18$. We also see that $11 + 7 = 18$. If the largest integer in the sum is 7 the largest the sum can be is $2 + 3 + 4 + 7 = 16$. Hence thee only sums that equal 18 are $16 + 2$, $13 + 3 + 2$,$11 + 5 + 2$, $11 + 4 + 3$, and $11 + 7$.

8.5.5. We multiply each element by 17 and reduce modulo 163 to get: (17, 51, 85, 7, 14, 45, 73).

8.5.6. We multiply each element of the sequence by 29 and reduce modulo 331 to get (162, 220, 80, 32, 6). Then we convert each letter to its binary equivalent: B =00001, U=10100, Y=11000, N=01101, O=01110, W=10110. Then B becomes 6, U becomes $162 + 80 = 242$, Y becomes $162 + 220 = 382$, N becomes $220 + 80 + 6 = 306$, O becomes $220 + 80 + 32 = 332$, and W becomes $162 + 80 + 32 = 274$.

8.5.7. 273 is the inverse of 17 modulo 464. We multiply each ciphertext element by 273 and reduce modulo 464 to get 242 59 280 101. Then $242 = 22 + 41 + 179$ and so corresponds to 01101 which is the binary equivalent for N. $59 = 18 + 41$ and 10100 stands for U. $280 = 18 + 83 + 179$ and 10011 stands for T. $101 = 18 + 83$ and 10010 stands for S. So the plaintext message is NUTS.

8.5.8. We multiply each element by 7 and reduce modulo 92 to get (21,28,56,27,47,9). Then we multiply by 11 and reduce modulo 95 to get (41,23,46,12,42,4). Finally, we multiply by 6 and reduce modulo 101 to get (44,37,74,72,50,24).

8.5.9. If the multipliers and moduli are $(w_1, m_1), (w_2, m_2), \ldots, (w_r, m_r)$, the inverse $\overline{w_1}, \overline{w_2}, \ldots, \overline{w_r}$ can be computed with respect to their corresponding moduli. Then we multiply and reduce succesively by $(\overline{w_r}, m_r), (\overline{w_{r-1}}, m_{r-1}), \ldots, (\overline{w_1}, m_1)$. The result will be the plaintext sequence of easy knapsack problems.

8.5.10. We have the following prime factorizations: $60 = 2^2 \cdot 3 \cdot 5$, $2 = 2$, $3 = 3$, $5 = 5$, $6 = 2 \cdot 3$, and $10 = 2 \cdot 5$. To obtain 60 by multiplying terms from $2, 3, 5, 6, 10$ we need to multiply integers so that the sum of the powers of 2 in these integers is 2 and so that there is one factor of 3 and one factor of 5 in these integers. This can be done as follows: $60 = 2 \cdot 3 \cdot 10$, $60 = 2 \cdot 5 \cdot 6$, and $60 = 6 \cdot 10$.

8.5.11. Since $5 \mid 15960$ the product must contain 95, the only element divisible by 5, so $15960 = 95 \cdot 168$. 8 is the only even element so we must have $95 \cdot 8 \cdot 21$ as the only possibility.

8.5.12. If p is a prime factor of $P = a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n}$, then p must divide one of the a_i's. Since the a_i's are pairwise relatively prime, at most one of them is divisible by p, and that one must be in the factorization. Thus, by examining all the prime factors of P, we can determine all a_i's that must be in the product. If the product of these a_i's actually equals P, then we have a unique solution. If they don't, there is no solution.

8.5.13. For $i = 1, 2, \ldots, n$, we have $b^{\alpha_i} \equiv a_i \pmod{m}$. Then $b^S \equiv P \equiv (b^{\alpha_1})^{x_1} (b^{\alpha_2})^{x_2} \cdots (b^{\alpha_n})^{x_n} \equiv b^{\alpha_1 x_1 + \cdots + \alpha_n x_n} \pmod{n}$. Then $S \equiv \alpha_1 x_1 + \cdots + \alpha_n x_n \pmod{\phi(m)}$. Since $S + k\phi(m)$ is also a logarithm of P to the base b we may take the congruence to be an equation. Since the $x_i = 0$ or 1, this becomes an

additive knapsack problem on the sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

8.5.14. Given the binary equivalent for a plaintext block, we can assign the value 1 or 0 to the x_i's accordingly. Then the knapsack cipher $S = \alpha_1 x_1 + \cdots + \alpha_n x_n$ is a generally difficult knapsack problem. But the receiver who knows b and m can easily convert the problem to $b^s \equiv P \equiv b^{\alpha_1 x_1 + \cdots + \alpha_n x_n} \equiv a^{x_1} \cdots a^{x_n} \pmod{n}$ as in Exercise 13. Then by Exercise 12, this is an easy multiplicative knapsack problem, so the receiver can recover the values of the x_i's and hence the plaintext.

8.6. Cryptographic Protocols and Applications

8.6.1. The first party, having chosen $k_1 = 27$, computes $y_1 \equiv 5^{27} \equiv 94 \pmod{103}$ and sends it to the second party. The second party, having chosen $k_2 = 31$, computes $K \equiv 94^{31} \equiv 90 \pmod{103}$.

8.6.2. The first party, having chosen $k_1 = 7$, computes $y_1 \equiv 2^7 \equiv 22 \pmod{53}$ and sends it to the second party. The second party, having chosen $k_2 = 8$, computes $K \equiv 22^8 \equiv 16 \pmod{53}$.

8.6.3. We compute $K = ((7^3)^{10})^5 \equiv 7^{150} \pmod{601}$. Using a calculator or computational software we find $K \equiv 7^{150} \equiv 476 \pmod{601}$.

8.6.4. We must compute $3^{11 \cdot 12 \cdot 17 \cdot 19} \pmod{1009}$. We find that $3^{504} \equiv 1 \pmod{1009}$ and $11 \cdot 12 \cdot 17 \cdot 19 \equiv 300 \pmod{504}$. Hence $3^{11 \cdot 12 \cdot 17 \cdot 19} \equiv 3^{300} \equiv 150 \pmod{1009}$.

8.6.5. Let $k_1, k_2, \ldots, k_n$ be the private keys for parties 1 through n respectively. There are n steps in this protocol. The first step is for each of the parties 1 through n to compute the least positive residue of $r^{k_i} \pmod{p}$ and send this value y_i to the $i+1$st party. (The nth party sends his value to the 1st party.) Now the ith party has the value y_{i-1} (where we take y_0 to be y_n.) The second step is for each party to compute the least positive residue of $y_{i-1}^{k_i} \pmod{p}$ and send this value to the $i+1$st party. Now the ith party has the least positive residue of $r^{k_{i-1}+k_{i-2}} \pmod{p}$. This process is continued for a total of n steps. However, at the nth step, the computed value is not sent on to the next party. Then the ith party will have the least positive residue of $r^{k_{i-1}+k_{i-2}+\cdots+k_1+k_n+k_{n-1}+\cdots k_{i+1}+k_i} \pmod{p}$, which is exactly the value of K desired.

8.6.6. a. We have $\phi(19 \cdot 67) = 1188$ and 713 an inverse for $5 \pmod{1188}$. So Romeo finds the numerical equivalents for his message: 0614 1403 0124 0418 2204 0419 1114 2104. Then he applies his decryption function to each block, that is he raises each block to the 713th power modulo $19 \cdot 67$, to get: 1100 0731 0945 0304 0285 0324 1046 1248. Then, since Juliet's modulus is smaller than his, he splits each block in two before encrypting them with Juliet's public key. He raises each block of size 2 to the 3rd power and reduces modulo $11 \cdot 71$ to get the signed ciphertext: 550, 000, 343, 113, 729, 529, 027, 064, 008, 259, 027, 547, 219, 492, 166, 471 which he sends to Juliet.

b. Juliet knows that 467 is an inverse for 3 modulo $\phi(11 \cdot 71) = 700$. So she applies her decryption functions to the numerical equivalents of her message: 00 03 08 04 20 05 14 17 04 21 04 17 to get 000 361 002 555 598 025 372 492 555 615 555 492. Since Romeo's modulus is larger than hers, she proceeds to encrypt the message with Romeo's key. She raises each block to the 5th power and reduces modulo $19 \cdot 67$ to get 000 266 32 1119 225 442 900 1127 1119 999 1119 1127 as the signed ciphertext.

8.6.7. a. We have $\phi(23 \cdot 47) = 1012$ and 675 an inverse for $3 \pmod{1012}$. The numerical equivalents for CHEERS HAROLD are 02 07 04 04 17 18 07 00 17 14 11 03, using blocks of 2 since $25 < 1081 < 2525$. The first step to perform the transformation $D \equiv P^{675} \pmod{1081}$ on each block, which gives us 0867 1003 0394 0394 0521 0625 1003 0000 0521 0477 1022 0357. Next we perform the transformation $C \equiv d^7 \pmod{1829}$ which gives 0371 0354 0858 0858 0087 1369 0354 0000 0087 1543 1797 0535.

b. We have $\phi(31 \cdot 59) = 1740$ and 1243 an inverse for 7 modulo 1740. The numerical equivalents for SINCERELY AUDREY are 18 08 13 02 04 17 04 11 24 0 20 03 17 04 24. We take each and perform the transformation $D \equiv P^{1243} \pmod{1829}$ and perform the transformation $C \equiv D^3 \pmod{1081}$. This

gives 0833 0475 0074 0323 0621 0105 0621 0865 0421 0000 0746 0803 0105 0621 0421.

8.6.8. a. The block size $2m$ is chosen so that every possible block of numerical equivalents is less than n_i. This ensures that each block will be unique modulo n. Since $n_i < H < n_{j^*}$, then each block will be unique modulo n_{j^*} as well.

b. Individual j knows $\overline{e_j}$ modulo n_{j^*}, so he can compute $D_{k_{j^*}}(E_{k_{j^*}}(d_{k_i}(P))) = D_{k_i}(P)$. Since he also knows e_i, he can compute $E_{k_i}(D_{k_i}(P)) = P$. Since only individual i knows $\overline{e_i}$ modulo n_i, only he could have produced $D_{k_i}(P)$, and thereby make $E_{k_i}(d_{k_i}(P))$ intelligible.

c. $\phi(781) = 700$. 467 is the inverse for 3 modulo 700, so $D_{k_i}(P) \equiv P^{467} \pmod{781}$. So the plaintext numbers for HELLO ADAM, 7 4 11 11 14 0 3 0 12, become 0138 0555 0033 0033 0372 0000 0361 0000 0419. Then apply $E_{k_{j^*}}(D) = D^7 \pmod{1147}$ to each 4-digit block gives 0360 0851 0562 0562 0868 0000 0576 0000 0194. $\phi(893) = 828$. 355 is the inverse for 7 modulo 828, so $D_{k_j}(P) \equiv P^{355} \pmod{828}$. Then $E_{k_{i^*}} \equiv D^3 \pmod{1189}$. This gives us 0921 0888 0888 0659 0001 0951 0575 0000 0890 1030 0700 0575 as the encryption for GOODBYE ALICE.

8.6.9. a. If $n_i < n_j$, the block sizes are chosen small enough so that each block is unique modulo n_i. Since $n_i < n_j$, each block will be unique modulo n_j after applying the transformation D_{k_j}. Therefore we can apply E_{k_j} to $D_{k_i}(P)$ and retain uniqueness of blocks. If $n_i > n_j$ the argument is similar.

b. If $n_i < n_j$, individual j receives $E_{k_j}(D_{k_i}(P))$ and know an inverse for e_j modulo $\phi(n_i)$. So he can apply $D_{k_j}(E_{k_j}(D_{k_i}(P))) = D_{k_i}(P)$. Since he also knows e_i, he can apply $E_{k_i}(D_{k_i}(P)) = P$ and discover the plaintext P. If $n_i > n_j$, individual j receives $D_{k_i}(E_{k_j}(P))$. Since he knows e_i he can apply $E_{k_i}(D_{k_i}(E_{k_j}(P))) = E_{k_j}(P)$. Since he also knows $\overline{e_j}$ he can apply $D_{k_j}(E_{k_j}(P)) = P$ and discover the plaintext P.

c. Since only individual i knows $\overline{e_i}$, only he can apply the transformation D_{k_i} and thereby make $E_{k_i}(D_{k_i}(P))$ intelligible.

d. $n_i = 2867 > n_j = 2537$, so we compute $D_{k_i}(E_{k_j}(P))$. Both n_i and $n_j > 2525$ so we use blocks of 4. REGARDS FRED becomes 1704 0600 1703 1805 1704 0323 (adding an X to fill out the last block.) $e_i = 11$ and $\phi(n_i) = 2760$, so $\overline{e_i} = 251$. We apply $E_{k_j} \equiv P^{e_j} \equiv P^{13} \pmod{2537}$ to each block and get 1943 0279 0847 0171 1943 0088. Then we apply $D_{k_i}(E) = E^{251} \pmod{2867}$ and get 0479 2564 0518 1571 0479 1064. Now since $n_j < n_i$ individual j must send $E_{k_i}(D_{k_j}(P)), e_j = 13, \phi(2537) = 2436$ and $\overline{e_j} = 937$. Then $D_{k_j}(P) \equiv P^{937} \pmod{2537}$ and $E_{k_i}(D) = D^{11} \pmod{2867}$. The cipher text is 1609 1802 0790 2508 1949 0267.

8.6.10. Since $t = 14$, we compute $K_0 = K + tp = 5 + 14 \cdot 7 = 103$. The three shadows are given by $k_1 \equiv 103 \equiv 4 \pmod{11}$, $k_2 \equiv 103 \equiv 7 \pmod{12}$ and $k_3 \equiv 103 \equiv 1 \pmod{17}$.

8.6.11. Suppose the master key $K = 3, p = 5, M_1 = 8, m_2 = 9, m_3 = 11$, and $t = 13$. Then $M = m_1 m2 = 72 > p \cdot m_3 = 5 \cdot 11 = 55$ and $t = 13 < \frac{M}{p} = \frac{72}{5}$. We have $K_0 = K + tp; = 3 + 13 \cdot 5 = 68$. The shadows k_1, k_2, and k_3 are given by $k_1 \equiv 68 \equiv 4 \pmod{8}, k_2 \equiv 68 \equiv 5 \pmod{9}$, and $k_3 \equiv 68 \equiv 2 \pmod{11}$.

8.6.12. The 3 shadows from Exercise 10 are $k_1 = 4, k_2 = 7$ and $k_3 = 1$. If k_1 and k_2 are known, we solve the system of congruences $x \equiv 4 \pmod{11}, x \equiv 7 \pmod{12}$ to get $x = 103$. If k_1 and k_3 are known, we solve the system of congruences $x \equiv 4 \pmod{11}, x \equiv 1 \pmod{17}$ to get $x = 103$. If k_2 and k_3 are known, we solve the system of congruences $x \equiv 7 \pmod{12}, x \equiv 1 \pmod{17}$ to get $x = 103$. In all three cases we recover K_0. Then $K = K_0 - tp = 103 - 14 \cdot 7 = 5$.

8.6.13. The 3 shadows from Exercise 11 are $k_1 = 4, k_2 = 5$ and $k_3 = 2$. If k_1 and k_2 are known, we solve the system of congruences $x \equiv 4 \pmod{8}, x \equiv 5 \pmod{9}$ to get $x = 68$. If k_1 and k_3 are known, we solve the system of congruences $x \equiv 4 \pmod{8}, x \equiv 2 \pmod{11}$ to get $x = 68$. If k_2 and k_3 are known, we solve the system of congruences $x \equiv 5 \pmod{9}, x \equiv 2 \pmod{11}$ to get $x = 68$. In all three cases we recover

K_0. Then $K = K_0 - tp = 68 - 13 \cdot 5 = 3$.

8.6.14. We choose $p = 23$ and mutually relatively prime moduli $m_1 = 41, m_2 = 43, m_3 = 45, m_4 = 47, m_5 = 49$. Then since $41 \cdot 43 \cdot 45 = 79335 > 52969 = 23 \cdot 47 \cdot 49$, the moduli satisfy inequality 8.7. Now $M/p = 41 \cdot 43 \cdot 45/23 = 3449.34\ldots$, so we may pick $t = 33$. Then $K_0 = 22 + 33 \cdot 23 = 781$. Then $k_1 \equiv 781 \equiv 2 \pmod{41}$, $k_2 \equiv 781 \equiv 7 \pmod{43}$, $k_3 \equiv 781 \equiv 16 \pmod{45}$, $k_4 \equiv 781 \equiv 29 \pmod{47}$, and $k_5 \equiv 781 \equiv 46 \pmod{49}$. Suppose we have the shadows $k_1 = 2$, $k_3 = 16$ and $k_5 = 46$. If we solve the system $x \equiv 2 \pmod{41}, x \equiv 16 \pmod{45}$, and $x \equiv 46 \pmod{49}$, the Chinese remainder theorem gives us $x = 781 = K_0$. Then $K = 781 - 33 \cdot 23 = 22$.

CHAPTER 9

Primitive Roots

9.1. The Order of an Integer and Primitive Roots

9.1.1. a. Since the order of an integer modulo 5 divides $\phi(5) = 4$, the order of an integer modulo 5 must equal 1, 2, or 4. Since $2^2 \equiv 4 \not\equiv 1 \pmod 5$ the order of 2 modulo 5 is 4.

b. Since the order of an integer modulo 10 divides $\phi(10) = 4$, the order of an integer modulo 10 must equal 1, 2, or 4. Since $3^2 \equiv 9 \not\equiv 1 \pmod{10}$, the order of 3 modulo 10 is 4.

c. Since the order of an integer modulo 13 divides $\phi(13) = 12$, the order of an integer modulo 12 must equal 1, 2, 3, 4, 6, or 12. We have $10^2 = (-3)^2 \equiv 9 \pmod{13}$, $10^3 \equiv 9 \cdot (-3) \equiv -1 \pmod{13}$, $10^4 \equiv (-1) \cdot (-3) \equiv 3 \pmod{13}$, and $10^6 = 10^3 \cdot 10^3 \equiv (-1)^2 = 1 \pmod{13}$. If follows that the order of 10 modulo 13 is 6.

d. Since the order of an integer modulo 10 divides $\phi(10) = 4$, the order of an integer modulo 10 must equal 1, 2, or 4. We have $7^2 = 49 \equiv 9 \equiv 1 \pmod{10}$, hence the order of 7 modulo 10 is 4.

9.1.2. a. Since $\text{ord}_{11}3$ must divide $\phi(11) = 10$, $3^2 \equiv -2 \pmod{11}$, and $3^5 \equiv 1 \pmod{11}$, we have $\text{ord}_{11}3 = 5$.

b. We have $\phi(17) = 16$. Then $2^4 \equiv -1 \pmod{17}$, so $2^8 \equiv 1 \pmod{17}$. Therefore, $\text{ord}_{17}2 = 8$.

c. We have $\phi(21) = 12$. Then $10^2 \equiv -5 \pmod{21}, 10^3 \equiv 13 \pmod{21}, 10^4 \equiv 4 \pmod{21}$, and $10^6 \equiv 1 \pmod{21}$. Therefore, $\text{ord}_{21}10 = 6$.

d. We have $\phi(25) = 20$. Then $9^2 \equiv 6 \pmod{25}, 9^4 \equiv 11 \pmod{25}, 9^5 \equiv -1 \pmod{25}$, and $9^{10} \equiv 1 \pmod{25}$. Therefore, $\text{ord}_{25}9 = 10$.

9.1.3. a. We have $\phi(6) = 2$, and $5^2 \equiv 1 \pmod 6$.

b. We have $\phi(11) = 10, 2^2 \equiv 4, 2^5 \equiv -1, 2^{10} \equiv 1 \pmod{11}$.

9.1.4. a. The order of 3 modulo 4 is $\phi(4) = 2$. Hence 3 is a primitive root of 4.

b. The order of 2 modulo 5 is $\phi(5) = 4$ since $2^2 \equiv -1 \pmod 4$. Hence 2 is a primitive root of 5.

c. The order of 3 modulo 10 is $\phi(10) = 3$ from Exercise 1(b). Hence 3 is a primitive root of 10.

d. We have $\phi(13) = 12$. The proper divisors of 12 are 1, 2, 3, 4, and 6. Then $2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 3$, and $2^6 \equiv -1 \pmod{13}$. So 2 is a primitive root modulo 13.

e. 3 is a primitive root modulo 14.

f. 2 is a primitive root modulo 18.

9.1.5. Only 1, 5, 7, 11 are prime to 12. Each one squared is congruent to 1, but $\phi(12) = 4$.

9.1.6. We have $\phi(20) = 8$. The proper divisors of 8 are 1, 2, and 4. The integers relatively prime to 20 are 1, 3, 7, 9, 11, 13, 17, and 19. Note that $1^4 \equiv 3^4 \equiv 7^4 \equiv 9^4 \equiv 11^4 \equiv 13^4 \equiv 17^4 \equiv 19^4 \equiv 1 \pmod{20}$. Therefore,

no element has order 8 and hence there are no primitive roots modulo 20.

9.1.7. Since $\phi(\phi(14)) = \phi(6) = 2$, there are 2: 3 and 5.

9.1.8. There are $\phi(\phi(13)) = \phi(12) = 4$ primitive roots modulo 13. The possible order for an integer modulo 13 is 1, 2, 3, 4, 6, or 12. We have $2^2 \equiv 4 \pmod{13}, 2^3 \equiv 8 \pmod{13}, 2^4 \equiv 16 \pmod{13}$, and $2^6 = 12 \pmod{13}$, so that 2 is a primitive root modulo 12. We know that the primitive roots of 12 are the least positive residues of 2^u where $(u, \phi(13)) = (u, 12) = 1$. Hence 2, $2^5 \equiv 6 \pmod{13}, 2^7 \equiv 11 \pmod{13}$, and $211 \equiv 7 \pmod{13}$ are a set of 4 incongruent primitive roots of 13.

9.1.9. That $\text{ord}_n a = \text{ord}_n \overline{a}$ follows from the fact that $a^t \equiv 1 \pmod{n}$ if and only if $\overline{a}^t \equiv 1 \pmod{n}$. To see this, suppose that $a^t \equiv 1 \pmod{n}$. Then $\overline{a}^t \equiv (\overline{a}^t a^t)(a^t) \equiv (a\overline{a})^t a^t \equiv 1^t \cdot 1 \equiv 1 \pmod{n}$. The converse is shown in a similar manner.

9.1.10. Let $r = \text{ord}_n a$, $s = \text{ord}_n b$, and $r = \text{ord}_n ab$. Then we have $(ab)^{rs} \equiv (a^r)^s (b^s)^r \equiv 1^s 1^r \equiv 1 \pmod{n}$. So $t \mid rs$. On the other hand, $1 \equiv (ab)^t \equiv (ab)^{rt} \equiv (a^r)^t b^{rt} \equiv b^{rt} \pmod{n}$. By Theorem 9.1, $s \mid rt$, but $(r, s) = 1$, so $s \mid t$. Similarly $r \mid t$. Again, since $(r, s) = 1$, we have $rs \mid t$. Therefore $rs = t$ as desired.

9.1.11. We have $[r, s]/(r, s) \leq \text{ord}_n ab \leq [r, s]$

9.1.12. This is false. For a counterexample, let $n = 8$, so that $\phi(n) = \phi(8) = 4$. Since $(a, 8) = 1$ implies that a is odd, and $a^2 \equiv 1 \pmod{8}$ whenever a is odd, the order of an integer modulo 8 is no more that two, and hence cannot equal 4.

9.1.13. Let $r = \text{ord}_m a^t$, then $a^{tr} \equiv 1 \pmod{m}$, hence $tr \geq ts$ and $r \geq s$. Since $1 \equiv a^{st} \equiv (a^t)^s \pmod{n}$, we have $s \geq r$.

9.1.14. Suppose that m was not prime. Then $\phi(m) < m - 1$. Since $\text{ord}_m a \mid \phi(m)$ it follows that $\text{ord}_m a$ is less than $m - 1$. This shows that if there is an integer a relatively prime to m such that $\text{ord}_m a = m - 1$ then m is prime.

9.1.15. Suppose that r is a primitive root modulo the odd prime p. Then $r^{(p-1)/q} \not\equiv 1 \pmod{p}$ for all prime divisors q of $p - 1$ since no smaller power than the $(p-1)$st of r is congruent to 1 modulo p. Conversely, suppose that $r^{(p-1)/q} \not\equiv 1 \pmod{p}$ for all prime divisors of $p - 1$. Suppose that r is not a primitive root of p. Then there is an integer t such that $r^t \equiv 1 \pmod{p}$ with $t < p - 1$. Since t must divide $p - 1$, we have $p - 1 = st$ for some positive integer s greater than 1. Then $(p - 1)/s = t$. Let q be a prime divisor of s. Then $(p - 1)/q = t(s/q)$, so that $r^{(p-1)/q} = r^{t(s/q)} = (r^t)^{s/q} \equiv 1 \pmod{p}$. This contradicts the original assumption, so r is a primitive root modulo p.

9.1.16. Suppose $\text{ord}_m \overline{r} = h$. Then $h \mid \phi(m)$. Note that $1 \equiv \overline{r}r \equiv (\overline{r}r)^h \equiv r^h \pmod{m}$. Therefore, $\text{ord}_m r = \phi(m) \mid h$. And so, $h = \text{ord}_m r$, and $\overline{r}$ is a primitive root for m.

9.1.17. Since $2^{2^2} + 1 \equiv 0 \pmod{F_n}$, then $2^{2^2} \equiv -1 \pmod{F_n}$. Squaring gives $(2^{2^2})^2 \equiv 1 \pmod{F_n}$. Thus, $\text{ord}_{F_n} 2 \leq 2^n 2 = 2^{n+1}$.

9.1.18. a. Let $h = \text{ord}_p 2$. Then $h \mid \phi(p) = p - 1$. Note that $2^{2^n} \equiv -1 \pmod{p}$, so $(2^{2^n})^2 \equiv 2^{2^{n+1}} \equiv 1 \pmod{p}$. Therefore, $h \mid 2^{n+1}$, say $h = 2^k$. But if $k < n + 1$ and $2^h \equiv 2^{2^k} \equiv 1 \pmod{p}$, then $2^{2^n} \equiv 1 \pmod{p}$, a contradiction. Therefore $h = 2^{n+1}$.

b. Since $2^{n+1} = \text{ord}_p 2 \mid \phi(p) = p - 1$, we have $2^{n+1}k = p - 1$ or $p = 2^{n+1}k + 1$.

9.1.19. Note that $a^t < m = a^n - 1$ whenever $1 \leq t < n$. Hence a^t cannot be congruent to 1 modulo m when t is a positive integer less than n. However, $a_n \equiv 1 \pmod{m}$ since $m = (a^n - 1) \mid (a^n - 1)$. It follows that $\text{ord}_m a = n$. Since $\text{ord}_m a \mid m$, we see that $n \mid \phi(m)$.

9.1.20. a. If $\text{ord}_q 2 \mid (p-1)$ and $\text{ord}_p 2 \mid (q-1)$, then $2^{pq} \equiv (2^p)^q \equiv 2^q \equiv 2 \pmod{q}$, and similarly, $2^{pq} \equiv 2 \pmod{q}$. By the Chinese Remainder Theorem, there exists a unique solution modulo pq to the system $x \equiv 2 \pmod{q}, x \equiv 2 \pmod{p}$. Since 2 and 2^{pq} are both solutions, we must have $2 \equiv 2^{pq} \pmod{pq}$. Therefore, pq is a pseudoprime to the base 2. Conversely, if pq is a pseudoprime to the base 2, then $2^{pq} \equiv 2 \pmod{pq}$ and so $2^{pq} \equiv 2 \pmod{p}$. But by Fermat's Little Theorem, $2^p \equiv 2 \pmod{p}$, so $(2^p)^2 \equiv 2^q \equiv 2 \pmod{p}$. Since $(2, p) = 1$, we have $2^{q-1} \equiv 1 \pmod{p}$ and so $\text{ord}_p 2 \mid (q-1)$. Similarly, $\text{ord}_q 2 \mid (p-1)$.

b. $19 \cdot 73$ and $23 \cdot 89$ are pseudoprimes to the base 2. The other numbers are not.

9.1.21. First suppose that pq is a pseudoprime to the base 2. By Fermat's Little Theorem, $2^p \equiv 2 \pmod{p}$, so there exists an integer k such that $2^p - 2 = kp$. Then $2^{M_p - 1} - 1 = 2^{2^p - 1} - 1 = 2^{kp} - 1$. This last expression is divisible by $2^p - 1 = M_p$ by Lemma 6.1. Hence, $2^{M_p - 1} \equiv 1 \pmod{M_p}$, or $2^{M_p} \equiv 2 \pmod{M_p}$. Since pq is a pseudoprime to the base 2, we have $2^{pq} \equiv 2 \pmod{pq}$, so $2^{pq} \equiv 2 \pmod{p}$. But $2^{pq} \equiv (2^p)^q \equiv 2^q \pmod{p}$. Therefore $2^q \equiv 2 \pmod{p}$. Then there exists an integer l such that $M_q - 1 = 2^q - 2 = lp$. Then $2^{M_q - 1} - 1 = 2^{2^q - 2} = 2^{lp} - 1$, so $2^p - 1 = M_p$ divides $2^{M_q - 1} - 1$. Therefore $2^{M_q} \equiv 2 \pmod{M_p}$. Then we have $2^{M_p M_q} \equiv (2^{M_p})^{M_q} \equiv 2^{M_q} \equiv 2 \pmod{M_p}$. Similarly, $2^{M_p M_q} \equiv 2 \pmod{M_q}$. By the Chinese remainder theorem, noting that M_p and M_q are relatively prime, we have $2^{M_p M_q} \equiv 2 \pmod{M_p M_q}$. Therefore $M_p M_q$ is a pseudoprime to the base 2. Conversely, suppose $M_p M_q$ is a pseudoprime to the base 2. From the reasoning in the proof of Theorem 6.6, we have that $2^{M_p} \equiv 2 \pmod{p}$. Therefore $2^{M_p M_q} \equiv 2^{(M_p - 1)M_q + M_q} \equiv 2^{M_q} \equiv 2 \pmod{p}$. But since $M_p = 2^p - 1 \equiv 0 \pmod{M_p}$, we have that the order of 2 modulo M_p is p. Therefore $p | M_q - 1$. In other words, $2^q \equiv 2 \pmod{p}$. Then $2^{pq} \equiv 2^q \equiv 2 \pmod{p}$. Similarly, $2^{pq} \equiv 2 \pmod{q}$. Therefore, by the Chinese remainder theorem, $2^{pq} \equiv 2 \pmod{pq}$. Therefore, since pq is composite, it is a pseudoprime to the base 2.

9.1.22. We prove that $C_j \equiv C^{e^j} \pmod{n}$ for every positive integer j using mathematical induction. For $j = 0$ we have $C_0 = C \equiv C^{e^0} = C^1 \pmod{n}$, so the basis step holds. Next we carry out the inductive step. Assume that $C_j \equiv C^{e^j} \pmod{n}$. Then $C_{j+1} \equiv (C^{e^j})^e = C^{e^{j+1}} \pmod{n}$. This completes the proof.

9.1.23. Let $j = \text{ord}_{\phi(n)} e$. Then $e^j \equiv 1 \pmod{\phi(n)}$. Since $\text{ord}_n P \mid \phi(n)$, we have $e^j \equiv 1 \pmod{\text{ord}_n P}$. Then by Theorem 8.2, $P^{e^j} \equiv P \pmod{n}$, so $C^{e^{j-1}} \equiv (P^e)^{e^{j-1}} \equiv P^{e^j} \equiv P \pmod{n}$ and $C^{e^j} \equiv P^e \equiv C \pmod{n}$.

9.1.24. Computing the sequence, we have $C_1 = 1504^{17} \equiv 2444 \pmod{47 \cdot 59}$, $C_2 = 2444^{17} \equiv 470 \pmod{47 \cdot 59}$, $C_3 = 470^{17} \equiv 2209 \pmod{47 \cdot 59}$, $C_4 = 2209^{17} \equiv 1504 \pmod{47 \cdot 59}$. Therefore $P = C - 3 = 2209$ which is the numerical equivalent of WJ.

9.2. Primitive Roots for Primes

9.2.1. a. By Lagrange's Theorem, there are at most 2 roots. Since $(\pm 3)^2 + 2 \equiv 0 \pmod{11}$, we have found all the roots.

b. By Lagrange's Theorem, there are at most 2 roots. Since $(\pm 1)^2 + 10 \equiv 0 \pmod{11}$, we have found all the roots.

c. By Lagrange's Theorem, there are at most 3 roots. Note that the polynomial factors thus $x^3 + x^2 + 2x + 2 = x^2(x+1) + 2(x+1) = (x^2+2)(x+1)$. So $x = -1$ is a solution, and the two solutions from part (a) are solutions, and this must be all.

d. By Lagrange's Theorem, there are at most 4 roots. Since the polynomial is an even function it suffices to check only the numbers $0, 1, 2, 3, 4$, and 5 as roots. We find that none of these work, so there are no roots of the polynomial modulo 11.

9.2.2. a. We find that $5^2 + 1 \equiv 8^2 + 1 \equiv 0 \pmod{13}$ are the only 2 solutions.

b. We find that $11^2 + 3 \cdot 11 + 2 \equiv 12^2 + 3 \cdot 12 + 2 \equiv 0 \pmod{13}$ are the only 2 solutions.

c. We find that $1^3 + 12 \equiv 3^3 + 12 \equiv 9^3 + 12 \equiv 0 \pmod{13}$ are the only 3 solutions.

d. We find that $7^4 + 7^2 + 7 + 1 \equiv 0 \pmod{13}$ is the only solution.

9.2.3. a. There are $\phi(7-1) = \phi(6) = 2$ primitive roots modulo 7.

b. There are $\phi(13-1) = \phi(12) = 4$ primitive roots of 13.

c. There are $\phi(17-1) = \phi(16) = 8$ primitive roots of 17.

d. There are $\phi(19-1) = \phi(18) = 6$ primitive roots of 19.

e. There are $\phi(29-1) = \phi(28) = 12$ primitive roots of 29.

f. There are $\phi(47-1) = \phi(46) = 22$ primitive roots of 47.

9.2.4. There must be $\phi(\phi(7)) = 2$ primitive roots modulo 7. Since 3 is one, the other must be 3 raised to a power relatively prime to $\phi(7) = 6$, so we take $3^5 \equiv 9 \cdot 9 \cdot 3 \equiv 2 \cdot 2 \cdot 3 \equiv 12 \equiv 4 \pmod{7}$. Thus 3 and 5 make a complete set of primitive roots modulo 7.

9.2.5. There must be $\phi(\phi(13)) = 4$ primitive roots modulo 13. Since 2 is one, the others must be 2 raised to a power relatively prime to $\phi(13) = 12$. So we take $2^5 \equiv 6 \pmod{13}$, $2^7 \equiv 6 \cdot 4 \equiv 11 \pmod{13}$, and $2^{11} \equiv 6 \cdot 6 \cdot 2 \equiv 7 \pmod{13}$. So a complete set of primitive roots is 2, 6, 7, 11

9.2.6. There must be $\phi(\phi(17)) = 8$ primitive roots modulo 17. Since 3 is one, the other ones must be 3 raised to powers relatively prime to $\phi(17) = 16$, so we take $3^3, 3^5, 3^7, 3^9, 3^{11}, 3^{13}$, and 3^{15} modulo 17. Reducing gives 10, 5, 11, 14, 7, 12, and 6.

9.2.7. Since $\phi(19) = 18$ and $\phi(18) = 6$, we seek 6 primitive roots for 19. Since 2 is one, we raise 2 to the powers which are relatively prime to 18, namely, $2^5, 2^7, 2^{11}, 2^{13}$, and 2^{17}. Reducing modulo 19 gives us 2, 3, 10, 13, 14, 15, as a complete set of primitive roots.

9.2.8. Suppose that r is a primitive root of the prime p where $p \equiv 1 \pmod{4}$. Let t be the order of $-r$ modulo p. We know that $t \mid (p-1)$. Let $tu = p - 1$. We first show that u cannot be odd. If u were odd then $t = (p-1)/u$ is even, so that $r^t = (-r)^t \equiv 1 \pmod{p}$ which is a contradiction since r is a primitive root of p. Now suppose that u is even. Then $(-r)^t = (-r)^{(p-1)/u} \equiv 1 \pmod{p}$. Since u is even, $(p-1)/u \mid (p-1)/2$ so that $(-r)^{(p-1)/2} \equiv 1 \pmod{p}$. But since $p \equiv 1 \pmod{4}$ it follows that $(p-1)/2$ is even. Hence $(-r)^{(p-1)/2} = (-1)^{(p-1)/2} r^{(p-1)/2} = r^{(p-1)/2} \equiv 1 \pmod{p}$. This is a contradiction since r is a primitive root of p.

9.2.9. By Lagrange's Theorem there are at most two solutions to $x^2 \equiv 1 \pmod{p}$, and we know $x \equiv \pm 1$ are the two solutions. Since $p \equiv 1 \pmod{4}, 4 \mid (p-1) = \phi(p)$ so there is an element x of order 4 modulo p. Then $x^4 = (x^2)^2 \equiv 1 \pmod{p}$, so $x^2 \equiv \pm 1 \pmod{p}$. If $x^2 \equiv 1 \pmod{p}$ then x does not have order 4. Therefore $x^2 \equiv -1 \pmod{p}$.

9.2.10. a. We have $0^2 - 0 \equiv 0 \pmod{6}$, $1^2 - 1 \equiv 0 \pmod{6}$, $2^2 - 2 \equiv 2 \pmod{6}$, $3^2 - 3 \equiv 0 \pmod{6}$, $4^2 - 4 \equiv 0 \pmod{6}$, and $5^2 - 5 \equiv 2 \pmod{6}$. Hence there are 4 incongruent roots modulo 6.

b. This does not contradict Lagrange's theorem since 6 is not prime.

9.2.11. a. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0$ and let k be the largest integer such p does not divide a_k. Let $g(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots a_0$. Then $f(x) \equiv g(x) \pmod{p}$ for every value of x. In particular $g(x)$ has the same set of roots as $f(x)$. Since the number of roots is greater than $n > k$, this contradicts

Lagrange's theorem. Therefore, no such k exists and p must divide every coefficient of $f(x)$.

b. Note that the degree of $f(x)$ is $p-2$. By Fermat's little theorem we have that $x^{p-1}-1 \equiv 0 \pmod{p}$, for $x = 1, 2, \ldots, p-1$. Further, each x in the same range is a zero for $(x-1)(x-2)\cdots(c-p+1)$. Therefore, each $x = 1, 2, \ldots, p-1$ is a root of $f(x)$. Since $f(x)$ has degree $p-2$ and $p-1$ roots, part (a) tells us that all the coefficients of $f(x)$ are divisible by p.

c. From part (b) we know that the constant term of $f(x)$ is divisible by p. The constant term is given by $f(0) = (-1)(-2)\cdots(-p+1)+1 \equiv (-1)^{p-1}(p-1)!+1 \equiv (p-1)!+1 \equiv 0 \pmod{p}$, which is Wilson's theorem.

9.2.12. This is clearly true if $p = 2$. Now suppose that $p > 2$. Note that r is a primitive root of p if and only if $\overline{r}$ is a primitive root of p where $\overline{r}$ is an inverse of r modulo p. Also note that r and $\overline{r}$ are incongruent modulo p sinc $r \equiv \overline{r} \pmod{p}$ if and only if $r \equiv \pm 1 \pmod{p}$. Hence the product of the $\phi(p-1)$ primitive roots of p is the product of $\phi(p-1)/2$ pairs of primitive roots r and $\overline{r}$, each pair of which has a product congruent to 1 modulo p. Hence the product of all these primitive roots is 1.

9.2.13. a. Since $q_i^{t_i} \mid \phi(p) = p-1$, by Theorem 9.8 there exists $\phi(q_i^{t_i})$ elements of order $q_i^{t_i}$ for each $i = 1, 2, \ldots, r$. Let a_i be a fixed element of this order.

b. Using induction and Exercise 10 of Section 9.1, we have $\text{ord}_p(a) = \text{ord}_p(a_1 a_2 \cdots a_r) = \text{ord}_p(a_1 \cdots a_{r-1})$ ord $_p(a_r) = \cdots = \text{ord}_p(a_1) \cdots$ ord$_p(a_r)$ since $\{\text{ord}_p(a_1), \text{ord}_p(a_2), \ldots, \text{ord}_p(a_r)\} = \{q_1^{t_1}, \ldots, q_r^{tr}\}$ are pairwise relatively prime.

c. $\phi(29) = 28 = 2^2 7$, and $12^4 \equiv 1 \pmod{29}$, so $\text{ord}_{29}(12) = 4$. Also, $16^7 \equiv 1 \pmod{29}$ so $\text{ord}_{29}(16) = 7$. Then by part (b), $\text{ord}_{29}(12 \cdot 16) = 4 \cdot 7 = 28$. Therefore $12 \cdot 16 = 192 \equiv 18 \pmod{29}$ is a primitive root modulo 29.

9.2.14. Let b be an integer such that n is a pseudoprime to the base b. Then $b^n \equiv b \pmod{n}$. For each i, let $d_i = (n-1, p_i - 1)$. Then $b^n \equiv b \pmod{p_i^{a_i}}$, and since $(b, p) = 1$, we have $b^{n-1} \equiv 1 \pmod{p_i^{a_i}}$. Therefore, $\text{ord}_{p_i^{a_i}} b \mid (n-1)$ and $\text{ord}_{p_i^{a_i}} b \mid \phi(p_i^{a_i})$. Hence, $\text{ord}_{p_i^{a_i}} b \mid (n-1, p_i^{a_i-1}(p_i-1)) = (n-1, p_i-1) = d_i$. Then $b^{d_i} \equiv 1 \pmod{p_i^{a_i}}$, and this last congrence has at most d solutions modulo $p_i^{a_i}$, by Lagrange's Theorem. Let s_i be a primitive root modulo $p_i^{a_i}$. Then $s^t \equiv b \pmod{p_i^{a_i}}$ for some t. Then $td = \phi(p_i^{a_i})$ and each of $s_i^t, s_i^{2t}, \ldots, s_i^{dt}$ is an incongruent solution to $b^{d_i} \equiv 1 \pmod{p_i^{a_i}}$. Therefore, there are exactly d_i solutions. If we choose a solution for each $i = 1, \ldots, r$, then the Chinese Remainder Theorem guarantees a unique solution modulo n. Since there are $\prod_{i=1}^{r} d_i$ ways to choose the solutions, we have this many different b's modulo n.

9.2.15. If n is odd, composite and not a power of 3, then the product in Exercise 14 is $\prod_{j=1}^{r}(n-1, p_j - 1) \geq (n-1, 3-1)(n-1, 5-1) \geq 2 \cdot 2 = 4$. So there must be two bases other than -1 and +1.

9.2.16. We have $\phi(p) = p-1 = 2q$. so the possible orders or $p-a^2$ are $1, 2, q$ and $2q$. Computing, we have $(p-a^2)^2 \equiv p^2 - 2pa^2 + a^4 \equiv a^4 \pmod{p}$. If $a^4 \equiv 1 \pmod{p}$, then $a^2 \equiv 1 \pmod{p}$ since a can not have order 4. Then $a \equiv \pm 1 \pmod{p}$, but $1 < a < p-1$ so this is a contradiction. and $p-a^2$ doesn't have order 2. Secondly, using the binomial theorem, $(p-a^2)^q \equiv -(a^{2q}) \equiv -1 \pmod{p}$, so $(p-a^2)$ doesn't have order q. Therefore, it has order $2q$ and must be a primitive root modulo p.

9.2.17. a. Suppose that $f(x)$ is a polynomial with integer coefficients of degree $n-1$. Suppose that $x_1, x_2, \cdots, x_n$ are incongruent modulo p where p is prime. Consider the polynomial $g(x) = f(x) - \sum_{j=1}^{n}\left(f(x_j)\prod_{i\neq j}(x-x_i)\overline{(x_j-x_i)}\right)$. Note that x_j, $j = 1, 2, \cdots, n$ is a root of this polynomial modulo p since its value at x_j is $f(x_j) - [0+0+\cdots+f(x_j)\prod_{i\neq j}(x_j-x_i)(x_j-x_i)+\cdots+0] \equiv f(x_j) - f(x_j) \cdot 1 \equiv 0 \pmod{p}$. Since $g(x)$ has n incongruent roots modulo p and since it is of degree $n-1$ or less, we can easily us Lagrange's theorem (Theorem 8.6) to see that $g(x) \equiv 0 \pmod{p}$ for every integer x.

b. By part (a) we have $f(5) \equiv f(1)(5-2)\overline{(1-2)}(5-3)\overline{(1-3)} + f(2)(5-1)\overline{(2-1)}(5-3)\overline{(2-3)} + f(3)(5-1)\overline{(3-1)}(5-2)\overline{(3-2)} \equiv 8 \cdot 3\overline{(-1)}2 \cdot \overline{(-2)} + 2 \cdot 4 \cdot 1 \cdot 2 \cdot \overline{(-1)} + 4 \cdot 4 \cdot \overline{2} \cdot 3 \cdot 1 \equiv 8 \cdot 3 \cdot 10 \cdot 2 \cdot 5 + 2 \cdot 4 \cdot 1 \cdot 2 \cdot 10 + 4 \cdot 4 \cdot 6 \cdot 3 \cdot 1 \equiv 10 \pmod{11}$.

9.2.18. a. Given r shadows, we have $f(x)$ evaluated at r incongruent integers modulo p. By Exercise 17, since $\deg f = r-1$, we can determine $f(0) \equiv K \pmod{p}$.

b. From Exercise 17 we have $f(0) \equiv K \equiv \sum_{j=1}^{r} k_j \prod_{\substack{i=1 \\ i \neq j}}^{n} (-x_i)(\overline{x_j - x_i}) \pmod{p}$. Solving for k_r gives $k_r \equiv \dfrac{K - \sum k_i \prod(-x_i)(\overline{x_j - x_i})}{\prod(-x_i)(\overline{x_j - x_i})} \pmod{p}$ from which we can see that k_r is determined by K and $k_1, k_2, \ldots, k_{r-1}$. If only $r-1$ shadows were needed, then $k-r$ could take on any value without effecting the value of K.

c. We have $f(1) = 69 \equiv 22, f(2) = 131 \equiv 37, f(3) = 243 \equiv 8, f(4) = 429 \equiv 6, f(5) = 713 \equiv 8, f(6) = 1119 \equiv 38$, and $f(7) = 1671 \equiv 26 \pmod{47}$.

d. $K = f(0) = 22[(-2)(\overline{1-2})(-3)(\overline{1-3})(-4)(\overline{1-4})] + 37[(-1)(\overline{2-1})(-3)(\overline{2-3})(-4)(\overline{2-4})] + 8[(-1)(\overline{3-1})(-2)(\overline{3-2})(-4)(\overline{3-4})] + 6[(-1)(\overline{4-1})(-2)(\overline{4-2})(-3)(\overline{4-3})] \equiv 33 \pmod{47}$.

9.2.19. By Exercise 23 of Section 9.1, $j \mid \text{ord}_{\phi(n)}e$. Here, $\phi(n) = \phi(pq) = 4p'q'$, so $j \mid \phi(4p'q') = 2(p'-1)(q'-1)$. Choose e to be a primitive root modulo p'. Then $p'-1 = \phi(p') | \phi(\phi(n))$, so $p'-1 | \text{ord}_{\phi(n)}e$. The decrypter needs $e^j \equiv 1 \pmod{n}$, but this choice of e forces $j = p'-1$, which will take quite some time to find.

9.3. The Existence of Primitive Roots

9.3.1. The positive integers that have a primitive root are 2,4 and integers of the form p^t, and $2p^t$ where p is prime and t is a positive integer. Hence the integers in the list that have a primitive root are 4, $10 = 2 \cdot 5$, and $22 = 2 \cdot 11$.

9.3.2. By Theorem 9.15, we only admit prime powers and twice prime powers. This leaves only 9, 26, 27, and 31.

9.3.3. a. First note that 2 is a primitive root modulo 3. Since $2^{3-1} = 2^2 = 4 \not\equiv 1 \pmod{3^2}$, 2 is also a primitive root modulo 3^2.

b. First note that 2 is a primitive root modulo 5. Since $2^{5-1} = 2^4 = 16 \not\equiv 1 \pmod{5^2}$, 2 is also a primitive root modulo 5^2.

c. First note that 5 is a primitive root modulo 23. Since $5^{23-1} = 5^{22} \equiv 323 \not\equiv 1 \pmod{23^2}$, 5 is also a primitive root modulo 23^2.

d. First note that 2 is a primitive root modulo 29. Since $2^{29-1} = 2^{28} \equiv 30 \not\equiv 1 \pmod{29^2}$, 2 is also a primitive root modulo 29^2.

9.3.4. a. First note that 2 is a primitive root modulo 11. Since $2^{11-1} = 2^{10} = 1024 \equiv 56 \neq 1 \pmod{11^2}$, 2 is also a primitive root modulo 11^2.

b. First note that 2 is a primitive root modulo 13. Since $2^{13-1} = 2^{12} = 4096 \equiv 40 \neq 1 \pmod{13^2}$, 2 is also a primitive root modulo 13^2.

c. First note that 3 is a primitive root modulo 17. Since $3^{17-1} = 3^8 = 6561 \equiv 203 \neq 1 \pmod{17^2}$, 3 is also a primitive root modulo 17^2.

d. First note that 2 is a primitive root modulo 19. Since $2^{19-1} = 2^{18} = 262144 \equiv 58 \neq 1 \pmod{19^2}$, 2 is also a primitive root modulo 19^2.

9.3.5. a. We know that 2 is a primitive root of 3 and also of 3^2 since $2^{(3-1)} = 4 \neq 1 \pmod 9$. It follows that 2 is also a primitive root of 3^k for all positive integers k.

b. From Exercise 2(a) we know that 2 is a primitive root modulo 11^2. It follows that 2 is a primitive root modulo 11^k for all positive integers k.

c. From Exercise 2(b) we know that 2 is a primitive root modulo 13^2. It follows that 2 is a primitive root modulo 13^k for all positive integers k.

d. From Exercise 2(c) we know that 3 is a primitive root modulo 17^2. It follows that 3 is a primitive root modulo 17^k for all positive integers k.

9.3.6. a. By Theorem 9.10, we need only find a primitive root for $k = 1, 2$. We find that 5 is a primitive root modulo 23. Then by Theorem 9.9, either 5 or $23 - 5 = 18$ is a primitive root modulo 23^2. We find that 5 is also a primitive root modulo 23^2, therefore it is a primitive root modulo 23^k for any positive integer k.

b. As in part (a) 2 is a primitive root modulo 29^k for any positive integer k.

c. As in part (a), 3 works.

d. As in part (a), 2 works.

9.3.7. a. Since 2 is even and primitive root for 5, we have by Theorem 9.14 that $5 + 2 = 7$ is a primitive root for 10.

b. Since 3 is odd and a primitive root for 17, we have by Theorem 9.14 that 3 is also a primitive root for 34.

c. Since 2 is even and a primitive root for 19, we have by Theorem 9.14 that $2 + 19 = 21$ is a primitive root for 38.

d. We have $50 = 2 \cdot 5^2$. By Exercise 3(b), 2 is a primitive root for 5^2. By Theorem 9.14, since 2 is even, $25 + 2 = 27$ is a primitive root for 50.

9.3.8. a. By Theorem 9.14, since 2 is an even primitive root for 3, then $2 + 3 = 5$ is a primitive root for 6.

b. By Theorem 9.14, and Exercise 3(a), since 2 is an even primitive root for 9, then $2 + 9 = 11$ is a primitive root for 18.

c. By Theorem 9.14, since 2 is an even primitive root for 13, then $2 + 13 = 15$ is a primitive root for 26.

d. We have $338 = 2 \cdot 13^2$. By Exercise 4(b), 2 is a primitive root for 13^2. By Theorem 9.14, since 2 is even, $169 + 2 = 171$ is a primitive root for 338.

9.3.9. First note that 2 is primitive root of 11. Since 2 is even, Theorem 9.14 tells us that $2 + 11 = 13$ is a primitive root of 22. Hence the primitive roots of 22 are the least positive residues of 13^k where $1 \leq k < \phi(22) = 10$ and $(k, \phi(12)) = (k, 10) = 1$. These are the integers $13^1 = 13$, $13^3 \equiv 19 \pmod{22}$, $13^7 \equiv 7 \pmod{22}$, and $13^9 \equiv 17 \pmod{22}$. Hence the primitive roots of 22 are 7, 13, 17, and 19.

9.3.10. 2 is a primitive root modulo 25. There must be $\phi(\phi(25)) = 8$ of them, given by raising 2 to powers relatively prime to $\phi(25) = 20$. It follows that $2^1, 2^3, 2^7, 2^9, 2^{11}, 2^{13}, 2^{17}$, and 2^{19} become 2, 8, 3, 12, 23, 17,

22, and 13 when reduced modulo 25.

9.3.11. By Exercise 7 in Section 9.1, a complete set of primitive roots modulo 19 is $2, 3, 10, 13, 14, 15$. By Theorem 9.14, the odd numbers in this set are primitive roots of 38, and if we add 19 to each of the even numbers in this set, we also have primitive roots of 38. Thus we have $2 + 19, 3, 10 + 19, 13, 14 + 19, 15$ as all the primitive roots of 38. Reducing gives us $3, 13, 15, 21, 29, 33$.

9.3.12. By Theorem 9.5, there are $\phi(\phi(p^t)) = \phi(p^t - p^{t-1})$ primitive roots modulo p^t, and $\phi(\phi(2p^t)) = \phi(p^t - p^{t-1})$ primitive roots modulo $2p^t$.

9.3.13. Suppose that r is a primitive root of m and suppose further that $x^2 \equiv 1 \pmod{m}$. Let $x \equiv r^t \pmod{m}$ where $0 \leq t \leq p - 1$. Then $r^{2t} \equiv 1 \pmod{m}$. Since r is a primitive root, it follows that $\phi(m) \mid 2t$ so that $2t = k\phi(m)$ and $t = k\phi(m)/2$ for some integer k. We have $x \equiv r^t = r^{k\phi(m)/2} = r^{(\phi(m)/2)k} \equiv (-1)^k \equiv \pm 1 \pmod{m}$, since $r^{\phi(m)/2} \equiv -1 \pmod{m}$. Conversely, suppose that m has no primitive root. Then m is not of one of the forms $2, 4, p^a$, or $2p^a$ with with p and odd prime. So either 2 distinct primes divide m or $m = 2^b M$ with M and odd integer and $b > 1$ or $m = 2^b$ with $c > 2$. In each of these cases we have $\phi(m) = 2^c N$ with N odd and $c \geq 3$. From Theorem 9.12, we know there are at least 3 solutions y_1, y_2, y_2 to $y^2 \equiv 1 \pmod{2^c}$ and certainly $z \equiv 1 \pmod{N}$ is a solution of $x^2 \equiv 1 \pmod{N}$. By the Chinese remainder theorem, there is a unique solution modulo $2^c N$ of the system $x \equiv y_i \pmod{2^c}, z \equiv 1 \pmod{N}$ for $i = 1, 2, 3$. Since these solutions are distinct modulo m, at least one of them is not $\pm 1 \pmod{m}$.

9.3.14. Let r be a primitive root modulo n. Note that $r^{\phi(n)/2} \equiv -1 \pmod{n}$. The integers $r, r^2, \ldots, r^{\phi(n)}$ reduced modulo n , is the set of integers less than and relatively prime to n. Their product is $r \cdot r^2 \cdots r^{\phi(n)} = r^{\sum_{i=1}^{\phi(n)} i} = r^{\phi(n)(\phi(n)-1)/2} \equiv (-1)^{\phi(n)-1} \equiv -1 \pmod{n}$ since $\phi(n)$ is even.

9.3.15. By Theorem 9.12 we know that $\text{ord}_{2^k} 5 = \phi(2^k)/2$. Hence the 2^{k-2} integers $5^j, j = 0, 1, \cdots, 2^{k-2} - 1$, are incongruent modulo $2k$. Similarly the 2^{k-2} integers $-5^j, j = 0, 1, \cdots, 2^{k-2} - 1$, are incongurent modulo 2^k. Note that 5^j cannot be congruent to -5^i modulo 2^k where i and j are integers since $5^j \equiv 1 \pmod{4}$ but $-5^i \equiv 3 \pmod{4}$. It follows that the integers $1, 5, \cdots, 5^{2^{k-2}-1}, -1, -5, \cdots, -5^{2^{k-2}-1}$ are 2^{k-1} incongruent integers modulo 2^k. Since $\phi(2^k) = 2^{k-1}$ and every integer of the form $(-1)^\alpha 5^\beta$ is relatively prime to 2^k, it follows that every odd integer is congruent to an integer of this form with $\alpha = 0$ or 1 and $0 \leq \beta = 2^{k-2} - 1$.

9.3.16. 2 is the smallest. To find the next case, we search tediously through the primes, in order, using Table E.3 in the back of the text. We do the case, $p = 19$ as an example. The table gives 2 as a primitive root modulo 19. Then all primitive roots are found by raising 2 to powers relatively prime to $\phi(19) = 18$. Reducing $2, 2^5, 2^7, 2^11, 2^13, 2^17$ modulo 19 gives us 2, 13, 14, 3, and 10 as all the primitive roots modulo 19. If one of these were not a primitive root modulo 19^2, then it would have order equal to $19 - 1 = 18$. (See the argument in the proof of Theorem 9.9.) We raise each of the primitive roots to the 18th power, and reduce modulo 19^2 to get 58, 343, 305, 210, 343, 286. Since we didn't get a 1, we continue our search with the next prime. We finally find that 14 is a primitive root modulo 29, but not for 29^2.

9.4. Index Arithmetic

9.4.1. We first compute the least positive residues of the powers of 5 modulo 23. We have $5^1 \equiv 5 \pmod{23}$, $5^2 \equiv 2 \pmod{23}$, $5^3 \equiv 10 \pmod{23}$, $5^4 \equiv 4 \pmod{23}$, $5^5 \equiv 20 \pmod{23}, 5^6 \equiv 8 \pmod{23}, 5^7 \equiv 17 \pmod{23}$, $5^8 \equiv 16 \pmod{23}$, $5^9 \equiv 11 \pmod{23}$, $5^{10} \equiv 9 \pmod{23}$, $5^{11} \equiv 22 \pmod{23}$, $5^{12} \equiv 18 \pmod{23}, 5^{13} \equiv 21 \pmod{23}$, $5^{14} \equiv 13 \pmod{23}$, $5^{15} \equiv 19 \pmod{23}$, $5^{16} \equiv 3 \pmod{23}, 5^{17} \equiv 15 \pmod{23}$, $5^{18} \equiv 6 \pmod{23}$, $5^{19} \equiv 7 \pmod{23}$, $5^{20} \equiv 12 \pmod{23}$, $5^{21} \equiv 14 \pmod{23}$, and $5^{22} \equiv 1 \pmod{23}$. Hence $\text{ind}_5 1 = 22$, $\text{ind}_5 2 = 2$, $\text{ind}_5 3 = 16$, $\text{ind}_5 4 = 4$, $\text{ind}_5 5 = 1$, $\text{ind}_5 6 = 18$, $\text{ind}_5 7 = 19$, $\text{ind}_5 8 = 6$, $\text{ind}_5 9 = 10$, $\text{ind}_5 10 = 3$, $\text{ind}_5 11 = 9$, $\text{ind}_5 12 = 20$, $\text{ind}_5 13 = 14$, $\text{ind}_5 14 = 21$, $\text{ind}_5 15 = 17$, $\text{ind}_5 16 = 8$, $\text{ind}_5 17 = 7$, $\text{ind}_5 18 = 12$, $\text{ind}_5 19 = 15$, $\text{ind}_5 20 = 5$, $\text{ind}_5 21 = 13$, and $\text{ind}_5 22 = 11$.

9.4.2. a. From Exercise 1, we can take indices base 5 modulo 23 and get $\text{ind}_5 3 + 5\text{ind}_5 x \equiv \text{ind}_5 1 \pmod{22}$. If $y = \text{ind}_5 x$, we have $16 + 5y \equiv 22 \pmod{22}$ which has solution $y \equiv 10 \pmod{22}$. Therefore, $x \equiv 9$

$\pmod{23}$.

b. Taking indices gives us $\text{ind}_5 3 + 14\text{ind}_5 x \equiv \text{ind}_5 2 \pmod{22}$ or $16 + 14y \equiv 2 \pmod{22}$ which has solution $y \equiv 10$ or $21 \pmod{22}$, so $x \equiv 9$ or $14 \pmod{23}$.

9.4.3. a. Suppose that $3^x \equiv 2 \pmod{23}$. We take indices with respect to the primitive root 5 of 23. This gives $\text{ind}_5(3^x) = \text{ind}_5 2$ which implies that $x\ \text{ind}_5 3 \equiv \text{ind}_5 2 \pmod{22}$. Since $\text{ind}_5(3^x) = 16$ and $\text{ind}_5 2 = 2$ it follows that $16 \equiv 2 \pmod{22}$. Hence $8x \equiv 1 \pmod{11}$. Since 7 is the inverse of 8 modulo 11, it follows that $x \equiv 7 \pmod{11}$, so that $x \equiv 7$ or $18 \pmod{22}$.

b. Taking indices gives us $x\ \text{ind}_5 13 \equiv \text{ind}_5 5 \pmod{22}$ or $14x \equiv 1 \pmod{22}$ which has no solutions since $(14, 22) = 2 \nmid 1$.

9.4.4. Suppose that $ax^4 \equiv 2 \pmod{13}$. Taking indices with respect to the primitive root 2 of 13, we have $\text{ind}_2(ax^4) \equiv \text{ind}_2 2 \pmod{12}$. Hence $\text{ind}_2 a + 4 \cdot \text{ind}_2 x \equiv 1 \pmod{12}$, or $4 \cdot \text{ind}_2 x \equiv 1 - \text{ind}_2 a \pmod{12}$. There is a solution x if and only if $(4, 12) = 4$ divides $1 - \text{ind}_2 a$. This is true if $\text{ind}_2 a = 1, 5$ or 9, which holds if and only if $a \equiv 2^1 \equiv 2 \pmod{13}$, $a \equiv 2^5 \equiv 6 \pmod{13}$, or $a \equiv 2^9 \equiv 5 \pmod{13}$. Hence this congruence has a solution if and only if $a \equiv 2, 5$, or $6 \pmod{13}$.

9.4.5. We use the table of indices on page 612 of the text. We see that 2 is a primitive root for 29. Taking indices base 2 of the congruence and expanding gives us $\text{ind}_2 8 + 7\text{ind}_2 x \equiv \text{ind}_2 b \pmod{28}$. From the table we have $3 + 7\text{ind}_2 x \equiv \text{ind}_2 b \pmod{28}$, which has a solution if and only if $(7, 28) = 7 \mid (\text{ind}_2 b - 3)$. So $\text{ind}_2 b - 3 = 0, 7, 14$, or 21, that is $\text{ind}_2 b = 3, 10, 17$, or 24. This corresponds to $b = 8, 9, 21$, or 20, respectively.

9.4.6. Suppose that $2^x \equiv x \pmod{13}$. Taking indices of both sides to the base 2 modulo 13 gives $\text{ind}_2(2^x) \equiv \text{ind}_2 x \pmod{12}$. Since $\text{ind}_2(2^x) \equiv x \pmod{12}$, this implies that $x \equiv \text{ind}_2 x \pmod{12}$. Since $\text{ind}_2 x$ depends on the remainder when x is divided by 13, and we also need the remainder when $\text{ind}_2 x$ is divided by 12. By the Chinese remainder theorem we need to consider the remainder when x is divided by $13 \cdot 12 = 156$. The solutions are given by the integers x such that $x \equiv 0 \pmod{12}$ and $\text{ind}_2 x = 0 \pmod{12}$; $x \equiv 1 \pmod{12}$ and $\text{ind}_2 x = 1 \pmod{12}$; $x \equiv 2 \pmod{12}$ and $\text{ind}_2 x = 2 \pmod{12}$; $x \equiv 3 \pmod{12}$ and $\text{ind}_2 x = 3 \pmod{12}$; $x \equiv 4 \pmod{12}$ and $\text{ind}_2 x = 4 \pmod{12}$; $x \equiv 5 \pmod{12}$ and $\text{ind}_2 x = 5 \pmod{12}$; $x \equiv 6 \pmod{12}$ and $\text{ind}_2 x = 6 \pmod{12}$; $x \equiv 7 \pmod{12}$ and $\text{ind}_2 x = 7 \pmod{12}$; $x \equiv 8 \pmod{12}$ and $\text{ind}_2 x = 8 \pmod{12}$; $x \equiv 9 \pmod{12}$ and $\text{ind}_2 x = 9 \pmod{12}$; $x \equiv 10 \pmod{12}$ and $\text{ind}_2 x = 10 \pmod{12}$; and $x \equiv 11 \pmod{12}$ and $\text{ind}_2 x = 11 \pmod{12}$. These are the solutions to $x \equiv 0 \pmod{12}$ and $x \equiv 1 \pmod{13}$; $x \equiv 1 \pmod{12}$ and $x \equiv 2 \pmod{13}$; $x \equiv 2 \pmod{12}$ and $x \equiv 4 \pmod{13}$; $x \equiv 3 \pmod{12}$ and $x \equiv 8 \pmod{13}$; $x \equiv 4 \pmod{12}$ and $x \equiv 3 \pmod{13}$; $x \equiv 5 \pmod{12}$ and $x \equiv 6 \pmod{13}$; $x \equiv 6 \pmod{12}$ and $x \equiv 12 \pmod{13}$; $x \equiv 7 \pmod{12}$ and $x \equiv 11 \pmod{13}$; $x \equiv 8 \pmod{12}$ and $x \equiv 9 \pmod{13}$; $x \equiv 9 \pmod{12}$ and $x \equiv 5 \pmod{13}$; $x \equiv 10 \pmod{12}$ and $x \equiv 10 \pmod{13}$; and $x \equiv 11 \pmod{12}$ and $x \equiv 7 \pmod{13}$. We solve each of these 12 systems of simultaneous congruences to see that all solutions, in order of which set of congruence they satisfy, are given by $x \equiv 144, 145, 134, 99, 16, 149, 90, 115, 152, 57, 10, 59 \pmod{156}$. Listing these in order, we see that all solutions are given by those integers x that satisfy $x \equiv 10, 16, 57, 59, 90, 99, 115, 134, 144, 145, 149$, or $152 \pmod{156}$.

9.4.7. Taking indices of the congruence gives us $x\text{ind}x \equiv \text{ind}x \pmod{22}$, so that $22 \mid (\text{ind}x)(x - 1)$. If $(\text{ind}x, 22) = 1$, then $22 \mid (x - 1)$, which is the case for $x = 5, 7, 10, 11, 14, 15, 17, 19, 20$, and 21, from the table on page 548. So any solution of the systems $x \equiv 1 \pmod{22}, x \equiv a \pmod{23}$, as a runs through the list above, is a solution to the congruence. If $(\text{ind}x, 22) = 2$, then x is one of $2, 3, 4, 6, 8, 9, 12, 13, 16$, or 18, and $11 \mid (x - 1)$, so any solution to the systems $x \equiv 1 \pmod{11}, x \equiv b \pmod{23}$, as b runs through this list, is also a solution to the congruence. If $(\text{ind}x, 22) = 11$, then $\text{ind}x = 11$, so $x \equiv 22 \pmod{23}$, but this is not a solution. Finally, if $(\text{ind}x, 22) = 22$, then $\text{ind}x = 22$, so $x \equiv 1 \pmod{23}$. Since $23 \cdot 22 = 506$, we list the solutions modulo 506: 1, 12, 23, 24, 45, 46, 47, 67, 69, 70, 78, 89, 91, 92, 93, 100, 111, 115, 116, 133, 137, 138, 139, 144, 155, 161, 162, 177, 183, 184, 185, 188, 199, 207, 208, 210, 221, 229, 230, 231, 232, 243, 253, 254, 265, 275, 276, 277, 287, 299, 300, 309, 321, 322, 323, 331, 345, 346, 353, 367, 368, 369, 375, 386, 391,

392, 397, 413, 414, 415, 419, 430, 437, 438, 441, 459, 460, 461, 463, 483, 484, 485, 496, 505.

9.4.8. Suppose that r is a primitive root modulo p. Then $r^{(p-1)/2} \equiv -1 \equiv p-1 \pmod{p}$ since $r^{(p-1)/2} \not\equiv 1 \pmod{p}$ and $(r^{(p-1)/2})^2 \equiv 1 \pmod{p}$. (This follows since the congruence $x^2 \equiv 1 \pmod{p}$ has exactly two incongruent solutions modulo p, namely $x \equiv 1 \pmod{p}$ and $x \equiv -1 \pmod{p}$.) Hence $\text{ind}_r(p-1) = (p-1)/2$.

9.4.9. Suppose that $x^4 \equiv -1 \pmod{p}$ and let $y = \text{ind}_r x$. Then, $-x$ is also a solution and by Exercise 8, $\text{ind}_r(-x) \equiv \text{ind}_r(-1) + \text{ind}_r(x) \equiv (p-1)/2 + y \pmod{p-1}$. So without loss of generality we may take $0 < y < (p-1)/2$, or $0 < 4y < 2(p-1)$. Taking indices of both sides of the congruence yields $4y \equiv \text{ind}_r(-1) \equiv (p-1)/2 \pmod{p-1}$, again using Exercise 8. So $4y = (p-1)/2 + m(p-1)$ for some m. But $4y < 2(p-1)$, so either $4y = (p-1)/2$ and so $p = 8y+1$ or $4y = 3(p-1)/2$. In this last case, 3 must divide y, so we have $p = 8(y/3)+1$. So in either case, p is of the desired form. Conversely, suppose $p = 8k+1$ and let r be a primitive root of p. Take $x = r^k$. Then $x^4 \equiv r^{4k} \equiv r^{(p-1)/2} \equiv -1 \pmod{p}$ by Exercise 8. So this x is a solution.

9.4.10. Suppose that $p_1, \ldots, p_n$ are all of the primes of the form $8k+1$. Let $Q = (p_1 \cdots P_n)^4 + 1$. Then $Q \equiv (1)^4 + 1 \equiv 2 \pmod{8}$. So Q has an odd prime factor q. Then $(p_1 \cdots P_n)^4 \equiv 1 \pmod{q}$. By Exercise 9, q is of the form $8k+1$.

9.4.11. We have $7 \equiv (-1)^1 5^2 \pmod{2^4}$ and $9 \equiv (-1)^0 5^2 \pmod{2^4}$. Hence the index systems of 7 and 9 modulo 16 are $(1,2)$ and $(0,2)$, respectively.

9.4.12. Let $x \equiv (-1)^\alpha 5^\beta$ and $y \equiv (-1)^\gamma 5^\delta$. Then $xy \equiv (-1)^{\alpha+\gamma} 5^{\beta+\delta} \pmod{2^k}$, and $x^n \equiv (-1)^{n\alpha} 5^{n\beta}$. So the index system for xy is $(\alpha+\gamma, \beta+\delta)$, and the index system for x^n is $(n\alpha, n\beta)$. Further, the first components of the indices are modulo 2, while the second are modulo 2^{k-2}.

9.4.13. Since $7 \equiv (-1)5^2 \pmod{32}$ and $11 \equiv (-1)5^5 \pmod{32}$, we have that the index systems for 7 and 11 are $(1,2)$ and $(1,5)$ respectively. Let the index system for x be (α, β). Then by the rules in Exercise 12, the index system for $7x^9$ is $(1+9\alpha, 2+9\beta)$, which must equal the index system for 11. Therefore $1+9\alpha \equiv 1 \pmod{2}$, so $\alpha = 0$. And $2+9\beta \equiv 5 \pmod{8}$, so $\beta = 3$. Then $x \equiv (-1)^0 5^3 \equiv 29 \pmod{32}$. For the second congruence, we note that the index system for 3 and 17 are $(1,3)$ and $(0,4)$ respectively. Then the index system for 3^x is $(x, 3x)$ and we must have $x \equiv 0 \pmod{2}$ while $3x \equiv 4 \pmod{8}$. A solution to the second congruence is necessarily a solution to the first. So all solutions are given by $x \equiv 4 \pmod{8}$.

9.4.14. We do only the case $t_0 \leq 2$. Suppose a and b have the same index system $(\gamma_0, \ldots, \gamma_m)$ modulo n. Then for each i, we have that a and b both solve the system given by $\gamma_i \equiv \text{ind}_{r_i} x \pmod{\phi(p_i^{t_i})}$, or $r_i^{\gamma_i} \equiv x \pmod{p_i^{t_i}}$. By the Chinese remainder theorem, there is a unique solution to this system modulo $p_0^{t_0} \cdots p_m^{t_m}$, therefore $a \equiv b \pmod{p_0^{t_0} \cdots p_m^{t_m}}$. The case $t_0 \geq 3$ is a concatenation of the case $t_0 \leq 2$ and the solution to Exercise 15 of Section 9.3.

9.4.15. We have $120 = 2^3 \cdot 3 \cdot 5$. The index system of 17 modulo 120 is $(\alpha, \beta, \gamma^1, \gamma^2)$ where $17 \equiv (-1)^\alpha 5^\beta \pmod{2^3}$, $17 \equiv 2^{\gamma_1} \pmod{3}$, and $17 \equiv 2^{\gamma_2} \pmod{5}$. We see that $17 \equiv 1 \equiv (-1)^0 5^0 \pmod{2^3}$, $17 \equiv 2^1 \pmod{3}$, and $17 \equiv 2^1 \pmod{5}$, so that $\alpha = 0$, $\beta = 0$, $\gamma_1 = 1$, and $\gamma_2 = 1$. Hence the index system of 17 modulo 120 is $(0,0,1,1)$. The index system of 41 modulo 120 is $(\alpha, \beta, \gamma_1, \gamma_2)$ where $41 \equiv (-1)^\alpha 5^\beta \pmod{2^3}$, $41 \equiv 2^{\gamma_1} \pmod{3}$, and $41 \equiv 2^{\gamma_2} \pmod{5}$. We see that $41 \equiv 1 \equiv (-1)^0 5^0 \pmod{2^3}$, $41 \equiv 2 \equiv 2^1 \pmod{3}$, and $41 \equiv 1 \equiv 2^4 \pmod{5}$. Hence $\alpha = 0$, $\beta = 0$, $\gamma_1 = 1$, and $\gamma_2 = 4$. Hence the index system of 41 modulo 120 is (0,0,1,4).

9.4.16. As in Exercises 12 and 14, we have $(\gamma_0, \ldots, \gamma_m) \cdot (\delta_0, \ldots, \delta_m) = (\gamma_0 + \delta_0, \ldots, \gamma_m + \delta_m)$, and $(\gamma_0, \ldots, \gamma_m)^n = (n\gamma_0, \ldots, n\gamma_m)$.

9.4.17. We have $60 = 4 \cdot 3 \cdot 5$. We take $3, 2$, and 2 as primitive roots for $4, 3$, and 5 respectively. Then we find that the index system for 11 is $(1,1,0)$, while the index system for 43 is $(1,0,3)$. Let the index system for x be (α, β, γ). Applying the rules from Exercise 16, we have $(1+7\alpha, 1+7\beta, 0+7\gamma) = (1,0,3)$. Therefore $1+7\alpha \equiv 1 \pmod{\phi(4)}$, so $\alpha = 0$. Next, $1+7\beta \equiv 0 \pmod{\phi(3)}$, so $\beta = 1$. Next, $0+7\gamma \equiv 3 \pmod{\phi(3)}$,

so $\gamma = 1$. Therefore the index system for x is $(0, 1, 1)$. Using the Chinese remainder theorem, we solve the system $x \equiv 3^0 \pmod 4, x \equiv 2^1 \pmod 3, x \equiv 2^1 \pmod 5$, to get that $x \equiv 17 \pmod{60}$.

9.4.18. Suppose that p is a prime greater that 3. Suppose that a is relatively prime to p. By Theorem 8.17 the congruence $x^3 \equiv a \pmod p$ has a solution if and only if $a^{\frac{(p-1)}{d}} \equiv 1 \pmod p$, where $d = (3, p-1)$. It follows that if $3 \mid (p-1)$, or equivalently, if $p \equiv 1 \pmod 3$, then $x^3 \equiv a \pmod p$ has a solution, that is a is a cubic residue of p, if and only if $a^{\frac{(p-1)}{3}} \equiv 1 \pmod p$. Also, if $(3, p-1) = 1$, or equivalently, if $p \equiv 2 \pmod p$, then $x^3 \equiv a \pmod p$ has a solution, that is a is cubic residue of p, if and only if $a^{p-1} \equiv 1 \pmod p$, but this is satisfied by every integer a relatively prime to p. Hence every integer a with $(a, p) = 1$ is a cubic residue of p when $p \equiv 2 \pmod p$.

9.4.19. We must have k odd for this exercise. We seek a solution to $x^k \equiv a \pmod{2^e}$. We take indices as described before Exercise 11. Suppose $a \equiv (-1)^\alpha 5^\beta$ and $x \equiv (-1)^\gamma 5^\delta$ Then we have ind$x^k = (k\gamma, k\delta)$ and ind$a = (\alpha, \beta)$, so $k\gamma \equiv \alpha \pmod 2$ and $k\delta \equiv \beta \pmod{2^{e-2}}$. Since k is odd, both congruences are solvable for γ and δ, which determine x.

9.4.20. Let $k = 2^n m$, where m is odd. Note that if $e \leq n+2$, then $b^k \equiv b^{2^n m} \equiv 1 \pmod{2^e}$, by Euler's Theorem, and the only solution to $b^k \equiv 1 \pmod{(4k, 2^e) = 2^e}$ is $b = 1$. Therefore, we may assume that $e > n+2$. We need to show first that if b is an odd integer then $a = b^k \equiv 1 \pmod{(4k, 2^e)}$. Note that $(4k, 2^e) = 2^{n+2}$. Now $\text{ord}_{2^{n+2}} b \mid \phi(2^{n+2}) = 2^{n+1}$, but there are no primitive roots modulo 2^{n+1}, so we have $\text{ord}_{2^{n+2}} b \mid 2^n$. Then $b^k \equiv b^{2^n m} \equiv 1 \pmod{2^{2+n}}$, as desired. To show the converse, note that there are $2^e/2^{n+2} = 2^{e-n-2}$ incongruent elements modulo 2^e, of the form $a = 1 + 2^{n+2}r$, that is $a \equiv 1 \pmod{2^{n+2}}$. From Exercise 15 of Section 8.3, we have that 5 has order 2^{e-2} modulo 2^e, and hence 5^k has order $2^e/(k, 2^{e-2}) = 2^{e-2-n}$. Therefore, the 2^{e-2-n} numbers $5^k, 5^{2k}, \ldots, 5^{2^{e-2-n}k}$, are incongruent modulo 2^e and each one is a kth power residue. Then from the first part of this proof, each of 5^{ki} is of the form $1+2^{n+2}r$, but since there are only 2^{e-n-2} of these, we must have them all. This completes the proof.

9.4.21. First we show that $\text{ord}_{2^e} 5 = 2^{e-2}$. Indeed, $\phi(2^e) = 2^{e-1}$, so it suffices to show that the highest power of 2 dividing $5^{2^{e-2}} - 1$ is 2^e. We proceed by induction. The basis step is the case $e = 2$, which is true. Note that $5^{2^{e-2}} - 1 = (5^{2^{e-3}} - 1)(5^{2^{e-3}} + 1)$. The first factor is exactly divisible by 2^{e-1} by the induction hypothesis. The second factor differs from the first by 2, so it is exactly divisible by 2, therefore $5^{2^{e-2}} - 1$ is exactly divisible by 2^e, as desired. Hence, if k is odd, the numbers $\pm 5^k, \pm 5^{2k}, \ldots, \pm 5^{2^{e-2}k}$ are 2^{e-1} incongruent kth power residues, which is the number given by the formula. If 2^m exactly divides k, then $5^k \equiv -5^k \pmod{2^e}$, so the formula must be divided by 2, hence the factor $(k, 2)$ in the denominator. Further, 5^{2^m} has order $2^{e-2}/2^m$ if $m \leq e-2$ and order 1 if $m > e-2$, so the list must repeat modulo 2^e every $\text{ord}_{2^e} 5^{2^m}$ terms, whence the other factor in the denominator.

9.4.22. Let r be a primitive root modulo p, and take indices base r to get $2^j u \equiv N\text{ind}_r x \equiv \text{ind}_r(-1) \equiv (p-1)/2 \equiv 2^{s-1}t \pmod{2^s t}$. By Theorem 3.10 This congruence has solutions if and only if $(2^j u, 2^s t) \mid 2^{s-1}$, that is, if and only if $j \leq s-1$. If there are solutions, then Theorem 3.10 gives us $((2^j u, 2^s t) = 2^j (u, t)$ of them, as desired.

9.4.23. a. From the first inequality in Case (*i*) of the proof of Theorem 6.10, if n is not square-free, the probability is strictly less than $2n/9$, which is substantially smaller than $(n-1)/4$ for large n. If n is square-free, the argument following inequality (9.6) shows that if n has 4 or more factors, then the probability is less than $n/8$. The next inequality shows that the worst case for $n = p_1 p_2$ is when $s_1 = s_2$ and s_1 is as small as possible, which is the case stated in this exercise.

b. We have $n - 1 = 2 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 29 \cdot 41 \cdot 197$, and $p_1 - 1 = 2 \cdot 3 \cdot 7 \cdot 29 \cdot 41$ and $p_2 - 1 = 2 \cdot 3 \cdot 7 \cdot 29 \cdot 41$. So that, using the notation in the proof of Case (*ii*) of Theorem 6.10, $t = 3^2 \cdot 7 \cdot 13^2 \cdot 29 \cdot 41 \cdot 197$, $t_1 = t_2 = \cdot 3 \cdot 7 \cdot 29 \cdot 41$, and $s_1 = 1$. Then the number of integers b with $1 \leq b \leq n-1$, for which n is a strong pseudoprime to the base b is $T_1 T_2 (1 + \sum_{j=0}^{0} 2^{j2}) = (3 \cdot 7 \cdot 29 \cdot 41)^2 (2)$. so the probability that n is a strong pseudoprime to the base b is $2(\cdot 3 \cdot 7 \cdot 29 \cdot 41)^2/(n-1) = 2(\cdot 3 \cdot 7 \cdot 29 \cdot 41)^2/(2 \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 29 \cdot 41 \cdot 197) = 7 \cdot 29 \cdot 41/(13^2 \cdot 197) = 0.24999\ldots$.

9.5. Primality Tests Using Orders of Integers and Primitive Roots

9.5.1. We have $2^2 \equiv 4 \pmod{101}$, $2^5 \equiv 32 \pmod{101}$, $2^{10} \equiv (2^5)^2 \equiv 32^2 \equiv 14 \pmod{101}$, $2^{20} \equiv (2^{10})^2 \equiv 14^2 \equiv 95 \pmod{101}$, $2^{25} \equiv (2^5)^5 \equiv 32^5 \equiv (32^2)^2 32 \equiv 1024^2 32 \equiv 14^2 32 \equiv 196 \cdot 32 \equiv -6 \cdot 32 \equiv -192 \equiv 10 \pmod{101}$, $2^{50} \equiv (2^{25})^2 \equiv 10^2 = 100 \equiv -1 \pmod{101}$, $2^{100} \equiv (2^{50})^2 \equiv (-1)^2 \equiv 1 \pmod{101}$. Since $2^{\frac{(101-1)}{q}} \not\equiv 1 \pmod{101}$ for every proper divisor q of 100, and $2^{\frac{(101-1)}{q}} \equiv 1 \pmod{101}$ it follows that 101 is prime.

9.5.2. Applying Theorem 9.18, we have $2^{210} \equiv 1 \pmod{211}$. The prime divisors of 210 are 2, 3, 5, and 7. Then $2^{(210/2)} \equiv -1 \pmod{211}, 2^{(210/3)} \equiv 196 \pmod{211}, 2^{(210/5)} \equiv 107 \pmod{211}$, and $2^{(210/7)} \equiv 171 \pmod{211}$. Therefore, 211 is prime.

9.5.3. Applying Corollary 9.18.1, we have $233-1 = 2^3 29, 3^{116} \equiv -1 \pmod{233}$, and $3^8 \equiv 27 \not\equiv 1 \pmod{233}$. So 233 is prime.

9.5.4. Applying Corollary 9.18.1, we have $3^{(256/2)} \equiv -1 \pmod{257}$. There are no odd prime divisors of 256, therefore 257 is prime.

9.5.5. The first condition implies $x^{F_n-1} \equiv 1 \pmod{F_n}$. The only prime dividing $F_n - 1 = 2^{2^n}$ is 2, and $(F_n-1)/2 = 2^{2^n-1}$, so the second condition implies $2^{(F_n-1)/2} \not\equiv 1 \pmod{F_n}$. Then by Theorem 9.18, F_n is prime.

9.5.6. Suppose that $n-1 = p_1^{a_1 \cdots} p_t^{a_t}$ and that there exist integers x_j, for $j = 1, 2, \cdots, t$, such that $x_j^{(n-1)/p_j} \not\equiv 1 \pmod{n}$ and $x_j^{n-1} \equiv 1 \pmod{n}$. Let $N = [\text{ord}_n x_1, \cdots, \text{ord}_n x_t]$. Then $N \mid (n-1)$ but N does not divide $(n-1)/p_j$ for $j = 1, 2, \cdots, t$. It follows from this observation that $N = n-1$. Since $x_j^{\phi(n)} \equiv 1 \pmod{n}$ for all j it follows that $\text{ord}_n x_j \mid \phi(n)$ for all j. We conclude that $\phi(n) \geq N$. It implies that n is prime since $\phi(n) < n-1 = N$ when n is not prime.

9.5.7. Let p be a prime dividing n. By the hypotheses, $x_j^{n-1} \equiv 1 \pmod{n}$, but $(x_j^{(n-1)/q_j} - 1, n) = 1$, so we know that $\text{ord}_p x_j$ divides $n-1$, but not $(n-1)/q_j$. Therefore $\text{ord}_p x_j$ is divisible by $p_j^{a_j}$ for some prime p_j dividing q_j. But since $\text{ord}_p x_j$ also divides $p-1$ and since the q_j are pairwise relatively prime, it follows that $\prod_{j=1}^r p_j^{a_j}$ divides $p-1$. Therefore, $p \geq 1 + \prod_{j=1}^r p_j^{a_j} \geq 1 + \prod_{j=1}^r b_j^{a_j} > \sqrt{n}$, by the last inequality of the hypotheses. Therefore, n can have only one such prime divisor, namely itself.

9.5.8. Since $n - 1 = 7056 = 2^4 3^2 7^2$, we take $F = 2^4 3^2 = 144$ and $R = 7^2 = 49$, noting that $F > R$. We apply Pocklington's test with $a = 2$. We check (using a calculator or computational software) that $2^{7056} \equiv 1 \pmod{7057}$ and $(2^{7056/2} - 1, 7057) = 1$ and $(2^{7056/3} - 1, 7057) = 1$, since 2 and 3 are the only primes dividing F. Therefore n passes Pocklington's test and so is prime.

9.5.9. Since $n - 1 = 9928 = 2^3 17 \cdot 73$, we take $F = 2^3 17 = 136$ and $R = 73$, noting that $F > R$. We apply Pocklington's test with $a = 2$. We check (using a calculator or computational software) that $2^{9928} \equiv 1 \pmod{9929}$ and $(2^{9928/2} - 1, 9929) = 1$ and $(2^{9928/17} - 1, 9929) = 1$, since 2 and 17 are the only primes dividing F. Therefore n passes Pocklington's test and so is prime.

9.5.10. Note that $449 = 2^6 7 + 1$ and $7 < 2^6$, so it is of the form which can be tested by Proth's test. We compute $2^{(449-1)/2} \equiv 2^{224} \equiv 1 \pmod{449}$ (using a calculator or computational software.) So Proth's test fails for $a = 2$. Next we try $a = 3$ and compute $3^{224} \equiv -1 \pmod{449}$, which shows that 449 is prime.

9.5.11. Note that $3329 = 2^8 13 + 1$ and $13 < 2^8$, so it is of the form which can be tested by Proth's test. We try $2^{(3329-1)/2} \equiv 2^{1664} \equiv 1 \pmod{3329}$ (using a calculator or computational software.) So Proth's test fails for $a = 2$. Next we try $a = 3$ and compute $3^{1664} \equiv -1 \pmod{3329}$, which shows that 3329 is prime.

9.5.12. Suppose p is a prime dividing n, and let F have prime-power factorization $F = \prod_{j=1}^r q_j^{c_j}$. Then, for each q_j, we have that $a_j^{n-1} \equiv 1 \pmod{n}$, for some integer a_j, and hence, $a_j^{n-1} \equiv 1 \pmod{p}$. So $\text{ord}_p a_j \mid (n-1)$. But $(a_j^{(n-1)/q_j} - 1, n) = 1$, so that $\text{ord}_p a_j \mid (n-1)/q_j$. Therefore, $q_j^{c_j} \mid \text{ord}_p a_j \mid (p-1)$. Since the q_j

are distinct primes, we have $F \mid (p-1)$. Likewise, $\text{ord}_p b \mid n-1$, and since $(b^F-1, n) = (b^{(n-1)/R}-1, n) = 1$, we have that at least one prime divisor Q of R divides $p-1$. Since $(F, Q) = 1$, we have $FQ \mid (p-1)$, and $Q \geq B$. Then $p > FQ \geq FB > \sqrt{n}$. Therefore all prime divisors of n are greater than $\sqrt{n}$. But this is only possible if n is prime.

9.5.13. We apply Pocklington's test to this situation. Note that $n-1 = hq^k$, so we let $F = q^k$ and $R = h$ and observe that by hypothesis $F > R$. Since q is the only prime dividing F, we need only check that there is an integer a such that $a^{n-1} \equiv 1 \pmod{n}$ and $(a^{(n-1)/q} - 1, n) = 1$. But both of these conditions are hypotheses, therefore n is prime by Pocklington's test.

9.5.14. Let $m = 78557 \cdot 2^n + 1$. Note that $78557 \equiv 2 \pmod 3$, so if $n = 2a$ is even, then $m \equiv 2 \cdot 2^{2a} + 1 \equiv 2 \cdot 1 + 1 \equiv 0 \pmod 3$. So $3 \mid m$ and $3 < m$, so m is not prime. If n is odd, there are two cases. First, if $n = 4a+1$, then $m = 78557 \cdot 2^{4a+1} + 1 \equiv 2 \cdot 2 + 1 \equiv 0 \pmod 5$, so again, m is not prime. Second, if $n = 4a+3$, there are 3 cases, either $n = 12a+3$, $n = 12a+7$, or $n = 12a+11$. If $n = 12a+7$, then $m = 78557 \cdot 2^{12a+7} + 1 \equiv 3 \cdot 2^7 + 1 \equiv 0 \pmod 7$, and so m is not prime. If $n = 12a+11$ we have $m = 78557 \cdot 2^{12a+11} + 1 \equiv 11 \cdot 2^{11} + 1 \equiv 0 \pmod{13}$, and so m is not prime. If $n = 12a+3$, there are 3 cases, either $n = 36a+3$, $36a+15$, or $36a+27$. If $n = 36a+3$ we have $m = 78557 \cdot 2^{36a+3} + 1 \equiv 9 \cdot 2^3 + 1 \equiv 0 \pmod{73}$, and so m is not prime. If $n = 36a+15$, we have $m = 78557 \cdot 2^{36a+15} + 1 \equiv 11 \cdot 2^{15} + 1 \equiv 0 \pmod{19}$. Finally, if $n = 36a+27$ we have $m = 78557 \cdot 2^{36a+27} + 1 \equiv 6 \cdot 2^{27} + 1 \equiv 6 \cdot (2^5)^5 2^2 + 1 \equiv 6 \cdot (-5)^5 4 + 1 \equiv 0 \pmod{37}$, and again, m is not prime. (Note that the congruence classes represented by the various arithmetic progressions for n constitute a system of covering congruences for the integers.)

9.6. Universal Exponents

9.6.1. a. Since the prime factorization of 100 is $100 = 2^2 5^2$ we have $\lambda(100) = [\lambda(2^2), \phi(5^2)] = [2, 20] = 20$.

b. Since the prime factorization of 144 is $144 = 2^4 3^2$ we have $\lambda(144) = [\lambda(2^4), \phi(3^2)] = [4, 6] = 12$.

c. Since the prime factorization of 22 is $222 = 2 \cdot 3 \cdot 37$, we have $\lambda(222) = [\lambda(2), \phi(3), \phi(37)] = [1, 2, 36] = 36$.

d. Since the prime factorization of 884 is $884 = 2^2 \cdot 13 \cdot 17$, we have $\lambda(884) = [\lambda(2^2), \phi(13), \phi(17)] = [2, 12, 16] = 48$.

e. We have $\lambda(2^4 \cdot 3^3 \cdot 5^2 \cdot 7) = [\lambda(2^4), \phi(3^3), \phi(5^2), \phi(7)] = [4, 18, 20, 6] = 180$.

f. We have $\lambda(2^5 \cdot 3^2 \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19) = [\lambda(2^5), \phi(3^2), \phi(5^2), \phi 7^3), \phi(11^2), \phi(13), \phi(17), \phi(19)] = [8, 6, 20, 294, 110, 12, 16, 18] = [2^3, 2 \cdot 3, 2^2 \cdot 5, 2 \cdot 3 \cdot 7^2, 2 \cdot 5 \cdot 11, 2^2 \cdot 3, 2^4, 2 \cdot 3^2] = 2^4 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 = 388080$.

g. Since $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot$ we have $\lambda(10!) = [\lambda(2^8), \phi(3^4), \phi(5^2), \phi(7)] = [64, 54, 20, 6] = 8640$.

h. Since $20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, it follows that $\lambda(20!) = [\lambda(2^{18}), \phi(3^8), \phi(5^4), \phi(72), \phi(11), \phi(13), \phi(17), \phi(19)] = [65536, 4374, 500, 42, 10, 12, 16, 18] = [2^{16}, 2 \cdot 3^7, 2^2 5^3, 2 \cdot 3 \cdot 7, 2 \cdot 5, 2^2 \cdot 3, 2^4, 2 \cdot 3^2] = 2^{16} \cdot 3^7 \cdot 5^3 \cdot 7 = 125411328000$.

9.6.2. a. We will use the following facts: $\phi(p^t)$ is even for an odd prime p. $\lambda(2^t) = 2^{t-2}$ is even for $t \geq 3$. $\lambda(4) = 2$, and $\lambda(2) = 1$. If $p^t \mid n$, then $\phi(p^t) \mid \lambda(n)$.
If $\lambda(n) = 1$ there can be no even components in the least common multiple. Therefore, $n = 1$ or 2.

b. Since $\lambda(p^t) \mid 2$, we can have only $p^t = 3^1, 2^1, 2^2$, or 2^3. Therefore $n = 8, 4, 3, 6, 12$, or 24.

c. The only integers n giving odd $\lambda(n)$ are given in part (a). Therefore, there are no solutions to $\lambda(n) = 3$.

d. We must have $\lambda(p^t) \mid 4$, so $p^t = 5, 3, 2, 2^2, 2^3$, or 2^4. Therefore $n = 5, 15, 16, 10, 30, 20, 60, 40, 120, 80,$ or 240.

e. As in part (c), there are no solutions to $\lambda(n) = 5$.

f. We must have $\lambda(p^t) \mid 4$, so $p^t = 2, 4, 8, 7$, or 9. Therefore $n = 7, 14, 28, 56, 9, 18, 36, 72$.

9.6.3. We seek $n = 2^{t_0}p_1^{t_1} \cdots p_m^{t_m}$ such that $\lambda(n) = [\lambda(2^{t_0}), \phi(p_1^{t_1}), \ldots, \phi(p_m^{t_m})] = 12$. So we must have $\lambda(2^{t_0}) \mid 12$. For $t_0 \geq 3$, we have $\lambda(2^{t_0}) = 2^{t_0-2} \mid 12$, so the largest t_0 can be is $t_0 = 4$. We also must have $\phi(p_i^{t_i}) = p_i^{t_i-1}(p_i - 1) \mid 12$, so $p_i - 1 = 1, 2, 3, 4, 6,$ or 12, and $p_i = 2, 3, 4, 5, 7,$ or 13. But p_i is an odd prime, so $p_i = 3, 5, 7,$ or 13 are the only possibilities for odd prime divisors of n. Also, $p_i^{t_i-1} \mid 12$, so if $t_i > 1$, we have that $p_i = 3$ and $t_i = 2$. Therefore the largest such n is $2^4 3^2 \cdot 5 \cdot 7 \cdot 13 = 65520$.

9.6.4. a. We have $\lambda(12) = [\lambda(4), \lambda(3)] = 2$, and $5^2 \equiv 1 \pmod{12}$.

b. We have $\lambda(15) = [\lambda(3), \lambda(5)] = 4$, and $2^4 \equiv 1 \pmod{15}$.

c. We have $\lambda(20) = [\lambda(4), \lambda(5)] = 4$, and $3^4 \equiv 1 \pmod{20}$.

d. We have $\lambda(36) = [\lambda(4), \lambda(9)] = 6$, and $5^6 \equiv 1 \pmod{36}$.

e. We have $\lambda(40) = [\lambda(8), \lambda(5)] = 4$, and $3^4 \equiv 1 \pmod{40}$.

f. We have $\lambda(63) = [\lambda(7), \lambda(9)] = 6$, and $5^6 \equiv 1 \pmod{63}$.

9.6.5. Suppose that $m = 2^{t_0}p_1^{t_1 \cdot} p_s^{t_s}$. Then $\lambda(m) = [\lambda(2^{t_0}), \phi(p_1^{t_1}), \ldots, \phi(p_{t_s})]$. Furthermore, $\phi(m) = \phi(2^{t_0})\phi(p_1^{t_1} \cdots \phi(p_s^{t_s})$. Since $\lambda(2^{t_0}) = 1, 2,$ or 2^{t_0-2} when $t_0 = 1, 2,$ or $t_0 \geq 3$, respectively, it follows that $\lambda(2^{t_0}) \mid \phi(2^{t_0}) = 2^{t_0-1}$. Since the least common multiple of a set of numbers divides the product of these numbers, or their multiples, we see that $\lambda(m) \mid \phi(m)$.

9.6.6. Let $M = [\lambda(m), \lambda(n)]$. Then for any integer a with $(a, m) = (a, n) = 1$, we have $a^M \equiv 1 \pmod{m}$ and $a^M \equiv 1 \pmod{n}$, since $\lambda(m) \mid M$ and $\lambda(n) \mid M$. By the Chinese remainder theorem, this system has a unique solution modulo mn, so we must have $a^M \equiv 1 \pmod{mn}$. Therefore, M is a universal exponent of mn and hence $\lambda(mn) \mid M$. Now let b be an element of order $\lambda(m)$ modulo m. Then $b^{\lambda(mn)} \equiv 1 \pmod{mn}$, so $b^{\lambda(mn)} \equiv 1 \pmod{m}$. So, $\lambda(m) \mid \lambda(mn)$. Similarly, $\lambda(m) \mid \lambda(mn)$, and so $M = \lambda(mn)$.

9.6.7. For any integer x with $(x, n) = (x, m) = 1$ we have $x^a \equiv 1 \pmod{n}$ and $x^a \equiv 1 \pmod{m}$. Then the Chinese remainder theorem gives us $x^a \equiv 1 \pmod{[n, m]}$. But since n is the largest integer with this property, we must have $[n, m] = n$, so $m|n$.

9.6.8. We count solutions of the system in the proof of Theorem 9.21. For each p_i there are $\phi(\phi(p_i^{t_i}))$ primitive roots, so there are this many choices for r_i for each i. Similarly, by Exercise 15 of Section 8.3, there are 2^{k-3} elements of maximal order modulo 2^{t_0}, so in all there are $2^{k-3}\prod_{i=1}^{m} \phi(\phi(p_i^{t_i}))$ ways to choose the system. Each system gives a unique and different element with maximal order.

9.6.9. Suppose that $ax \equiv b \pmod{m}$. Multiplying both sides of this congruence by $a^{\lambda(m)-1}$ gives $a^{\lambda(m)}x \equiv a^{\lambda(m)-1}b \pmod{m}$. Since $a^{\lambda(m)} \equiv 1 \pmod{m}$, it follows that $x \equiv a^{\lambda(m)-1}b \pmod{m}$. Conversely, let $x_0 \equiv a^{\lambda(m)-1}b \pmod{m}$. then $ax_0 \equiv aa^{\lambda(m)-1}b \equiv a^{\lambda(m)}b \equiv b \pmod{m}$, so x_0 is a solution

9.6.10. Suppose $p^2 \mid m$, where p is prime. We show that for no a is $a^c \equiv p \pmod{m}$. If this congruence holds, then $p \mid a$. However, if $c > 1, p^2 \mid a^c$, but $p^2 \nmid p$, so since $p^2 \mid m$ this congruence is impossible modulo p^2, and hence, modulo m. If $(c, \lambda(m)) > 1$, let $g = \lambda(m)/(c, \lambda(m))$. We know that g is not a universal exponent for m, because $g < \lambda(m)$. So, if $1^c, 2^c, ..., (m-1)^c$ is a complete residue system, one of these numbers raised to the g power is not congruent to 1 modulo m. This, however, is impossible, because $\lambda(m) \mid cg$. Conversely, suppose $a^c \equiv b^c \pmod{m}$, and let q be a prime dividing m. If $q \mid (a - b)$ then $a \equiv b \pmod{q}$.If $q \nmid (a - b)$, then we note that $\phi(q) = (q - 1) \mid \lambda(m)$. Since $(c, \lambda(m)) = 1, (c, q - 1) = 1$.

Therefore, $a^c \equiv b^c \pmod{q}$, and hence $a \equiv b \pmod{q}$. Since this is true for all such primes q dividing m, we have $a \equiv b \pmod{m}$, by the Chinese remainder theorem.

9.6.11. a. First suppose that $m = p^a$. Then we have $x(x^{c-1} - 1) \equiv 0 \pmod{p^a}$. Let s be a primitive root for p^a, then the solutions to $x^{c-1} \equiv 1$ are exactly the powers s^k with $(c-1)k \equiv 1 \pmod{\phi(p^a)}$, and there are $(c-1, \phi(p^a))$ of these. Also, 0 is a solution, so we have $1 + (c-1, \phi(p^a))$ solutions all together. Now if $m = p_1^{a_1} \cdots p_r^{a_r}$, we can count the number of solutions modulo $p_i^{a_i}$ for each i. There is a one-to-one correspondence between solutions modulo m and the set of r-tuples of solutions to the system of congruences modulo each of the prime powers. The correspondence is given by the Chinese Remainder Theorem.

b. Suppose $(c-1, \phi(m)) = 2$, then $c-1$ is even. Since $\phi(p^a)$ is even for all prime powers, except 2, we have $(c-1, \phi(p_i^{a_i})) = 2$ for each i. Then by part (a), we have the number of solutions $= 3^r$. If 2^1 is a prime factor, then $\phi(m) = \phi(m/2)$, and since x^c and x have the same parity, x is a solution modulo m if and only if it is a solution modulo $m/2$, so the proposition still holds.

9.6.12. In an RSA cipher, $m = pq$, so $r = 2$. By part (b) of Exercise 11, there are exactly $3^3 = 9$ solutions to $P \equiv P^e \pmod{m}$. These are the 9 plaintext messages which stay unchanged.

9.6.13. Let $n = 3pq$, with $p < q$ odd primes, be a Carmichael number. Then by Theorem 9.27, $p-1 | 3pq - 1 = 3(p-1)q + 3q - 1$, so $p-1 | 3q-1$, say $(p-1)a = 3q - 1$. Since $q > p$, we must have $a \geq 4$. Similarly, there is an integer b such that $(q-1)b = 3p - 1$. Solving these two equations for p and q yields $q = (2a + ab - 3)/(ab - 9)$, and $p = (2b + ab - 3)/(ab - 9) = 1 + (2b + 6)/(ab - 9)$. Then since p is an odd prime greater than 3, we must have $4(ab - 9) \leq 2b + 6$, which reduces to $b(2a - 1) \leq 21$. Since $a \geq 4$, this implies that $b \leq 3$. Then $4(ab - 9) \leq 2b + 6 \leq 12$, so $ab \leq 21/4$, so $a \leq 5$. Therefore $a = 4$ or 5. If $b = 3$, then the denominator in the expression for q is a multiple of 3, so the numerator must be a multiple of 3, but that is impossible since there is no choice for a which is divisible by 3. Thus $b = 1$ or 2. The denominator of q must be positive, so $ab > 9$, which eliminates all remaining possibilities except $a = 5$, $b = 2$, in which case $p = 11$ and $q = 17$. So the only Carmichael number of this form is $561 = 3 \cdot 11 \cdot 17$.

9.6.14. The inequalities in the solution to Exercise 15, give us $p, q \leq (5^3 + 5^2 - 5 + 1)/(2) = 73$, so we have finitely many cases to check. Only 3 of the possible numbers are pseudoprimes to the base 2: $5 \cdot 13 \cdot 17, 5 \cdot 17 \cdot 29$, and $5 \cdot 29 \cdot 73$. Of these, $5 \cdot 17 \cdot 29$ is not a pseudoprime to the base 3, and $5 \cdot 13 \cdot 17$ one fails to be a pseudoprime to the base 6. However, $5 \cdot 29 \cdot 73$ is a Carmichael number, as was shown in Exercise 16 of Section 6.2.

9.6.15. Assume $q < r$. By Theorem 9.23, $q - 1 | pqr - 1 = (q-1)pr + pr - 1$. Therefore $q - 1 | pr - 1$, say $a(q-1) = pr - 1$. Similarly $b(r-1) = pq - 1$. Since $q < r$, we must have $a > b$. Solving these two equations for q and r yields $r = (p(a-1) + a(b-1))/(ab - p^2)$ and $q = (p(b-1) + b(a-1))/(ab - p^2) = 1 + (p^2 + pb - p - b)/(ab - p^2)$. Since this last fraction must be an integer we have $ab - p^2 \leq p^2 + pb - p - b$ which reduces to $a(b-1) \leq 2p^2 + p(b-1)$ or $a - 1 \leq 2p^2/b + p(b-1)/b \leq 2p^2 + p$. So there are only finitely many values for a. Likewise, the same inequality gives us $b(a-1) \leq 2p^2 + pb - p$ or $b(a - 1 - p) \leq 2p^2 - p$. Since $a > b$ and the denominator of the expression for q must be positive, we have that $a \geq p + 1$. If $a = p + 1$, we have $(p+1)(q-1) = pq - p + q - 1 = pr - 1$, which implies that $p | q$, a contradiction. Therefore $a > p + 1$, and so $a - 1 - p$ is a positive integer. The last inequality gives us $b \leq b(a - 1 - p) \leq 2p^2 - p$. Therefore there are only finitely many values for b. Since a and b determine q and r, we see that there can be only finitely many Carmichael numbers of this form.

9.6.16. Since $(e, \phi(n)) = 1$ and $\lambda(n) \mid \phi(n)$, we have $(e, \lambda(n)) = 1$. Therefore, and inverse d of e modulo $\lambda(n)$ exists. Then $ed = k\lambda(n) + 1$ for some integer k. Then, if P is a plaintext block, the cipher text is $C \equiv P^e \pmod{n}$. Then $C^d \equiv P^{ed} \equiv P^{k\lambda(n)+1} \equiv 1^k P \equiv P \pmod{n}$, since $\lambda(n)$ is a universal exponent.

9.6.17. We have $q_n(ab) \equiv ((ab)^{\lambda(n)} - 1)/n = (a^{\lambda(n)} b^{\lambda(n)} - a^{\lambda(n)} - b^{\lambda(n)} + 1 + a^{\lambda(n)} + b^{\lambda(n)} - 2)/n = (a^{\lambda(n)} - 1)(b^{\lambda(n)} - 1)/n + ((a^{\lambda(n)} - 1) + (b^{\lambda(n)} - 1))/n \equiv q_n(a) + q_n(b) \pmod{n}$. At the last step, we use the fact that n^2 must divide $(a^{\lambda(n)} - 1)(b^{\lambda(n)} - 1)$, since $\lambda(n)$ is the universal exponent.

9.6.18. First, note that $a^{\lambda(n)-1}a \equiv a^{\lambda(n)} \equiv 1 \pmod{n}$, so $\overline{a} \equiv a^{\lambda(n)-1} \pmod{n}$. Then by the Binomial Theorem, we have, $q_n(a+nc) \equiv ((a+nc)^{\lambda(n)}-1)/(n) \equiv (a^{\lambda(n)} + \lambda(n)a^{\lambda(n)-1}nc + \binom{\lambda(n)}{2}a^{\lambda(n)-2}(nc)^2 + \cdots + (nc)^{\lambda(n)}-1)/(n) \equiv a^{\lambda(n)}/n + \lambda(n)\overline{a}c - 1/n \equiv q_n(a) + \lambda(n)\overline{a}c \pmod{n}$.

CHAPTER 10

Applications of Primitive Roots and the Order of an Integer

10.1. Pseudorandom Numbers

10.1.1. First term: 69; second term: 76, since $69^2 = 4761$; third term: 77, since $76^2 = 5776$; fourth term: 92, since $77^2 = 5929$; fifth term: 46, since $92^2 = 8464$; sixth term: 11, since $46^2 = 2116$; seventh term: 12, since $11^2 = 0121$; eighth term: 14, since $12^2 = 0144$; ninth term: 19, since $14^2 = 0196$; tenth term: 36, since $19^2 = 0361$; eleventh term: 29, since $36^2 = 1296$; twelfth term: 84, since $29^2 = 0841$; thirteenth term: 05, since $84^2 = 7056$; fourteenth term: 02, since $5^2 = 0025$; fifteenth term: 00, since $02^2 = 0004$; sixteenth term and all remaining terms are 00, since $0^2 = 0000$.

10.1.2. We have $x_0 = 6, x_1 \equiv 5 \cdot 6 + 2 = 32 \equiv 13 \pmod{19}, x_1 \equiv 5 \cdot 13 + 2 = 67 \equiv 10 \pmod{19}, x_2 \equiv 5 \cdot 10 + 2 = 52 \equiv 14 \pmod{19}, x_3 \equiv 5 \cdot 14 + 2 = 72 \equiv 15 \pmod{19}, x_4 \equiv 5 \cdot 15 + 2 = 77 \equiv 1 \pmod{19}, x_5 \equiv 5 \cdot 1 + 2 = 7 \pmod{19}, x_6 \equiv 5 \cdot 7 + 2 = 37 \equiv 18 \pmod{19}, x_7 \equiv 5 \cdot 18 + 2 = 92 \equiv 16 \pmod{19}, x_8 \equiv 5 \cdot 16 + 2 = 82 \equiv 6 \pmod{19}, x_9 \equiv 5 \cdot 6 + 2 = 32 \equiv 1, 3 \pmod{19}, x_{10} \equiv 5 \cdot 13 + 2 = 67 \equiv 10 \pmod{19}, x_{11} \equiv 5 \cdot 10 + 2 = 52 \equiv 14 \pmod{19}$, and since $x_{11} = x_2$, it follows that $x_k = x_{k-9}$ for $k \geq 11$. Hence the sequence is $13, 10, 14, 15, 1, 7, 18, 16, 6, 13, 10, 14, 15, 1, 7, 18, \ldots$ and the period length is 9.

10.1.3. We compute $x_0 = 2, x_1 = 15, x_2 = 17, x_3 = 0, x_4 = 7, x_5 = 10, x_6 = 22, x_7 = 20, x_8 = 12, x_9 = 15$, and $x_1 0 = 2 = x_0$. So the period length is 10.

10.1.4. If $a = 0$ we have $x_{n+1} \equiv c \pmod{m}$ which means that the sequence is constant for $n \geq 1$, clearly not a good choice for a sequence of pseudorandom numbers. If $a = 1$ we have $x_{n+1} \equiv x_n + c \pmod{m}$, which shows that the terms of the sequence differ by a constant modulo m, also not a good choice for a sequence of pseudorandom numbers.

10.1.5. a. From Theorem 10.2, we must have $a \equiv 1 \pmod{4}$ since $4 \mid 1000$, and since 5 is the only odd prime dividing 1000, we must also have $a \equiv 1 \pmod{5}$. By the Chinese remainder theorem, we have $a \equiv 1 \pmod{20}$.

b. We have $30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. So we solve the system $a \equiv 1 \pmod{m}$, for $m = 2, 3, 5, 7, 11, 13$, to get $a \equiv 1 \pmod{30030}$.

c. We have $10^6 - 1 = 3^2 7 \cdot 11 \cdot 13 \cdot 37$, so we must have $a \equiv 1 \pmod{m}$ for $m = 3, 7, 11, 13, 37$. This system has solutions $a \equiv 1 \pmod{111111}$, by the Chinese remainder theorem.

d. We have $2^{25} - 1 = 31 \cdot 601 \cdot 1801$, so we must have $a \equiv 1 \pmod{m}$ for $m = 31, 601, 1801$. By the Chinese remainder theorem, we have $a \equiv 1 \pmod{2^{25} - 1}$.

10.1.6. We proceed by induction. $y_0 = 0$, so $y_1 = 1$, and $by_1 + x_0 \equiv (a-1)x_0 + c + x_0 \equiv ax_0 + c \equiv x_1 \pmod{m}$, so the basis step holds. Suppose $x_{n-1} \equiv by_{n-1} + x_0 \pmod{m}$. Then we have $by_n + x_0 \equiv b(ay_{n-1} + 1) + x_0 \equiv aby_{n-1} + b + x_0 \equiv a(x_{n-1} - x_0) + b + x_0 \equiv ax_{n-1} + b - (a-1)x_0 \equiv ax_{n-1} + c \equiv x_n \pmod{m}$, which completes the induction step.

10.1.7. a. Since $2^{31} \equiv 1 \pmod{M_{31}}$, the order of 2 must be a divisor of 31. Since 31 is prime, the order must be 31, which is the period length.

b. Using computational software, we compute $3^{(M_{31}-1)/p} \pmod{M_{31}}$ for each prime power divisor p^k of $M_{31}-1$. The residue is 1 only for $p=3$, but not for $p^k=3^2$. Therefore the period length is $(M_{31}-1)/3 = 715827882$.

c. From part (a) we have $4^{31} = (2^{31})^2 \equiv 1 \pmod{M_{31}}$, so the order of 4, and hence the period length, must be 31.

d. Using computational software, we compute $5^{(M_{31}-1)/p} \pmod{M_{31}}$ for each prime power divisor p^k of $M_{31}-1$. The residue is 1 only for $p=11$. Therefore the period length is $(M_{31}-1)/11 = 195225786$.

e. Using computational software, we compute $13^{(M_{31}-1)/p} \pmod{M_{31}}$ for each prime power divisor p^k of $M_{31}-1$. The residue is 1 only for $p=2$. Therefore the period length is $(M_{31}-1)/2 = 1073741823$.

f. Using computational software, we compute $17^{(M_{31}-1)/p} \pmod{M_{31}}$ for each prime power divisor p^k of $M_{31}-1$. The residue is 1 only for $p=2$. Therefore the period length is $(M_{31}-1)/2 = 1073741823$.

10.1.8. From Exercise 15 of Section 9.3, we know that 5 has order 2^{e-2} modulo 2^e, but no element has higher order, since that order would have to be 2^{e-1} which would imply the existence of a primitive root, contradicting Theorem 9.15. Therefore, we have $x_{2^{e-2}} \equiv a^{2^{e-2}} x_0 \equiv x_0 \pmod{2^e}$, and hence the maximum period length is 2^{e-2}. We also see that this is achieved for $a = 5 \equiv -3 \pmod 8$. Then it is also achieved for $a = -5 \equiv 3 \pmod 8$.

10.1.9. We compute $x_1 \equiv 8^2 \equiv 64 \pmod{77}$, $x_2 \equiv 64^2 \equiv 15 \pmod{77}$, $x_3 \equiv 15^2 \equiv 71 \pmod{77}$, $x_4 \equiv 71^2 \equiv 36 \pmod{77}$, and $x_5 \equiv 36^2 \equiv 64 \equiv x_1 \pmod{77}$. So the sequence of numbers is $8, 64, 15, 71, 36, 64, \ldots$.

10.1.10. We compute $x_1 \equiv 5^2 \equiv 25 \pmod{1001}$, $x_2 \equiv 25^2 \equiv 625 \pmod{1001}$, $x_3 \equiv 625^2 \equiv 235 \pmod{1001}$, $x_4 \equiv 235^2 \equiv 170 \pmod{1001}$, $x_5 \equiv 170^2 \equiv 872 \pmod{1001}$, and $x_6 \equiv 872^2 \equiv 625 \equiv x_2 \pmod{1001}$. So the sequence of numbers is $5, 25, 625, 235, 170, 872, 625, \ldots$.

10.1.11. First we compute $\text{ord}_{77}8$. Since $8 \equiv 1 \pmod 7$ and $8^{10} \equiv 1 \pmod{11}$ by Fermat's Little Theorem, the Chinese remainder theorem shows that $8^{10} \equiv 1 \pmod{77}$. Since $8^5 \equiv 43 \pmod{77}$, we know that $\text{ord}_{77}8 = 10$. Therefore $t=1$ and $s=5$. Since 2 is a primitive root modulo 5, we know that $\text{ord}_5 2 = 4$. So by Theorem 10.4, the period length is 4.

10.1.12. First we compute $\text{ord}_{1001}5$. Since $5^6 \equiv 1 \pmod 7$, $5^5 \equiv 1 \pmod{11}$, and $5^4 \equiv 1 \pmod{13}$, then by the Chinese remainder theorem, $5^{60} \equiv 1 \pmod{1001}$. Since $5^{30} \equiv 155 \pmod{1001}$, we have $\text{ord}_{1001}5 = 60$. Therefore $t=2$ and $s=15$ in Theorem 10.4. Since $2^4 \equiv 1 \pmod{15}$, we know the period length is 4.

10.1.13. Using the notation of Theorem 10.4, we have $\phi(77) = 60$, so $\text{ord}_{77}x_0$ is a divisor of $60 = 2^2 3 \cdot 5$. Then the only possible values for s are the odd divisors of 60, which are $3, 5$, and 15. Then we note that $2^2 \equiv 1 \pmod 3$, $2^4 \equiv 1 \pmod 5$, and $2^4 \equiv 16 \equiv 1 \pmod{15}$. In each case we have shown that $\text{ord}_s 2 \leq 4$. Hence by Theorem 10.4 the maximum period length is 4.

10.1.14. Using the notation of Theorem 10.4, we have $\phi(989) = 924 = 2^2 3 \cdot 7 \cdot 11$. So $\text{ord}_{1001}x_0$ is an odd divisor of 924. Then the only possible values for s are $3, 7, 11, 21, 33, 77$, and 231. We compute that $\text{ord}_{231}2 = 30$. Hence by Theorem 10.4 the maximum period length is 30.

10.1.15. We have $x_0 = 1$ and $x_1 = 24$. Using the definition of the Fibonacci generator, it follows that $x_2 \equiv x_1 + x_0 \equiv 1 + 24 = 25 \pmod{31}$. Hence $x_2 = 25$. Continuing, we find that $x_3 \equiv x_2 + x_1 \equiv 25 + 24 = 49 \equiv 18 \pmod{31}$, so $x_3 = 18$. We compute successive terms in the same manner: $x_4 \equiv x_3 + x_2 \equiv 18 + 25 = 43 \equiv 12 \pmod{31}$, so $x_4 = 12$; $x_5 \equiv x_4 + x_3 \equiv 12 + 18 = 30 \pmod{31}$, so $x_5 = 30$; $x_6 \equiv x_5 + x_4 \equiv 30 + 12 = 42 \equiv 11 \pmod{31}$, so $x_6 = 11$; $x_7 \equiv x_6 + x_5 \equiv 11 + 30 = 41 \equiv 10 \pmod{31}$, so $x_7 = 10$; and $x_8 \equiv x_7 + x_6 \equiv 10 + 11 = 21 \pmod{31}$, so $x_8 = 21$. The terms x_i with $i = 0, 1, 2, \ldots, 8$ are 1, 24, 25, 18,

12, 30, 11, 10, and 21.

10.1.16. From Table E.3 in the back of the text, we find that 2 is a primitive root modulo 101. Now $(17, \phi(101)) = 1$, so $2^{17} \equiv 75$ is also a primitive root modulo 101. Since it is so large, it will make a good multiplier.

10.1.17. Check that 7 has maximal order 1800 modulo $2^{25} - 1$. To make a large enough multiplier, raise 7 to a power relatively prime to $\phi(2^{25} - 1) = 32400000$, for example, to the 11th power.

10.1.18. We have $402 \equiv a + c \pmod{1003}$ and $361 \equiv 402a + c \pmod{1003}$. Multiply the first congruence by 402 and subtract the second to get $402^2 - 361 \equiv 402c - c \pmod{1003}$, or $401c \equiv 763 \pmod{1)003}$, which has solution $c \equiv 197 \pmod{1003}$. Then the first congruence gives us $a \equiv 402 - 197 \equiv 205 \pmod{1003}$.

10.1.19. We must have $313a \equiv 145 \pmod{1000}$. Solving this congruence yields $a = 665$.

10.1.20. a. We have $x_1 \equiv 3^2 \equiv 9 \pmod{17}$, $x_2 \equiv 3^9 \equiv 14 \pmod{17}$, $x_3 \equiv 3^1 4 \equiv 2 \equiv x_0 \pmod{17}$.

b. We have $x_1 \equiv 5^3 \equiv 31 \pmod{47}$, $x_2 \equiv 5^3 1 \equiv 39 \pmod{47}$, and the sequence continues: 39, 30, 36, 4, 14, 27, 33, 35, 29, 26, 16, 17, 38, 6, 21, 15, 41, 45, 19, 10, 12, 18, 2, 25, 22, 28, 24, 42, 37, 20, $3 = x_{33} = x_0$.

c. If we have a table of indices for the primitive root g modulo p, then we have $\text{ind}_g x_n \equiv x_{n-1} \pmod{p-1}$. Since each $x_n < p$, this will determine x_{n-1}.

10.1.21. a. We compute $x_1 \equiv 2^3 \equiv 8 \pmod{15}$, and $x_2 \equiv 8^3 \equiv 64 \cdot 8 \equiv 4 \cdot 8 \equiv 32 \equiv 2 \pmod{15}$. Since $x_2 = x_0$, the sequence is $8, 2, 8, 2, 8, 2, \ldots$

b. We compute $x_1 \equiv 3^2 \equiv 9 \pmod{23}$, $x_2 \equiv 9^2 \equiv 81 \equiv 12 \pmod{23}$, $x_3 \equiv 12^2 \equiv 6 \pmod{23}$, $x_4 \equiv 6^2 \equiv 13 \pmod{23}$, $x_5 \equiv 13^2 \equiv 8 \pmod{23}$, $x_6 \equiv 8^2 \equiv 18 \pmod{23}$, $x_7 \equiv 18^2 \equiv 2 \pmod{23}$, $x_8 \equiv 2^2 \equiv 4 \pmod{23}$, $x_9 \equiv 4^2 \equiv 16 \pmod{23}$, $x_{10} \equiv 16^2 \equiv 3 \pmod{23}$. Since $x_{10} = x_0$, the sequence is $9, 12, 6, 13, 8, 18, 2, 4, 16, 3, 9, 12, 6, \ldots$

10.2. The ElGamal Cryptosystem

10.2.1. We select $k = 1234$ for our random integer. Converting the plaintext into numerical equivalents results in 0700 1515 2401 0817 1907 0300 2423, where we filled out the last block with an X. Using a calculator or computational software, we find $\gamma \equiv r^k \equiv 6^{1234} \equiv 517 \pmod{2551}$. Then for each block P we compute $\delta \equiv P \cdot b^k \equiv P \cdot 33^{1234} \equiv P \cdot 651 \pmod{2551}$. The resulting blocks are $0700 \cdot 651 \equiv 1622 \pmod{2551}$, $1515 \cdot 651 \equiv 1579 \pmod{2551}$, $2401 \cdot 651 \equiv 1839 \pmod{2551}$, $0817 \cdot 651 \equiv 1259 \pmod{2551}$, $1907 \cdot 651 \equiv 1671 \pmod{2551}$, $0300 \cdot 651 \equiv 1424 \pmod{2551}$ and $2423 \cdot 651 \equiv 855 \pmod{2551}$. Therefore, the ciphertext is $(517, 1622), (517, 1579), (517, 1839), (517, 1259), (517, 1671), (517, 1424), (517, 855)$. To decrypt this ciphertext, we compute $\gamma^{p-1-a} \equiv 517^{2551-1-13} \equiv 517^{2537} \equiv 337 \pmod{2551}$. Then for each block of the cipher text we compute $P \equiv 337 \cdot \delta \pmod{2551}$. For the first block we have $337 \cdot 1622 \equiv 0700 \pmod{2551}$ which was the first block of the plaintext. The other blocks are decrypted the same way.

10.2.2. We select $k = 1007$ for our random integer. Converting the plaintext into numerical equivalents results in 0314 1314 1915 0018 1806 1423, where we filled out the last block with an X. Using a calculator or computational software, we find $\gamma \equiv 7^k \equiv 7^{1007} \equiv 1423 \pmod{2591}$. Then for each block P we compute $\delta \equiv P \cdot b^k \equiv P \cdot 591^{1007} \equiv P \cdot 1313 \pmod{2591}$. The resulting blocks are $0314 \cdot 1313 \equiv 0313 \pmod{2591}$, $1314 \cdot 1313 \equiv 2267 \pmod{2591}$, $1915 \cdot 1313 \equiv 1125 \pmod{2591}$, $0018 \cdot 1313 \equiv 0315 \pmod{2591}$, $1806 \cdot 1313 \equiv 0513 \pmod{2591}$, and $1423 \cdot 1313 \equiv 0288 \pmod{2591}$. Therefore, the ciphertext is $(1423, 0313), (1423, 2267), (1423, 1125), (1423, 0315), (1423, 0513), (1423, 0288)$. To decrypt this ciphertext, we compute $\gamma^{p-1-a} \equiv 1423^{2591-1-99} \equiv 1423^{2491} \equiv 2443 \pmod{2591}$. Then for each block of the cipher text we compute $P \equiv 2443 \cdot \delta \pmod{2591}$. For the first block we have $2443 \cdot 0313 \equiv 0314 \pmod{2591}$ which was the first block of the plaintext. The other blocks are decrypted the same way.

10.2.3. We start by computing $\overline{\gamma^a} \equiv 2161^{2713-1-17} \equiv 2161^{2695} \equiv 167 \pmod{2713}$. Then multiplying the second number of each block and reducing yields $167 \cdot 660 \equiv 1700 \pmod{2713}$, $167 \cdot 1284 \equiv 0101$

$\pmod{2713}$, and $167 \cdot 1467 \equiv 0819 \pmod{2713}$. So the plaintext is 170001010819 which is equivalent to RABBIT.

10.2.4. We start by computing $\overline{\gamma^a} \equiv 1061^{2677-1-133} \equiv 1061^{2543} \equiv 1759 \pmod{2677}$. Then multiplying the second number of each block and reducing yields $1759 \cdot 2185 \equiv 1920 \pmod{2677}$, $1759 \cdot 0733 \equiv 1710 \pmod{2677}$, and $1759 \cdot 1096 \equiv 0424 \pmod{2677}$. So the plaintext is 192017100424 which is equivalent to TURKEY.

10.2.5. First we compute $\gamma \equiv 3^{101} \equiv 2022 \pmod{2657}$. Using the Euclidean algorithm we can compute $\overline{101} \equiv 973 \pmod{2656}$ then the signature is given by $s \equiv (823 - 211 \cdot 2022)973 \equiv 833 \pmod{2656}$. To verify this signature, we compute $V_1 \equiv 2022^{833}801^{2022} \equiv 1014 \pmod{2657}$ and $V_2 \equiv 3^{823} \equiv 1014 \pmod{2657}$. Since $V_1 = V_2$, the signature is verified.

10.2.6. First we compute $\gamma \equiv 5^{257} \equiv 1344 \pmod{2543}$. Using the Euclidean algorithm we can compute $\overline{257} \equiv 999 \pmod{2542}$ then the signature is given by $s \equiv (2525 - 99 \cdot 1344)999 \equiv 1589 \pmod{2542}$. To verify this signature, we compute $V_1 \equiv 1344^{1589}1615^{1344} \equiv 614 \pmod{2543}$ and $V_2 \equiv 5^{2525} \equiv 614 \pmod{2543}$. Since $V_1 = V_2$, the signature is verified.

10.2.7. Let $\delta_1 = P_1 b^k$ and $\delta_2 = P_2 b^k$ as in the ElGamal cryptosystem. If P_1 is known, it is easy to compute an inverse for P_1 modulo p. Then $b^k \equiv \overline{P_1}\delta_1 \pmod{p}$. Then it is also easy to compute an inverse for $b^k \pmod{p}$. Then $P_2 \equiv \overline{b^k}\delta_2 \pmod{p}$. Hence the plaintext P_2 is recovered.

10.2.8. We have knowledge of $P_1, P_2, s_1, s_2, \gamma_1, \gamma_2$, and the public key information. Note that $\gamma_1 \equiv r^k \equiv \gamma_2 \pmod{p}$, so we call this common value γ. Then we compute $s_1 - s_2 \equiv (P_1 - a\gamma)\overline{k} - (P_2 - a\gamma)\overline{k} \equiv (P_1 - P_2)\overline{k} \pmod{p-1}$. There are $(p-1, P_1 - P_2)$ solutions for $\overline{k}$, which is hopefully a small number of solutions. If $\overline{k}$ is a solution, then it is easy to find k by solving $\overline{k}x \equiv 1 \pmod{p-1}$. Then it is easy to recover a by solving $s_1 \equiv (P_1 - a\gamma) \pmod{p-1}$ for a, that is $a \equiv -\overline{\gamma}(S_1\overline{k} - P_1) \pmod{p-1}$.

10.3. An Application to the Splicing of Telephone Cables

10.3.1. a. Since 17 is prime it has a primitive root. Hence the maximal ± 1-exponent of 17 is $\phi(17)/2 = 8$.

b. Since 22 is of the form $2p$ where p is prime it has a primitive root. Hence the maximal ± 1-exponent of 22 is $\phi(22)/2 = 10/2 = 5$.

c. We see that $24 = 2^3 \cdot 3$ does not have a primitive root since it is not a power of a prime nor twice a power of a prime. Hence the maximal ± 1-exponent of 24 is $\lambda(24) = [\lambda(2^3), \phi(3)] = [2, 2] = 2$.

d. We see that $36 = 2^2 \cdot 3^2$ does not have a primitive root since it is not a power of a prime nor twice a power of a prime. Hence the maximal ± 1-exponent of 36 is $\lambda(36) = [\lambda(2^2), \phi(3^2)] = [2, 6] = 6$.

e. We see that $99 = 3^2 \cdot 11$ does not have a primitive root since it is not a power of a prime not twice a power of a prime. Hence the maximal ± 1-exponent of 99 is $\lambda(99) = [\phi(3^2), \phi(11)] = [6, 10] = 30$.

f. We see that $100 = 2^2 5^2$ does not have a primitive root since it is not a power of a prime nor twice a power of a prime. Hence the maximal ± 1-exponent of 100 is $\lambda(100) = [\lambda(2^2), \phi(5^2)] = [2, 20] = 20$.

10.3.2. a. Since 2 is a primitive root modulo 13, $2^6 \equiv -1 \pmod{13}$. So 2 has maximal ± 1-exponent.

b. Since 3 is a primitive root modulo 14, $3^6 \equiv -1 \pmod{14}$. So 3 has maximal ± 1-exponent.

c. We have $\lambda(15) = 4$, and 2 has order 4 modulo 15 without any lower power being congruent to -1.

d. Since 2 is a primitive root modulo 25, it does the job.

e. We have $\lambda(36) = 6$, and 5 has order 6.

f. We have $\lambda(60) = 4$, and 7 has order 4.

10.3.3. a. By Theorems 10.6 and 9.23, the maximal $\pm 1-$exponent of 50 is $\lambda_0(50) = \lambda(50) = [\lambda(2), \phi(25)] = 20$. We seek an integer with order $20 \pmod{50}$ to be the spread. Since 3 is a primitive root for 5, either 3 or 8 is a primitive root for 25. It turns out that $\text{ord}_{25}3 = 20 = \phi(25)$. So it follows that $\text{ord}_{50}3 = 20$. So we choose $s = 3$ for our spread.

b. We compute $\lambda_0(76) = [\lambda(4), \phi(19)] = [2, 18] = 18$. Since 2 is a primitive root modulo 19, we consider $s = 19 + 2 = 21$ for our spread. A quick computation shows that $\text{ord}_{76}21 = 18$.

c. We compute $\lambda_0(125) = \phi(125) = 100$. Since 2 is a primitive root for 5, we start there. Now $2^{50} \equiv -1 \pmod{125}$, so $\text{ord}_{100}2 = 100$. Thus we use $s = 2$ as our spread.

10.3.4. In a section of cable, there are m adjacent pairs of wires. After $[(m-1)/2]$ sections of cable, we have generated $m[(m-1)/2]$ pairs of adjacent wires. But from a set of size m there are only $\binom{m}{2} = m(m-1)/2$ possible different pairs, so the above is the maximum. If two wires are adjacent in the first section and in the kth section, then we have $S_k(j) \equiv S_k(j \pm 1) + 1 \pmod{m}$. Using Theorem 10.7, and assuming s has maximal ± 1-exponent, we have $1 + (j-1)s^{k-1} \equiv 1 + (j \pm 1 + 1)s^{k-1} + 1 \pmod{m}$ or $s^{k-1} \equiv \pm 1 \pmod{m}$, which implies that $k = \lambda_0(m) + 1 = \phi(m)/2 = (m+1)/2$. So we can have $k - 1 = (m-1)/2$ sections of cable before we repeat a pair.

CHAPTER 11

Quadratic Residues

11.1. Quadratic Residues and Nonresidues

11.1.1. a. We have $1^2 \equiv 2^2 \equiv 1 \pmod 3$. Hence the quadratic residues of 3 are those integers congruent to 1 modulo 3.

b. We have $1^2 \equiv 4^2 \equiv 1 \pmod 5$ and $2^2 \equiv 3^2 \equiv 4 \pmod 5$. Hence the quadratic residues of 5 are those integers congruent to 1 or 4 modulo 5.

c. We have $1 \equiv 12^2 \equiv 1 \pmod{13}$, $2^2 \equiv 11^2 \equiv 4 \pmod{13}$, $3^2 \equiv 10^2 \equiv 9 \pmod{13}$, $4^2 \equiv 9^2 \equiv 3 \pmod{13}$, $5^2 \equiv 8^2 \equiv 12 \pmod{13}$, and $6^2 \equiv 7^2 \equiv 10 \pmod{13}$. Hence the quadratic residues of 13 are those integers congruent to 1, 3, 4, 9, 10, or 12 modulo 13.

d. We have $1^2 \equiv 18^2 \equiv 1 \pmod{19}$, $2^2 \equiv 17^2 \equiv 4 \pmod{19}$, $3^2 \equiv 16^2 \equiv 9 \pmod{19}$, $4^2 \equiv 15^2 \equiv 16 \pmod{19}$, $5^2 \equiv 14^2 \equiv 6 \pmod{19}$, $6^2 \equiv 13^2 \equiv 17 \pmod{19}$, $7^2 \equiv 12^2 \equiv 11 \pmod{19}$, $8^2 \equiv 11^2 \equiv 7 \pmod{19}$, and $9^2 \equiv 10^2 \equiv 5 \pmod{19}$. Hence the quadratic residues of 19 are those integers congruent to 1, 4, 5, 6, 7, 9, 11, 16, or 17 modulo 19.

11.1.2. a. We have $1^2 \equiv 6^2 \equiv 1 \pmod 7$, $2^2 \equiv 5^2 \equiv 4 \pmod 7$, and $3^2 \equiv 4^2 \equiv 2 \pmod 7$. Hence, the quadratic residues of 7 are those integers congruent to 1, 2, or 4 modulo 7.

b. A reduced residue system modulo 8 is 1, 3, 5, and 7. We have $1^2 \equiv 3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \pmod 8$. Hence 1 is the only quadratic residue modulo 8.

c. A reduced residue system modulo 15 is 1, 2, 4, 7, 8, 11, 13, and 14. We have $1^2 \equiv 14^2 \equiv 4^2 \equiv 11^2 \equiv 1 \pmod{15}$, and $2^2 \equiv 13^2 \equiv 7^2 \equiv 8^2 \equiv 4 \pmod{15}$. Hence 1 and 4 are the only quadratic residues modulo 15.

d. A reduced residue system modulo 18 is 1, 5, 7, 11, 13, and 17. We have $1^2 \equiv 17^2 \equiv 1 \pmod{18}$, $5^2 \equiv 13^2 \equiv 7 \pmod{18}$, and $7^2 \equiv 11^2 \equiv 13 \pmod{18}$. Hence, the quadratic residues of 18 are those integers congruent to 1, 7, or 13 modulo 18.

11.1.3. From Exercise 1 (b) we have $\left(\frac{1}{5}\right) = \left(\frac{4}{5}\right) = 1$ and $\left(\frac{2}{5}\right) = \left(\frac{3}{5}\right) = -1$.

11.1.4. We have $1^2 = 6^2 \equiv 1 \pmod 7$, $2^2 \equiv 5^2 \equiv 4 \pmod 7$, and $3^2 \equiv 4^2 \equiv 2 \pmod 7$. Hence the quadratic residues of 7 are those integers congruent to 1,2,or 4 modulo 7. It follows that $\left(\frac{1}{7}\right) = 1$, $\left(\frac{2}{7}\right) = 1$, $\left(\frac{3}{7}\right) = -1$, $\left(\frac{4}{7}\right) = 1$, $\left(\frac{5}{7}\right) = -1$, and $\left(\frac{6}{7}\right) = -1$.

11.1.5. a. We compute $\left(\frac{7}{11}\right) \equiv 7^{(11-1/2} \equiv 7^5 \equiv 49^2 \cdot 7 \equiv 5^2 \cdot 7 \equiv 3 \cdot 7 \equiv -1 \pmod{11}$

b. We compute $(7, 14, 21, 28, 35) \equiv (7, 3, 10, 6, 2) \pmod{11}$ and three of these are greater than $11/2$, so $\left(\frac{7}{11}\right) = (-1)^3 = -1$

11.1.6. Note that $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$. It follows that if either both or neither of a and b is a quadratic residue of p then ab is a quadratic residue of p. On the other hand, if exactly one of a and b is a quadratic residue of p then it follows that ab is also a nonresidue. We conclude that either one or all three of a, b, and ab is

a quadratic residue of p.

11.1.7. We know that $\left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)$ by Theorem 11.4 Using Theorems 11.5 and 11.6 we have: If $p \equiv 1 \pmod 8$ then $\left(\frac{-2}{p}\right) = (1)(1) = 1$. If $p \equiv 3 \pmod 8$ then $\left(\frac{-2}{p}\right) = (-1)(-1) = 1$. If $p \equiv -1 \pmod 8$ then $\left(\frac{-2}{p}\right) = (-1)(1) = -1$. If $p \equiv -3 \pmod 8$ then $\left(\frac{-2}{p}\right) = (1)(-1) = -1$.

11.1.8. Theorem 11.4 gives $\left(\frac{n}{q}\right) = \left(\frac{p_1^{2t_1+1}}{q}\right)\cdots\left(\frac{p_k^{2t_k+1}}{q}\right) = \left(\frac{p_1^{2t_1}}{q}\right)\left(\frac{p_1}{q}\right)\cdots\left(\frac{p_k^{2t_k}}{q}\right) \leq p_k q$. Since $p_i^{2t_i}$ is a square, we have $\left(\frac{n}{q}\right) = 1 \cdot \left(\frac{p_1}{q}\right) \cdot 1\left(\frac{p_2}{q}\right)\cdots 1 \cdot \left(\frac{p_k}{q}\right)$.

11.1.9. Since $p-1 \equiv -1, p-2 \equiv -2, \ldots, (p+1)/2 \equiv (p-1)/2 \pmod p$, we have$((p-1)/2)!^2 \equiv -(p-1)! \equiv 1 \pmod p$ by Wilson's theorem. (since $p \equiv 3 \pmod 4$ the minus signs cancel). By Euler's criterion, $((p-1)/2)!^{(p-1)/2} \equiv \left(\frac{1}{p}\right)\left(\frac{2}{p}\right)\cdots\left(\frac{(p-1)/2}{p}\right) \equiv (-1)^t \pmod p$, by definition of the Legendre symbol. Since $(p-1)/2! \equiv \pm 1 \pmod p$, and $(p-1)/2$ is odd, we have the result.

11.1.10. Suppose that $(b,p) = 1$. Then $\left(\frac{b}{p}\right) + \left(\frac{2b}{p}\right) + \cdots + \left(\frac{(p-1)b}{p}\right) = \left(\frac{b}{p}\right)\left[\left(\frac{1}{p}\right) + \left(\frac{2}{p}\right) + \cdots + \left(\frac{(p-1)}{p}\right)\right] = \left(\frac{b}{p}\right)\cdot 0 = 0$, because $\left(\frac{1}{p}\right) + \left(\frac{2}{p}\right) + \cdots + \left(\frac{(p-1)}{p}\right) = 0$ since it is the sum of an equal number of 1's and -1's. (This follows since there are an equal number of quadratic residues and nonresidues modulo p among the integers 1,2,$\cdots, p-1$).

11.1.11. If $p \equiv 1 \pmod 4$, $\left(\frac{-a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a}{p}\right) = 1 \cdot 1 = 1$. If $p \equiv 3 \pmod 4$, $\left(\frac{-a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a}{p}\right) = (-1)\cdot 1 = -1$.

11.1.12. a. If $c \equiv 0 \pmod 2$ then $x \equiv 0$ is a solution. $x \equiv 1$ is a solution of $x^2 + 1 \equiv 0 \pmod 2$. $1^2 + 1 + 1 \equiv 1$ and $0^2 + 0 + 1 \equiv 1 \pmod 2$ so $x^2 + x + 1 \equiv 0 \pmod 2$ has no solution.

b. $y^2 \equiv d \pmod p$ if and only if $(2ax+b)^2 \equiv b^2 - 4ac \pmod p$ if and only if $4a^2x^2 + 4abx + b^2 - b^2 + 4ac \equiv 0 \pmod p$ if and only if $a^2x^2 + abx + ac \equiv 0 \pmod p$ since $(4,p) = 1$, if and only if $ax^2 + bx + c \equiv 0$ since $(a,p) = 1$. The conclusion follows.

11.1.13. a. We will use properties of congruence to complete the square. Suppose that $x^2 + x + 1 \equiv 0 \pmod 7$. Adding $-7x + 8$ to both sides give $x^2 - 6x + 9 \equiv -7x + 8 \equiv 1 \pmod 7$. Hence $(x-3)^2 \equiv 1 \pmod 7$. Since the solutions of $y^2 \equiv 1 \pmod 7$ are $y \equiv 1$ and $y \equiv -1 \pmod 7$, this implies that $x - 3 \equiv 1 \pmod 7$ or $x - 3 \equiv -1 \pmod 7$. It follows that the solutions are those x satisfying $x \equiv 4 \pmod 7$ or $x \equiv 2 \pmod 7$.

b. Suppose that $x^2 + 5x + 1 \equiv 0 \pmod 7$. Adding $-7x$ to both sides gives $x^2 - 2x + 1 \equiv 7x \equiv 0 \pmod 7$. Hence $(x-1)^2 \equiv 0 \pmod 7$. It follows that $x - 1 \equiv 0 \pmod 7$, so all solutions are given by $x \equiv 1 \pmod 7$.

c. Suppose that $x^2 + 3x + 1 \equiv 0 \pmod 7$. Then adding $-7x + 3$ to both sides gives $x^2 - 4x + 4 \equiv -7x + 3 \equiv 3 \pmod 7$. Hence $(x-2)^2 \equiv 3 \pmod 7$. But 3 is a quadratic nonresidue of 7. Hence there are no solutions.

11.1.14. Suppose that p is a prime that is at least 7. At least one of the three incongruent integers 2,5, and 10 is a quadratic residue of p, because if neither 2 nor 5 is a quadratic residue of p then $10 = 2 \cdot 5$ is a quadratic residue of p. If 2 is a quadratic residue of p then 1 and 2 are consecutive quadratic residues, if 5 is a quadratic residue of p then 4 and 5 are consecutive quadratic residues, while if 10 is a quadratic residue of p then 9 and 10 are consecutive quadratic residues.

11.1.15. Suppose that p is a prime that is at least 7. At least one of the three incongruent integers 2,3, and 6 is a quadratic residue of p, because if neither 2 nor 3 is a quadratic residue of p then $2 \cdot 3 = 6$ is a quadratic residue of p. If 2 is a quadratic residue, then 2 and 4 are quadratic residues that differ by 2; if 3 is a quadratic residue, then 1 and 3 are quadratic residues that differ by 2; while if 6 is a quadratic residue then

4 and 6 are quadratic residues that differ by 2.

11.1.16. Since 1 and 4 are quadratic residues for all primes greater than 3, and $4 - 1 = 3$, we're done.

11.1.17. a. Since $p = 4n + 3, 2n + 2 = (p + 1)/2$. Then $x^2 \equiv (\pm a^{n+1})^2 \equiv a^{2n+2} \equiv a^{(p+1)/2} \equiv a^{(p-1)/2}a \equiv 1 \cdot a \equiv a \pmod{p}$ using the fact that $a^{(p-1)/2} \equiv 1 \pmod{p}$ since a is a quadratic residue of p.

b. From Lemma 11.1, there are exactly two solutions to $y^2 \equiv 1 \pmod{p}$, namely $y \equiv \pm 1 \pmod{p}$. Since $p \equiv 5 \pmod{8}, -1$ is a quadratic residue of p and 2 is a quadratic nonresidue of p. Since $p = 8n + 5$, we have $4n + 2 = (p - 1)/2$ and $2n + 2 = (p + 3)/4$. Then $(\pm a^{n+1})^2 \equiv a^{(p+3)/4} \pmod{p}$ and $(\pm 2^{2n+1}a^{n+1})^2 \equiv 2^{(p-1)/2}a^{(p+3)/4} \equiv -a^{(p+3)/4} \pmod{p}$ by Euler's criterion. We must show that one of $a^{(p+3)/4}$ or $-a^{(p+3)/4} \equiv a \pmod{p}$. Now a is a quadratic residue of p, so $a^{(p-1)/2} \equiv 1 \pmod{p}$ and therefore $a^{(p-1)/4}$ solves $x^2 \equiv 1 \pmod{p}$. But then $a^{(p-1)/4} \equiv \pm 1 \pmod{p}$, that is $a^{(p+3)/4} \equiv \pm a \pmod{p}$ or $\pm a^{(p+3)/4} \equiv a \pmod{p}$ as desired.

11.1.18. Since $4n < p - 1$, and r is a primitive root modulo p, we have $r^{4n} \not\equiv 1 \pmod{p}$, and $r^{8n} \equiv r^{p-1} \equiv 1 \pmod{p}$. Then $r^{4n}(r^{6n} + r^{2n}) \equiv r^{10n} + r^{6n} \equiv r^{8n}r^{2n} + r^{6n} \equiv r^{2n} + r^{6n} \pmod{p}$. Since $r^{4n} \not\equiv 1 \pmod{p}$ we must have $r^{6n} + r^{2n} \equiv 0 \pmod{p}$. Then $(\pm(r^{7n} \pm r^n))^2 \equiv r^{14n} \pm 2r^{8n} + r^{2n} \equiv r^{6n} \pm 2 + r^{2n} \equiv \pm 2 \pmod{p}$ as desired.

11.1.19. Note $x^2 \equiv 1 \pmod{15}$ if and only if $x^2 \equiv 1 \pmod{3}$ and $x^2 \equiv 1 \pmod{5}$. The solutions to $x^2 \equiv 1 \pmod{3}$ are $x \equiv 1$ and $x \equiv 2 \pmod{3}$ and the solutions to $x^2 \equiv 1 \pmod{5}$ are $x \equiv 1$ and $x \equiv 4 \pmod{5}$. We use the Chinese remainder theorem to solve the four sets of simultaneous congruences: $x \equiv 1 \pmod{3}$ and $x \equiv 1 \pmod{5}, x \equiv 1 \pmod{3}$ and $x \equiv 4 \pmod{5}, x \equiv 2 \pmod{3}$ and $x \equiv 1 \pmod{5}$, and $x \equiv 2 \pmod{3}$ and $x \equiv 4 \pmod{5}$. This yields the four incongruence solutions $x \equiv 1, 4, 11$, and $14 \pmod{15}$.

11.1.20. Solving $x^2 \equiv 58 \equiv 2 \pmod{7}$ yields $x \equiv 3$ or $4 \pmod{7}$. Solving $x^2 \equiv 58 \equiv 3 \pmod{11}$ yields $x \equiv 5$ or $6 \pmod{7}$. This gives 4 possibilities: If $x \equiv 3 \pmod{7}$ and $x \equiv 5 \pmod{11}$, then the Chinese Remainder Theorem gives us $x \equiv 38 \pmod{77}$. Similarly, if $x \equiv 3 \pmod{7}$ and $x \equiv 6 \pmod{11}$, then $x \equiv 17 \pmod{77}$. Similarly, if $x \equiv 4 \pmod{7}$ and $x \equiv 5 \pmod{11}$, then $x \equiv 60 \pmod{77}$. Similarly, if $x \equiv 4 \pmod{7}$ and $x \equiv 6 \pmod{11}$, then $x \equiv 39 \pmod{77}$. So the solutions are 38, 17, 39, and 60 modulo 77.

11.1.21. Note that $1001 = 7 \cdot 11 \cdot 13$, so we solve the congruence modulo each of these primes. First we have $x^2 \equiv 207 \equiv 4 \pmod{7}$, so $x \equiv 2$ or $5 \pmod{7}$. Next we have $x^2 \equiv 207 \equiv 9 \pmod{11}$, so $x \equiv 3$ or $8 \pmod{11}$. Next we have $x^2 \equiv 207 \equiv -1 \pmod{13}$, so $x \equiv 5$ or $8 \pmod{13}$. There are now 8 systems of three congruences each to solve via the Chinese remainder theorem. The solution of $x \equiv 2 \pmod{7}, x \equiv 3 \pmod{11}, x \equiv 5 \pmod{13}$ is $x \equiv 135 \pmod{1001}$. The solution of $x \equiv 2 \pmod{7}, x \equiv 3 \pmod{11}, x \equiv 8 \pmod{13}$ is $x \equiv 905 \pmod{1001}$. The solution of $x \equiv 2 \pmod{7}, x \equiv 8 \pmod{11}, x \equiv 5 \pmod{13}$ is $x \equiv 954 \pmod{1001}$. The solution of $x \equiv 2 \pmod{7}, x \equiv 8 \pmod{11}, x \equiv 8 \pmod{13}$ is $x \equiv 723 \pmod{1001}$. The solution of $x \equiv 5 \pmod{7}, x \equiv 3 \pmod{11}, x \equiv 5 \pmod{13}$ is $x \equiv 278 \pmod{1001}$. The solution of $x \equiv 5 \pmod{7}, x \equiv 3 \pmod{11}, x \equiv 8 \pmod{13}$ is $x \equiv 47 \pmod{1001}$. The solution of $x \equiv 5 \pmod{7}, x \equiv 8 \pmod{11}, x \equiv 5 \pmod{13}$ is $x \equiv 96 \pmod{1001}$. The solution of $x \equiv 5 \pmod{7}, x \equiv 8 \pmod{11}, x \equiv 8 \pmod{13}$ is $x \equiv 866 \pmod{1001}$. In order, the solutions modulo 1001 are 47, 96, 135, 278, 723, 866, 905, and 954.

11.1.22. If x_0 is a solution to $x^2 \equiv a \pmod{p^e}$, then $(-x_0)^2 \equiv a \pmod{p^e}$ and $x_0 \not\equiv -x_0 \pmod{p^e}$ since $p^e \nmid 2x_0$. So if there is one solution, there must be two. Now suppose x_0 and x_1 are solutions. Then $x_0^2 \equiv x_1^2 \pmod{p^e}$ so $p^e \mid (x_0^2 - x_1^2) = (x_0 - x_1)$. If $p \mid x_0 - x_1$ and $p \mid x_0 + x_1$. Then $p \mid (x_0 - x_1) + (x_0 + x_1) = 2x_0$ which is impossible since $p \nmid a \equiv x_0^2$ so $p^e \mid x_0 - x_1$ or $p^e \mid x_0 + x_1$ and hence $x_0 \equiv \pm x_1 \pmod{p^e}$. So there are at most 2 solutions.

11.1.23. If $x_0^2 \equiv a \pmod{p^{e+1}}$ then $x_0^2 \equiv a \pmod{p^e}$. Conversely, if $x_0^2 \equiv a \pmod{p^e}$ then $x_0^2 = a + bp^e$ for some integer b. We can solve the linear congruence $2x_0y \equiv -b \pmod{p}$, say $y = y_0$. Let $x_1 = x_0 + y_0p^e$. Then $x_1^2 \equiv x_0^2 + 2x_0y_0p^e = a + p^e(b + 2x_0y_0) \equiv a \pmod{p^{e+1}}$ since $p \mid 2x_0y_0 + b$. This is the induction step in

showing that $x^2 \equiv a \pmod{p^e}$ has solutions if and only if $(\frac{a}{p}) = 1$.

11.1.24. Finding a solution to $x^2 \equiv a \pmod{n}$ is equivalent to finding a solution to the system

$$\begin{aligned} x^2 &\equiv a \pmod{p_1^{t_1}} \\ x^2 &\equiv a \pmod{p_2^{t_2}} \\ &\vdots \\ x^2 &\equiv a \pmod{p_m^{t_m}} \end{aligned}$$

So we count the solutions to the system. $x^2 \equiv a \pmod{p_i^{t_i}}$ has two solutions if $\left(\frac{a}{p_i}\right) = 1$ and no solutions if $\left(\frac{a}{p_i}\right) = -1$. So if any $\left(\frac{a}{p_i}\right) = -1$, there are no solutions to $x^2 \equiv a \pmod{n}$. Otherwise there are 2^m solutions.

11.1.25. a. $75 = 5^2 3$ and $\left(\frac{31}{5}\right) = \left(\frac{1}{5}\right) = 1$ and $\left(\frac{31}{3}\right) = \left(\frac{1}{3}\right) = 1$, so there are $2^2 = 4$ solutions.

b. $105 = 3 \cdot 5 \cdot 7$ and 16 is a quadratic residue of 3, 5, and 7, so there are $2^3 = 8$ solutions.

c. $231 = 3 \cdot 7 \cdot 11$ and $\left(\frac{46}{3}\right) = \left(\frac{1}{3}\right) = 1, \left(\frac{46}{7}\right) = \left(\frac{4}{7}\right) = 1, \left(\frac{46}{11}\right) = \left(\frac{2}{11}\right) = 1$ so there are $2^3 = 8$ solutions.

d. $\left(\frac{1156}{3}\right) = \left(\frac{1}{3}\right) = 1, \left(\frac{1156}{5}\right) = \left(\frac{1}{5}\right) = 1, \left(\frac{1156}{7}\right) = \left(\frac{1}{7}\right) = 1$, and $\left(\frac{1156}{11}\right) = \left(\frac{1}{11}\right) = 1$ so there are $2^4 = 16$ solutions.

11.1.26. If $x^2 \equiv \pmod{2^e}$ has a solution x_0, then $-x_0$ is also a solution, and $(2^{e-1} \pm x_0)^2 \equiv (2^e \pm 2 \cdot 2^{e-1} x_0 + x_0^2) \equiv x_0^2 \pmod{2^e}$. If $x_0 \equiv -x_0 \pmod{2^e}$, then $2^e \mid 2x_0$, but x_0 is odd, so $x_0 \nmid -x_0$. If $x_0 \equiv 2^{e-1} + x_0 \pmod{2^e}$ then $2^e \mid 2^{e-1}$, so $x_0 \not\equiv 2^{e-1} + x_0$. If $x_0 \equiv 2^{e-1} - x_0 \pmod{2^e}$ then $2^e \mid 2^{e-1} + 2x_0$, so $2e - 1 \mid 2x_0$ which is impossible, so $x \not\equiv 2^{e-1} - x_0$. Since any of the four solutions could have been x_0, we have shown there are four distinct solutions. Suppose x_1 is a fifth solution. Then $x_0^2 - x_1^2 \equiv 0 \pmod{2^e}$. Then $2^e \mid (x_0^2 - x_1^2) = (x_0 - x_1)(x_0 + x_1)$. If $4 \mid (x_0 - x_1)$ and $(x_0 + x_1)$ then $4 \mid (x_0 - x_1) + (x_0 - x_1) = 2x_0$, but x_0 is odd, so $\gcd((x_0 - x_1), (x_0 + x_1)) = 2$ or 1. So either $2^e \mid x_0 - x_1$ and so $x_1 \equiv x_0 \pmod{2^e}$, or $2^e \mid x_0 + x_1$ and so $x_1 \equiv -x_0 \pmod{2^e}$, or $2^{e-1} \mid x_0 - x_1$ and so $x_1 \equiv 2^{e-1} + x_0 \pmod{2^e}$ or $2^{e-1} \mid x_0 + x_1$ and $x_1 \equiv 2^{e-1} - x_0 \pmod{2^e}$, so x_1 is one of the four solutions.

11.1.27. Suppose $p_1, p_2, \ldots, p_n$ are the only primes of the form $4k + 1$. Let $N = 4(p_1 p_2 \cdots p_n)^2 + 1$. Let q be an odd prime factor of N. Then $q \neq p_i, i = 1, 2, \ldots, n$, but $N \equiv 0 \pmod{q}$, so $4(p_1 p_2 \cdots p_n)^2 \equiv -1 \pmod{q}$ and therefore $(\frac{-1}{q}) = 1$, so $q \equiv 1 \pmod{4}$ by Theorem 11.5.

11.1.28. a. Let $N = (p_1, p_2, \ldots, p_n)^2 = 2$. Then $N \equiv 3 \pmod{8}$ since$(p_1, p_2, \ldots, p_n)^2$ is an odd square. The product of integers of the form $8k + 1$ is another integer of the same form. Therefore, N has an odd prime divisor q not of the form $8k + 1$. Then $-2 \equiv (p_1, p_2 \ldots, p_n)^2 \pmod{q}$ and so $(\frac{-2}{q}) = 1$. By Exercise 7, $q \equiv 1$ or $3 \pmod{8}$ and we have excluded $q \equiv 1$, so q is of the form $8k + 3$. If $q = p_i$ for some i, then $q \mid N$ and $q \mid (p_1, p_2, \ldots p_n)^2$ so $q \mid 2$, a contradiction. Therefore q is a new prime of the form $8k + 3$.

b. Let $N = (p_1, p_2, \ldots p_n)^2 + 4$. Then $N \equiv 5 \pmod{8}$. As in part (a), there must be an odd prime divisor q not of the form $8k + 1$, and $q \neq p_i$ for any i. Then we have $(p_1, p_2 \ldots, p_n)^2 \equiv -4 \pmod{q}$, and since 4 is a quadratic residue, -1 must be. Therefore $q \equiv 1 \pmod{4}$, but $q \not\equiv 1 \pmod{8}$ so $q \equiv 5 \pmod{8}$ as desired.

c. Let $N = (4p_1, p_2, \ldots, p_n)^2 - 2$. Then $\frac{N}{2} \equiv 7 \pmod{8}$, and must have an odd divisor q not of the form $8k + 1$. We have $2 \equiv (4p_1, p_2 \ldots, p_n)^2 \pmod{q}$, so $(\frac{2}{q}) = 1$ and $q \equiv \pm 1 \pmod{8}$. Therefore $q \equiv -1 \equiv 7 \pmod{8}$ as desired.

11.1.29. Let b_1, b_2, b_3, and b_4 be the four modular square roots of a modulo pq. Then each b_i is a solution to exactly one of the four systems of congruences given in the text. For convenience, let the subscripts

correspond to the lower case Roman numerals of the systems. Suppose two of the b_i's were quadratic residues modulo pq. Without loss of generality, say $b_1 \equiv y_1^2 \pmod{pq}$ and $b_2 \equiv y_2^2 \pmod{pq}$. Then from systems (i) and (ii), we have that $y_1^2 \equiv b_1 \equiv x_2 \pmod{q}$ and $y_2^2 \equiv b_2 \equiv -x_2 \pmod{q}$. Therefore both x_2 and $-x_2$ are quadratic residues modulo q, but this is impossible since $q \equiv 3 \pmod{4}$. The other cases are identical.

11.1.30. Let r be a primitive root modulo p and $k = \text{ind}_r a$. If $\left(\frac{a}{p}\right) = 1$, then $k = 2m$ for some integer m. Then $\text{ind}_r a^{(p-1)/2} \equiv ((p-1)/2)2m \equiv 0 \pmod{p-1}$. Since only 1 has index 0, we must have $a^{(p-1)/2} \equiv 1 \pmod{p}$. If $\left(\frac{a}{p}\right) = -1$, then $k = 2m+1$ for some integer m. Then $\text{ind}_r a^{(p-1)/2} \equiv ((p-1)/2)(2m+1) \equiv (p-1)/2 \pmod{p-1}$. Since only -1 has index $(p-1)/2$, we must have $a^{(p-1)/2} \equiv -1 \pmod{p}$.

11.1.31. Let r be a primitive root for p and let $a \equiv r^s \pmod{p}$ and $b \equiv r^t \pmod{p}$ with $1 \le s, t \le p-1$. If $a \equiv b \pmod{p}$, then $s = t$ and so s and t have the same parity. By Theorem 11.2, we have part (i). Further, we have $ab \equiv r^{s+t} \pmod{p}$. Then the right hand side of (ii) is 1 exactly when s and t have the same parity, which is exactly when the left hand side is 1. This proves part (ii). Finally, since $a^2 \equiv r^{2s} \pmod{p}$ and $2s$ is even, we must have that a^2 is a quadratic residue modulo p, proving part (iii).

11.1.32. By Exercise 31 we know that every primitive root of p is a quadratic nonresidue of p. Since there are $\phi(p-1)$ primitive roots of p and $(p-1)/2 - \phi(p-1)$ quadratic nonresidues of p that are not primitive roots of p.

11.1.33. If r is a primitive root of q, then the set of all primitive roots is given by $\{r^k : (k, \phi(q)) = (k, 2p) = 1\}$. So the $p-1$ numbers $\{r^k : k \text{ is odd and } k \ne p, 1 \le k < 2p\}$ are all the primitive roots of q. On the other hand, q has $(q-1)/2 = p$ quadratic residues, which are given by $\{r^2, r^4, \ldots, r^{2p}\}$. This set has no intersection with the first one.

11.1.34. Let r be a primitive root of q and let $r^k = a$. Since a is a nonresidue, k is odd. Since $4 \ne \text{ord}_q a = 4p/(k, 4p)$, we have $(k, \phi(q)) = (k, 4p) = 1$ and hence a is a primitive root.

11.1.35. First suppose $p = 2^{2^n} + 1$ is a Fermat prime and let r be a primitive root for p. Then $\phi(p) = 2^{2^n}$. Then an integer a is a nonresidue if and only if $a = r^k$ with k odd. But then $(k, \phi(p)) = 1$, so a is also a primitive root. Conversely, suppose that p is an odd prime and every quadratic nonresidue of p is also a primitive root of p. Let r be a particular primitive root of p. Then, r^k is a quadratic nonresidue and hence a primitive root for p if and only if k is odd. But this implies that every odd number is relatively prime to $\phi(p)$, so $\phi(p)$ must be a power of 2. Thus $p = 2^b + 1$ for some b. If b had a nontrivial odd divisor, then we could factor p as a difference of b powers, contradicting the primality of of p. Therefore b is a power of 2 and so p is a Fermat prime.

11.1.36. Note that $2^{2^n} \equiv -1 \pmod{p}$ and $2^{2^{n+1}} \equiv 1 \pmod{p}$ so $\text{ord}_p 2 = 2^{n+1}$. Now $2^{2^n} + 1 \equiv 1 \pmod{8}$, so $2^{(p-1)/2} \equiv 1 \pmod{p}$ by Theorem 11.6. Therefore $2^{n+1} \mid (p-1)/2$, say $k2^{n+1} = (p-1)/2$. Then $p = 2^{n+2}k + 1$.

11.1.37. a. We have $q = 2p+1 = 2(4k+3)+1 = 8k+7$, so $(\frac{2}{q}) = 1$ by Theorem 11.6. Then by Euler's criterion, $2^{(q-1)/2} \equiv 2^p \equiv 1 \pmod{q}$. Therefore $q \mid 2^p - 1$.

b. $11 = 4(2)+3$ and $23 = 2(11)+1$, so $23 \mid 2^{11} - 1 = M_{11}$, by part (a); $23 = 4(5)+3$ and $47 = 2(23)+1$, so $47 \mid M_{23}$; $251 = 4(62)+3$ and $503 = 2(251)+1$, so $503 \mid M_{251}$.

11.1.38. If $n \equiv 0 \pmod{4}$ then $2n+1 \equiv 1 \pmod{8}$. If $n \equiv 3 \pmod{4}$ then $2n+1 \equiv 7 \pmod{8}$. In either case, $\left(\frac{2}{2n+1}\right) = 1$ by Theorem 11.6. Therefore $2^{(2n+1-1)/2} \equiv 2^n \equiv 1 \pmod{2n+1}$, and hence $2n+1 \mid 2^n - 1 = M_n$. If $n \equiv 1 \pmod{4}$ then $2n+1 \equiv 3 \pmod{8}$. If $n \equiv 2 \pmod{4}$ then $2n+1 \equiv 5 \pmod{8}$. In either case, $\left(\frac{2}{2n+1}\right) = -1$ by Theorem 11.6. Therefore $2^{2n+1-1}2 \equiv 2^n \equiv -1 \pmod{2n+1}$, and hence $2n+1 \mid 2^n + 1 = M_n + 2$.

11.1.39. Let $q = 2k + 1$. Since q does not divide $2^p + 1$, we must have, by Exercise 38, that $k \equiv 0$ or $3 \pmod 4$. That is, $k \equiv 0, 3, 4$ or $7 \pmod 8$. Then $q \equiv 2(0, 3, 4 \text{ or } 7) + 1 \equiv \pm 1 \pmod 8$.

11.1.40. By Theorem 7.12, every prime divisor of $M_{17} = 2^{17} - 1$ must be of the form $2 \cdot 17k + 1$, where k is a positive integer. Further, by Exercise 39, every prime divisor must be of the form $8l \pm 1$, where l is a positive integer. Therefore $2 \cdot 17k + 1 \equiv \pm 1 \pmod 8$. Whence, $k \equiv 0$ or $3 \pmod 4$. Therefore we would need only check prime divisors of the forms $2 \cdot 4 \cdot 17m + 1 = 136m + 1$ and $2(4m + 3)17 + 1 = 136m + 103$.

11.1.41. Note that $\left(\frac{j(j+1)}{p}\right) = \left(\frac{j \cdot j(1+\bar{j})}{p}\right) = \left(\frac{j^2(1+\bar{j})}{p}\right) = \left(\frac{(1+\bar{j})}{p}\right)$ since j^2 is a perfect square. Then, $\sum_{j=1}^{p-2}\left(\frac{j(j+1)}{p}\right) = \sum_{j=1}^{p-2}\left(\frac{\bar{j}+1}{p}\right) = \sum_{j=2}^{p-1}\left(\frac{j}{p}\right) = \sum_{j=1}^{p-1}\left(\frac{j}{p}\right) - 1 = -1$. Here we have used the method in the solution to Exercise 10 to evaluate the last sum, and the fact that as j runs through the values 1 through $p - 2$, so does $\bar{j}$.

11.1.42. a. If we add the number of pairs of consecutive quadratic residues among the integers 1,2,. . . , $p - 1$ and the number of pairs of a quadratic residue followed by a nonresidue among these integers, we obtain the number of quadratic residues other than perhaps $p - 1$ among the integers 1,2,. . . , $p - 1$. When $p \equiv 1 \pmod 4 - 1$ is a quadratic residue of p and the number of quadratic residues of p among the first $p - 1$ positive integers is $(p - 1)/2 - 1 = (p - 3)/2 = (p - 2 - (-1)^{((p-1))/2})/2$. When $p \equiv -1 \pmod 4, -1$ is a quadratic nonresidue of p and the number of quadratic residues of p among the first $p - 1$ positive integers is $(p - 1)/2 = (p - 2 - (-1)^{(p-1)/2}))/2$. Hence **(RR)+(RN)**$= (p - 2 - (-1)^{((p-1))/2})/2$. Similarly, if we add the number of pairs of consecutive quadratic nonresidues and the number of pairs of a quadratic nonresidue followed by a quadratic residue, we obtain the number of quadratic nonresidues other than perhaps $p - 1$ among the integers 1,2,. . . , $p - 1$. An analysis similar to that given above shows that **(NR)+(NN)** $= (p - 2 + (-1)^{(p-1)/2})/2$. If we add the number of pairs of consecutive quadratic residues and the number of pairs of a quadratic nonresidue followed by a quadratic residue is the number of quadratic residues other than 1. Since there are $(p - 1)/2 - 1$ quadratic residues other than 1, we have **(RR)+(NR)** $= (p - 1)/2 - 1$. If we add the number of pairs of a quadratic residue followed by a quadratic nonresidue and the number of pairs of quadratic nonresidues, we obtain the number of pairs of quadratic nonresidues among the integers 2,. . . , $p - 1$.

b. Note that $\left(\frac{j(j+1)}{p}\right) = 1$ if and only if $\left(\frac{j}{p}\right) = \left(\frac{j+1}{p}\right) = 1$ or if $\left(\frac{j}{p}\right) = \left(\frac{j+1}{p}\right) = -1$. Hence $\sum_{j=1}^{p-2}\left(\frac{j(j+1)}{p}\right) = (\mathbf{RR}) + (\mathbf{NN}) - (\mathbf{RN}) - (\mathbf{NR}) = -1$, since this sum is -1 from Exercise 35.

c. The system of 5 equations in 4 unknown determines the solution **(RR)** $= \frac{1}{4}\left(p - 4 - (-1)^{(p-1)/2}\right)$, **(RN)** $= \frac{1}{4}\left(p - (-1)^{(p-1)/2}\right)$, **(NR) = (NN)**$= \frac{1}{4}\left(p - 2 + (-1)^{(p-1)/2}\right)$.

11.1.43. Let r be a primitive root of p. Then $x^2 \equiv a \pmod p$ has a solution if and only if $2 \operatorname{ind}_r x \equiv \operatorname{ind}_r a \pmod{p-1}$ has a solution in $\operatorname{ind}_r x$. Since $p - 1$ is even, the last congruence is solvable if and only if $\operatorname{ind}_r a$ is even, which happens when $a = r^2, r^4, \ldots, r^{p-1}$,i.e. $(p - 1)/2$ times.

11.1.44. If p is of the form $4k + 1$, then $q = 4(4k + 1) + 1 = 8n + 5$. If p is of the form $4k + 3$, then $q = 4(4k + 3) + 1 = 8n + 5$. So by Theorem 11.6, 2 is a quadratic nonresidue of q. Therefore $2^{(q-1)/2} \equiv 2^{2p} \equiv -1 \pmod q$, and so $2^2 \not\equiv 1, 2^p \not\equiv 1$ and $2^{2p} \not\equiv 1 \pmod q$. Hence $\operatorname{ord}_q 2 = 4p$.

11.1.45. $q = 2(4k + 1) + 1 = 8k + 3$, so 2 is a quadratic nonresidue of q. By Exercise 33, 2 is a primitive root.

11.1.46. Since $q = 2(4k - 1) + 1 = 8k - 1$, -2 is a quadratic nonresidue of q, by Exercise 7. By Exercise 33, -2 is a primitive root.

11.1.47. Check that $q \equiv 3 \pmod 4$, so -1 is a quadratic nonresidue of q. Since $4 = 2^2$, we have $\left(\frac{-4}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{2^2}{q}\right) = (-1)(1) = -1$. Therefore -4 is a nonresidue of q. By Exercise 33, -4 is a primitive root.

11.1.48. Note that $482 \equiv -49 \pmod{59}$, and $\left(\frac{-49}{59}\right) = \left(\frac{-1}{59}\right)\left(\frac{7^2}{59}\right) = \left(\frac{-1}{59}\right) = -1$ since $59 \equiv 3 \pmod 4$, so there are no solutions.

11.1.49. a. By adding $(\overline{2}b)^2$ to both sides of the congruence $C \equiv P(P+b) \pmod n$, we have $C + a \equiv P^2 + Pb + (\overline{2}b)^2 \equiv (P + \overline{2}b)^2 \pmod n$.

b. There are 4 solutions to $x^2 \equiv C + a \pmod{pq}$. From each, subtract $\overline{2}b$, which gives the 4 messages.

c. First we solve $2x \equiv 1 \pmod{2773}$ to get $\overline{2} = 1387$. Then $\overline{2}b \equiv 1338 \pmod{2773}$, and $(\overline{2}b)^2 \equiv 2082 \pmod{2773}$. For the first block of ciphertext, we have $1819 \equiv P(P+3) \pmod{2773}$. we add 2082 to both sides to get $1128 \equiv (P + 1388)^2 \pmod{2773}$. We solve $x^2 \equiv 1128 \pmod{2773}$ to find the two solutions 1692 and 1081. Subtracting 1388 from both of these and reducing gives us the two possible values for P, 0304 and 2466. Since 66 is not the numerical equivalent of a letter, we know the solution must be 0304 which is equivalent to DE. Similarly we find that 0459 has $P = 0856, 1796, 1914$, or 0974. Of these, only 1914 has letter equivalents, so the second digraph is TO. Finally, we find that 0803 has $P = 2346, 2017, 0424$, or 0753. Two of these have letter equivalents: 0424 is equivalent to EY, which makes the message DETOEY; 2017 is equivalent to UR, which makes the message DETOUR. We guess that the message is DETOUR.

11.1.50. Suppose that P is a quadratic residue of p. Then there is an integer x such that $x^2 \equiv P \pmod p$. Hence $C \equiv P^e \equiv (x^2)^e = (x^e)^2 \pmod p$. It follows that C is also a quadratic residue of p. Now suppose that C is a quadratic residue of p. Then there is an integer y such that $y^2 \equiv C \pmod p$. Then $P \equiv C^d \equiv (y^2)^d \pmod p$, where d is an inverse of e modulo $p-1$. Hence P is also a quadratic residue of p.

11.1.51. a. By noting this, the second player can tell which cards dealt are quadratic residues, since the ciphertext will also be quadratic residues modulo p.

b. All ciphers will be quadratic residues modulo p.

11.1.52. We complete the square in j. Since $(b, m) = 1$, and inverse $\overline{b}$ of b exists modulo m. Then $4h_j(K) \equiv 4h(K) + 4aj + bj^2 \pmod m$. Then $4h_j(K) - 4h(K) + \overline{b}a^2 \equiv b(2j - \overline{b}a)^2 \pmod m$ or $(2j - \overline{b}a)^2 \equiv 4\overline{b}h_j(K) - 4\overline{b}h(K) + \overline{b}^2 \pmod m$. Since there are only $m/2$ quadratic residues modulo m the right-hand side can take on only $m/2$ values. For each of these values, the congruence is linear in $h_j(K)$, and so has only one solution for $h_j(K)$, giving only $m/2$ locations searched.

11.1.53. The quadratic residues modulo 11 are $1, 3, 4, 5$, and 9. So there are several chains: $1 + 3 = 4$, $4 + 5 = 9$, and $9 + 5 = 14 \equiv 3 \pmod{11}$, for example.

11.1.54. The quadratic residues modulo 7 are 1, 2, and 4. Since the sum of no two of these is congruent to the other modulo 7, there is no chain of quadratic residues modulo 7.

11.2. The Law of Quadratic Reciprocity

11.2.1. a. Since $53 \equiv 1 \pmod 4$ the law of quadratic reciprocity shows that $\left(\frac{3}{53}\right) = \left(\frac{53}{3}\right)$. We have $\left(\frac{53}{3}\right) = \left(\frac{2}{3}\right) = -1$. Hence $\left(\frac{3}{53}\right) = -1$.

b. Since $79 \equiv 1 \pmod 4$ the law of quadratic reciprocity shows that $\left(\frac{7}{79}\right) = \left(\frac{79}{7}\right)$. We have $\left(\frac{79}{7}\right) = \left(\frac{2}{7}\right) = 1$ since 2 is a quadratic residue of 7.

c. We have $\left(\frac{15}{101}\right) = \left(\frac{3}{101}\right)\left(\frac{5}{101}\right)$. Since $101 \equiv 1 \pmod 4$, the law of quadratic reciprocity shows that $\left(\frac{3}{101}\right) = \left(\frac{101}{3}\right)$ and $\left(\frac{5}{101}\right) = \left(\frac{101}{5}\right)$. We have $\left(\frac{101}{3}\right) = \left(\frac{2}{3}\right) = -1$ and $\left(\frac{101}{5}\right) = \left(\frac{1}{5}\right) = 1$. Hence

$\left(\frac{15}{101}\right) = -1 \cdot 1 = 1$.

d. $\left(\frac{31}{641}\right) = \left(\frac{641}{31}\right) = \left(\frac{21}{31}\right) = \left(\frac{3}{31}\right)\left(\frac{7}{31}\right) = \left(-\left(\frac{31}{3}\right)\right)\left(-\left(\frac{31}{7}\right)\right) = \left(\frac{1}{3}\right)\left(\frac{3}{7}\right) = 1\left(-\left(\frac{7}{3}\right)\right) = -\left(\frac{2}{3}\right) = 1$.

e. $\left(\frac{111}{991}\right) = \left(\frac{3}{991}\right)\left(\frac{37}{991}\right) = -\left(\frac{991}{3}\right)\left(\frac{991}{37}\right) = -\left(\frac{1}{3}\right)\left(\frac{29}{37}\right) = -\left(\frac{29}{37}\right) = -\left(\frac{37}{29}\right) = -\left(\frac{8}{29}\right) = -\left(\frac{2}{29}\right)\left(\frac{4}{29}\right) = -\left(\frac{2}{29}\right) = -(-1) = 1$.

f. $\left(\frac{105}{1009}\right) = \left(\frac{3}{1009}\right)\left(\frac{5}{1009}\right)\left(\frac{7}{1009}\right) = \left(\frac{1009}{3}\right)\left(\frac{1009}{5}\right)\left(\frac{1009}{7}\right) = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{7}\right) = 1$.

11.2.2. First suppose that p is a prime with $p \equiv 1 \pmod 4$. Then by the law of quadratic reciprocity it follows that $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right)$. We see that if $p \equiv 1 \pmod 3$, so that $p \equiv 1 \pmod{12}$, then $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = 1$. If $p \equiv 2 \pmod 3$, so that $p \equiv 5 \pmod{12}$, then $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = -1$. Next suppose that p is a prime with $p \equiv 3 \pmod 4$. Then by the law of quadratic reciprocity it follows that $\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right)$. We see that if $p \equiv 1 \pmod 3$ so that $p \equiv 7 \equiv -5 \pmod{12}$ then $\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right) = -1$. If $p \equiv 2 \pmod 3$, so that $p \equiv 11 \equiv -1 \pmod{12}$, then $\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right) = 1$.

11.2.3. If $p \equiv 1 \pmod 6$ there are 2 cases: If $p \equiv 1 \pmod 4$ then $\left(\frac{-1}{p}\right) = 1$ and $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{1}{3}\right) = 1$. So $\left(\frac{-3}{p}\right) = 1$. If $p \equiv 3 \pmod 4$ then $\left(\frac{-1}{p}\right) = -1$ and $\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right)$, so $\left(\frac{-3}{p}\right) = (-1)(-1) = 1$. If $p \equiv -1 \pmod 6$ and $p \equiv 1 \pmod 4$, then $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = 1 \cdot \left(\frac{p}{3}\right) = \left(\frac{-1}{3}\right) = -1$. If $p \equiv 3 \pmod 4$, then $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)\left(-\left(\frac{p}{3}\right)\right) = \left(\frac{p}{3}\right) = \left(\frac{-1}{3}\right) = -1$.

11.2.4. By the law of quadratic reciprocity it follows that $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$. We find that $\left(\frac{1}{5}\right) = \left(\frac{4}{5}\right) = 1$ and $\left(\frac{2}{5}\right) = \left(\frac{3}{5}\right) = -1$. It follows that 5 is a quadratic residue of the odd prime p if and only if $p \equiv 1 \pmod 5$ or $p \equiv 4 \pmod 5$.

11.2.5. Suppose that p is an odd prime. By the law of quadratic reciprocity it follows that $\left(\frac{7}{p}\right) = \left(\frac{p}{7}\right)$ if $p \equiv 1 \pmod 4$ and $\left(\frac{7}{p}\right) = -\left(\frac{p}{7}\right)$ if $p \equiv 3 \pmod 4$. So, 7 is a quadratic residue of a prime p with $p \equiv 1 \pmod 4$ if $\left(\frac{p}{7}\right) = 1$. This is the case when $p \equiv 1, 2$ or $4 \pmod 7$. Using the Chinese remainder theorem we see that 7 is a quadratic residue of p when $p \equiv 1, 9$ or $25 \pmod{28}$, and 7 is a quadratic nonresidue of p when $p \equiv 5, 13$ or $17 \pmod{28}$. Also, 7 is a quadratic residue of a prime p with $p \equiv 3 \pmod 4$ if $\left(\frac{p}{7}\right) = -1$. This is the case when $p \equiv 3, 5$ or $6 \pmod 7$. Using the Chinese remainder theorem we see that 7 is a quadratic residue of p when $p \equiv 3, 19$ or $27 \pmod{28}$ and 7 is a quadratic nonresidue of p when $p \equiv 11, 15$ or $23 \pmod{28}$. It follows that 7 is a quadratic residue of p if and only if $p \equiv 1, 3, 9, 19, 25$ or $27 \pmod{28}$.

11.2.6. If every prime divisor of $Q = 5(n!) - 1$ were of the form $5k + 1$, then $Q \equiv 1 \pmod 5$, which it isn't, so Q has a prime divisor p not of the form $5k + 1$. Also, if $p \leq n$, then $p \mid 5(n!)^2$ and $p \mid 5(n!)^2 - Q = 1$, a contradiction, so $p > n$. Now $5(n!)^2 \equiv 1 \pmod p$, so $1 = \left(\frac{1}{p}\right) = \left(\frac{5(n!)^2}{p}\right) = \left(\frac{5}{p}\right)\left(\frac{(n!)^2}{p}\right) = \left(\frac{5}{p}\right)\left(\frac{p}{5}\right)$. Therefore $p \equiv 4$ or $1 \pmod 5$ and we have excluded the latter case.

11.2.7. a. We have $F_1 = 2^{2^1} + 1 = 5$. We find that $3^{(F_1-1)/2} = 3^{(5-1)/2} = 3^2 = 9 \equiv -1 \pmod{F_1}$. Hence by Pepin's test we come (to the already obvious) conclusion that $F_1 = 5$ is prime.

b. We have $F_3 = 2^{2^3} + 1 = 257$. We find that $3^{(F_3-1)/2} = 3^{(257-1)/2} = 3^{128} \equiv (3^8)^{16} \equiv 136^{16} \equiv (136^4)^4 \equiv 64^4 \equiv (64^2)^2 \equiv 241^2 \equiv 256 \equiv -1 \pmod{257}$. Hence by Pepin's test we see that $F_3 = 257$ is prime.

c. Using a calculator we find $3^{255} \equiv 94 \pmod{F_4}$. $3^{32768} \equiv 3^{255 \cdot 128} 3^{128} \equiv 94^{128} 3^{128} \equiv -1 \pmod{F_4}$.

11.2.8. If $F_m = 2^{2^m} + 1$ is prime, then $\phi(F_m) = 2^{2^m}$, so $\text{ord}_{F_m} 3$ is a power of 2, say 2^k. Then $3^{2^n} \equiv 3^{2^k 2^{n-k}} \equiv (1)^{2^{n-k}} \equiv 1 \pmod{F_m}$ if $n \geq k$. But by Pepin's test, $3^{(f_m-1)/2} \equiv -1 \pmod{F_m}$ and $(F_m - 1)/2 = 2^{2^m-1}$. So $\text{ord}_{F_m} 3 = 2^{2^m} = \phi(F_m)$.

11.2.9. a. The lattice points in the rectangle are the points (i, j) where $0 < i < p/2$ and $0 < j < q/2$. There are the lattice points (i, j) with $i = 1, 2, \ldots, (p-1)/2$ and $j = 1, 2, \ldots, (q-1)/2$. Consequently, there are $(p-1)/2 \cdot (q-1)/2$ such lattice points.

b. The points on the diagonal connecting **O** and **C** are the points (x, y) where $y = (q/p)x$. Suppose that x and y are integers with $y = (q/p)x$. Then $py = qx$. Since $(p, q) = 1$ it follows that $p \mid x$ which is impossible if $0 < x < p/2$. Hence there are no lattice points on this diagonal.

c. The number of lattice points in the triangle with vertices **O,A,** and **C** is the number of lattice points (i, j) with $i = 1, 2, \ldots, (p-1)/2$ and $1 \leq j \leq iq/p$. For a fixed value of i in the indicated range, there are $[iq/p]$ lattice points (i, j) in the triangle. Hence the total number of lattice points in the triangle is $\sum_{i=1}^{(p-1)/2} [iq/p]$.

d. The number of lattice points in the triangle with vertices **O,B,** and **C** is the number of lattice points (i, j) with $j = 1, 2, \ldots, (q-1)/2$ and $1 \leq i < jp/q$. For a fixed value of j in the indicated range, there are $[jp/q]$ lattice points (i, j) in the triangle. Hence the total number of lattice points in the triangle is $\sum_{j=1}^{(q-1)/2} [jp/q]$.

e. Since there are $(p-1)/2 \cdot (q-1)/2$ lattice points in the rectangle, and no points on the diagonal **OC,** the sum of the numbers of lattice points in the triangles **OBC** and **OAC** is $(p-1)/2 \cdot (q-1)/2$. By parts (b) and (c) it follows that $\sum_{j=1}^{(p-1)/2} [jq/p] + \sum_{j=1}^{(q-1)/2} [jp/q] = (p-1)/2 \cdot (q-1)/2$. By Lemma 11.3 it follows that $\left(\frac{p}{q}\right) = (-1)^{T(p,q)}$ and $\left(\frac{q}{p}\right) = (-1)^{T(q,p)}$ where $T(p, q) = \sum_{j=1}^{(p-1)/2} [jp/q]$ and $T(q, p) = \sum_{j=1}^{(q-1)/2} [jq/p]$. We conclude that $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)/2} \cdot (q-1)/2$. This is the law of quadratic reciprocity.

11.2.10. Without loss of generality, assume $p > q$ are odd primes. First assume that $p \equiv q \pmod 4$. Then $p = q + 4a$ for some positive integer a, and $p \equiv q \pmod{4a}$. Then $\left(\frac{p}{q}\right) = \left(\frac{q+4a}{q}\right) = \left(\frac{4a}{q}\right) = \left(\frac{a}{q}\right)$. Likewise, $\left(\frac{q}{p}\right) = \left(\frac{p-4a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{4}{p}\right)\left(\frac{a}{p}\right) = (-1)^{(p-1)/2}\left(\frac{a}{p}\right)$. From these two equations we have $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \left(\frac{a}{q}\right)(-1)^{(p-1)/2}\left(\frac{a}{p}\right)$. By Theorem 11.8, we have $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$, so $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)/2}$. But since $(p-1)/2 = (q-1)/2 = 2a$, we know that $(p-1)/2$ and $(q-1)/2$ have the same parity. Therefore $(-1)^{(p-1)/2} = (-1)^{((p-1)/2)((q-1)/2)}$. This proves the first case. Now assume $p \not\equiv q \pmod 4$. Then we must have $p \equiv -q \pmod 4$, so that $p = -q + 4a$ for some positive integer a, and so $p \equiv -q \pmod{4a}$. Then $\left(\frac{p}{q}\right) = \left(\frac{-q+4a}{q}\right) = \left(\frac{a}{q}\right)$, and $\left(\frac{q}{p}\right) = \left(\frac{-p+4a}{p}\right) = \left(\frac{a}{p}\right)$. By Theorem 11.8, $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$, so $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \left(\frac{a}{q}\right)\left(\frac{a}{p}\right) = 1$. But $(p-1)/2 + (q-1)/2 = 2a - 1$, so $(p-1)/2$, and $(q-1)/2$ have opposite parity, so at least one of them is even. Noting that then $(-1)^{((p-1)/2)((q-1)/2)} = 1$ completes the second case.

11.2.11. First suppose $a = 2$. Then we have $p \equiv \pm q \pmod 8$ and so $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$ by Theorem 11.6 Now suppose a is an odd prime. If $p \equiv q \pmod{4a}$, then $p \equiv q \pmod a$ and so $\left(\frac{q}{a}\right) = \left(\frac{p}{a}\right)$. And since $p \equiv q \pmod 4$, $(p-1)/2 \equiv (q-1)/2 \pmod 2$. Then by Theorem 11.7, $\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right)(-1)^{(p-1)/2 \cdot (a-1)/2} = \left(\frac{q}{a}\right)(-1)^{(q-1)/2 \cdot (a-1)/2} = \left(\frac{a}{q}\right)$. But if $p \equiv -q \pmod{4a}$, then $p \equiv -q \pmod a$ and so $\left(\frac{-q}{a}\right) = \left(\frac{p}{a}\right)$. And since $p \equiv -q \pmod 4$, $(p-1)/2 \equiv (q-1)/2 + 1 \pmod 2$. Then by Theorem 11.7, $\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right)(-1)^{(p-1)/2 \cdot (a-1)/2} = \left(\frac{-q}{a}\right)(-1)^{((q-1)/2+1) \cdot (a-1)/2} = \left(\frac{-1}{a}\right)(-1)^{(a-1)/2}\left(\frac{a}{q}\right) = \left(\frac{a}{q}\right)$. The general case follows from the multiplicativity of the Legendre symbol.

11.2.12. To apply Gauss's lemma to compute $\left(\frac{a}{p}\right)$ we need to find the parity of the number of $a, 2a, \ldots, ((p-1)/2)a$ which have least positive residue between $p/2$ and p. If ka is such a number, with $1 \leq k \leq (p-1)/2$, then $(2t-1)p/2 \leq ka \leq tp$ for some integer t. Since $1 \leq k \leq (p-1)/2$, the range for t is $1, 2, \ldots, [a/2]$. Let $u = [a/2] = a/2$ if a is even and $(a-1)/2$ if a is odd. For $t = 1, 2, \ldots u$, divide each inequality by a to get $(2t-1)p/a \leq k \leq tp/a$. We must find the parity of the number of integers k satisfying these last conditions. Suppose $p = 4am + r$, with $0 < r < 4a$. Then the conditions become $2(2t-1)m + (2t-1)r/(2a) \leq k \leq 4mt + tr/a$, for $t = 1, 2, \ldots u$. Since we are only concerned with the parity of the number of k, we can drop the even integers in each of these inequalities, (thereby reducing the count of k by even numbers and preserving parity.) Thus, the conditions reduce to $(2t-1)r/(2a) \leq k \leq tr/a$, which depend only on r. Therefore, if $p \equiv q \equiv r \pmod{4a}$, then $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$. Now if $p \equiv -q \equiv r \pmod{4a}$, then $q \equiv 4a - r \pmod{4a}$. Then substituting $4a - r$ in for r in the conditions we get $(2t-1)(4a-r)/2a \leq k \leq t(4a-r)/a$ which reduces to $(2t-1)2 - (2t-1)r/(2a) \leq 4t - tr/a$. Again we may drop the even integers in each inequality. Multiplying through by -1 doesn't change the number of k, but it makes the conditions identical to the conditions for p above. Therefore, $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$.

11.2.13. a. Recall that $e^{xi} = 1$ if and only if x is a multiple of 2π. First we compute $(e^{(2\pi i/n)k})^n = e^{(2\pi i/n)nk} = (e^{(2\pi i)})^k = 1^k = 1$, so $e^{(2\pi i/n)k}$ is an nth root of unity. Now if $(k, n) = 1$, then $((2\pi i/n)k)a$ is a multiple of $2\pi i$ if and only if $n|a$. Therefore $a = n$ is the least positive integer for which $(e^{(2\pi i/n)k})^a = 1$. Therefore $e^{(2\pi i/n)k}$ is a primitive nth root of unity. Conversely, suppose $(k, n) = d > 1$. Then $(e^{(2\pi i/n)k})^{(n/d)} = e^{(2\pi i)k/d} = 1$, since k/d is an integer, and so in this case $e^{(2\pi i/n)k}$ is not a primitive nth root of unity.

b. Let $m = l + kn$ where k is an integer. Then $\zeta^m = \zeta^{l+kn} = \zeta^l\zeta^{kn} = \zeta^l$. Now suppose ζ is a primitive nth root of unity and that $\zeta^m = \zeta^l$, and without loss of generality, assume $m \geq l$. From the first part of this exercise, we may take $0 \leq l \leq m < n$. Then $0 = \zeta^m - \zeta^l = \zeta^l(\zeta^{m-l} - 1)$. Hence, $\zeta^{m-l} = 1$. Since n is the least positive integer such that $\zeta^n = 1$, we must have $m - l = 0$.

c. First, $f(z+1) = e^{2\pi i(z+1)} - e^{-2\pi i(z+1)} = e^{2\pi iz}e^{2\pi i} - e^{-2\pi iz}e^{-2\pi i} = e^{2\pi iz}1 - e^{-2\pi iz}1 = f(z)$. Next, $f(-z) = e^{-2\pi iz} - e^{2\pi iz} = -(e^{2\pi iz} - e^{-2\pi iz}) = -f(z)$. Finally, suppose $f(z) = 0$. Then $0 = e^{2\pi iz} - e^{-2\pi iz} = e^{-2\pi iz}(e^{4\pi iz} - 1)$, so $e^{4\pi iz} = 1$. Therefore $4\pi iz = 2\pi in$ for some integer n, and so $z = n/2$.

d. Fix y and consider $g(x) = x^n - y^n$ and $h(x) = (x-y)(\zeta x - \zeta^{-1}y)\cdots(\zeta^{n-1}x - \zeta^{-(n-1)}y)$ as polynomials in x. Both polynomials have degree n. The leading coefficient in $h(x)$ is $\zeta^{1+2+\cdots+n-1} = \zeta^{n(n-1)/2} = (\zeta^n)^{(n-1)/2} = 1$, since $n-1$ is even. So both polynomials are monic. Further, note that $g(\zeta^{-2k}y) = (\zeta^{-2k}y)^n - y^n = y^n - y^n = 0$ for $k = 0, 1, 2, \ldots, n-1$. Also $h(\zeta^{-2k}y)$ has $(\zeta^k\zeta^{-2k}y - \zeta^{-k}y) = (\zeta^{-k}y - \zeta^{-k}y) = 0$ as one of its factors. So g and h are monic polynomials sharing these n distinct zeros (since $-2k$ runs through a complete set of residues modulo n). By the Fundamental Theorem of Algebra, g and h are identical.

e. Let $x = e^{2\pi iz}$ and $y = e^{-2\pi iz}$ in the identity from part (d). Then the right hand side becomes $\prod_{k=0}^{n-1}\left(\zeta^k e^{2\pi iz} - \zeta^{-k}e^{-2\pi iz}\right) = \prod_{k=0}^{n-1}\left(e^{2\pi i(z+k/n)} - e^{-2\pi i(z+k/n)}\right) = \prod_{k=0}^{n-1} f\left(z + \frac{k}{n}\right) =$ $f(z)\prod_{k=1}^{(n-1)/2} f\left(z+\frac{k}{n}\right)\prod_{k=(n+1)/2}^{n-1} f\left(z+\frac{k}{n}\right)$. From the identities in part (c), this last product becomes $\prod_{k=(n+1)/2}^{n-1} f\left(z+\frac{k}{n}\right) = \prod_{k=1}^{(n-1)/2} f\left(z+\frac{n-k}{n}\right) = \prod_{k=1}^{(n-1)/2} f\left(z+1-\frac{k}{n}\right) = \prod_{k=1}^{(n-1)/2} f\left(z-\frac{k}{n}\right)$. So the product above is equal to $f(z)\prod_{k=1}^{(n-1)/2} f\left(z+\frac{k}{n}\right)\prod_{k=1}^{(n-1)/2} f\left(z-\frac{k}{n}\right) =$ $f(z)\prod_{k=1}^{(n-1)/2} f\left(z+\frac{k}{n}\right)f\left(z-\frac{k}{n}\right)$. Then noting that the left side of the identity in part (d) is $(e^{2\pi iz})^n -$

$(e^{-2\pi iz})^n = e^{2\pi inz} - e^{-2\pi inz} = f(nz)$ finishes the proof.

f. For $l = 1, 2, \ldots, (p-1)/2$, let k_l be the least positive residue of la modulo p. Then $\prod_{l=1}^{(p-1)/2} f\left(\frac{la}{p}\right) = \prod_{l=1}^{(p-1)/2} f\left(\frac{k_l}{p}\right)$ by the perodicity of f established in part (c). We break this product into two pieces $\prod_{k_l<p/2} f\left(\frac{k_l}{p}\right) \prod_{k_l>p/2} f\left(\frac{k_l}{p}\right) = \prod_{k_l<p/2} f\left(\frac{k_l}{p}\right) \prod_{k_l>p/2} -f\left(\frac{-k_l}{p}\right) = \prod_{k_l<p/2} f\left(\frac{k_l}{p}\right) \prod_{k_l>p/2} -f\left(\frac{p-k_l}{p}\right) = \prod_{l=1}^{(p-1)/2} f\left(\frac{l}{p}\right)(-1)^N$, where N is the number of k_l exceeding $p/2$. But by Gauss' lemma, $(-1)^N = \left(\frac{a}{p}\right)$. This establishes the identity.

g. Let $z = l/p$ and $n = q$ in the identities in parts (e) and (f). Then we have $\left(\frac{q}{p}\right) = \prod_{l=1}^{(p-1)/2} f\left(\frac{lq}{p}\right) \Big/ f\left(\frac{l}{p}\right) = \prod_{l=1}^{(p-1)/2} \prod_{k=1}^{(q-1)/2} f\left(\frac{l}{p}\right) + \frac{k}{q} f\left(\frac{l}{p}\right) - \frac{k}{q} = \prod_{l=1}^{(p-1)/2} \prod_{k=1}^{(q-1)/2} f\left(\frac{k}{q}\right) + \frac{l}{p} f\left(\frac{k}{q}\right) - \frac{l}{p}(-1)^{(p-1)/2 \cdot (q-1)/2}$, where we have used the fact that $f(-z) = -f(z)$ and the fact that there are exactly $(p-1)/2 \cdot (q-1)/2$ factors in the double product. But, by symmetry, this is exactly the expression for $\left(\frac{q}{p}\right)(-1)^{(p-1)/2 \cdot (q-1)/2}$ which completes the proof.

11.2.14. Assume that $n = k2^m + 1$ with $k < 2^m$ and $m \geq 2$. Then $n \equiv 1 \pmod 4$. If n is prime, then by the Law of Quadratic Reciprocity, we have $\left(\frac{p}{n}\right) = \left(\frac{n}{p}\right) = -1$. Then by Euler's Criterion, $p^{(n-1)/2} \equiv \left(\frac{p}{n}\right) \equiv -1 \pmod n$ as desired. Conversely, suppose $p^{(n-1)/2} \equiv -1 \pmod n$. Then n satisfies all the hypotheses of Proth's Primality Test, Theorem 9.20, and hence n is prime.

11.2.15. Since $p \equiv 1 \pmod 4$, we have $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$. And since $p \equiv 1 \pmod q$ for all primes $q \leq 23$, then $\left(\frac{p}{q}\right) = \left(\frac{1}{q}\right) = 1.$. Then if a is an integer with $0 < a < 29$ and prime factorization $a = p_1 p_2 \cdots p_k$, then each $p_i < 29$ and $\left(\frac{a}{p}\right) = \left(\frac{p_1}{p}\right) \cdots \left(\frac{p_k}{p}\right) = 1^k = 1$. So there are no quadratic nonresidues modulo p less than 29. Further, since a quadratic residue must be an even power of any primitive root r, then r^1 can not be less than 29.

11.2.16. a. Since $p \equiv 1 \pmod 8$, we have by Theorems 11.5 and 11.6 that $\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = 1$. Since $p \equiv 4 \pmod p$, by the Law of Quadratic Reciprocity and Theorem 11.4, we have, for each $i = 2, 3, \ldots, n$ that $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right) = \left(\frac{1+8q_1q_2\cdots q_n}{q_i}\right) = \left(\frac{1}{q_i}\right) = 1$.

b. Let t be an arbitrary integer and let k be an integer such that $-M \leq t + kp \leq M$. Since $q_1, q_2, \ldots, q_n$ are all the primes not exceeding M, $t + kp$ has a prime power factorization of the form $(-1)^{e_0} q_1^{e_1} q_2^{e_2} \ldots q_n^{e_n}$, where $0 \leq e_i$. Then by the complete multiplicativity of the Legendre symbol and by part (a), we have $\left(\frac{t+kp}{p}\right) = \left(\frac{-1}{p}\right)^{e_0} \left(\frac{q_1}{p}\right)^{e_1} \cdots \left(\frac{q_n}{p}\right)^{e_n} = 1$. Therefore $t + kp$ is a quadratic residue modulo p and hence not a primitive root modulo p. Now let r_p be the least primitive root modulo p. If $0 \leq r_p \leq M$. By the above, $r_p + 0p = r_p$ is a quadratic residue modulo p, a contradiction. If $p - M \leq r_p \leq p$, then $-M \leq r_p - p \leq 0$, and from the above, $r_p - p$ is a quadratic residue modulo p. But $r_p \equiv r_p - p \pmod p$, so r_p is also a quadratic residue modulo p, again a contradiction. Therefore, we must have $M < r_p < p - M$.

11.2.17. a. If $a \in T$ then $a = qk$ for some $k = 1, 2, \ldots (p-1)/2$. So $1 \leq a \leq q(p-1)/2 < (pq-1)/2$. Further, since $k \leq (p-1)/2$, and p is prime, we have $(p, k) = 1$. Since $(q, p) = 1$, then $(a, p) = (qk, p) = 1$, so $a \in S$, and hence $T \subset S$. Now suppose $a \in S - T$. Then $1 \leq a \leq (pq-1)/2$ and $(a, p) = 1$, and

since $a \notin T$, then $a \neq qk$ for any k. Thus $(a,q) = 1$, so $(a, pq) = 1$, and so $a \in R$. Thus $S - T \subset R$. Conversely, if $a \in R$, then $1 \leq a \leq (pq-1)/2$ and $(a, pa) = 1$, so certainly $(a,q) = 1$, and so a is not a multiple of q and hence $a \notin T$. Hence $a \in S - T$. Thus $R \subset S - T$. Therefore $R = S - T$.

b. Since by part (a), $R = S - T$ we have $\prod_{a \in S} a = \prod_{a \in R} a \prod_{a \in T} a = A(q \cdot 2q \cdots ((p-1)/2)a) = Aq^{(p-1)/2}((p-1)/2)! \equiv A\left(\frac{q}{p}\right)((p-1)/2)! \pmod{p}$ by Euler's criterion. Note that $(pq-1)/2 = p(q-1)/2 + (p-1)/2$, so that we can evaluate $\prod_{a \in S} a \equiv ((p-1)!)^{(q-1)/2}((p-1)/2)! \equiv (-1)^{(q-1)/2}((p-1)/2)! \pmod{p}$ by Wilson's Theorem. When we set these two expressions congruent to each other modulo p and cancel we get $A \equiv (-1)^{(q-1)/2}\left(\frac{q}{p}\right)$ as desired.

c. Since the roles of p and q are identical in the hypotheses and in parts (a) and (b), the result follows by symmetry.

d. Assume that $(-1)^{(q-1)/2}\left(\frac{q}{p}\right) = (-1)^{(p-1)/2}\left(\frac{p}{q}\right)$. Then $A = \pm 1$ so certainly $A \equiv \pm 1 \pmod{pq}$. Conversely, suppose $A \equiv 1 \pmod{pq}$. Then $A \equiv 1 \pmod{p}$ and $A \equiv 1 \pmod{q}$. Then by parts (b) and (c) we have $(-1)^{(q-1)/2}\left(\frac{q}{p}\right) = A = (-1)^{(p-1)/2}\left(\frac{p}{q}\right)$. The same argument works if $A \equiv -1 \pmod{pq}$.

e. If a is an integer in R, it is in the range $1 \leq a \leq (pq-1)/2$ and therefore its additive inverse modulo pq is in the range $(pq+1)/2 \leq -a \leq pq - 1$ in the set of reduced residue classes. By the Chinese Remainder Theorem, the congruence $a^2 \equiv 1 \pmod{pq}$ has exactly 4 solutions, $1, -1, b$, and $-b \pmod{pq}$ and the congruence $a^2 \equiv -1 \pmod{pq}$ has solutions if and only $p \equiv q \equiv 1 \pmod{4}$, and in this case it has exactly 4 solutions $i, -i, ib$, and $-ib \pmod{pq}$. Now for each element $a \in R$, $(a, pq) = 1$, so a has a multiplicative inverse v. By the remark above, exactly one of $v, -v$ is in R. We let U be the set of those elements which are their own inverse or their own negative inverse, that is let $U = \{a \in R | a^2 \equiv \pm 1 \pmod{pq}\}$. Then when we compute A, all other elements will be paired with another element which is either its inverse or the negative of its inverse. Thus we have $A \prod_{a \in R} a \equiv \pm \prod_{a \in U} a \pmod{pq}$. So if $p \equiv q \equiv 1 \pmod{pq}$, then $A \equiv \pm \prod_{a \in U} a \equiv \pm(1 \cdot c \cdot i \cdot ic) \equiv c^2 i^2 \equiv \mp 1 \pmod{pq}$. Conversely, in the other case, $A \equiv \prod_{a \in U} a \equiv \pm(1 \cdot c) \not\equiv \pm 1 \pmod{pq}$, which completes the proof.

f. By parts (d) and (e) we have that $(-1)^{(q-1)/2}\left(\frac{q}{p}\right) = (-1)^{(p-1)/2}\left(\frac{p}{q}\right)$ if and only if $p \equiv q \equiv 1 \pmod{4}$. So if $p \equiv q \equiv 1 \pmod{4}$ we have $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$. But if $p \equiv 1 \pmod{4}$ while $q \equiv 3 \pmod{4}$ then we must have $-\left(\frac{q}{p}\right) \neq \left(\frac{p}{q}\right)$ which means we must change the sign and have $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$. The case where $p \equiv 3 \pmod{4}$ but $q \equiv 1 \pmod{4}$ is identical. If $p \equiv q \equiv 3 \pmod{4}$, then we must have $-\left(\frac{q}{p}\right) \neq -\left(\frac{p}{q}\right)$ so that we must have $-\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$, which concludes the proof.

11.3. The Jacobi Symbol

11.3.1. a. By the reciprocity law for Jacobi symbols, since $5 \equiv 1 \pmod{4}$ we have $\left(\frac{5}{21}\right) = \left(\frac{21}{5}\right) = \left(\frac{1}{5}\right) = 1$.

b. We have $\left(\frac{27}{101}\right) = \left(\frac{3}{101}\right)^3 = \left(\frac{101}{3}\right)^3 = \left(\frac{2}{3}\right)^3 = (-1)^3 = -1$, where we have used the law of quadratic reciprocity to replace $\left(\frac{101}{3}\right)$ by $\left(\frac{3}{101}\right)$ since $101 \equiv 1 \pmod{4}$.

c. Since $1001 \equiv 1 \pmod 4$, by the reciprocity law of Jacobi symbols we have $\left(\frac{11}{1001}\right) = \left(\frac{1001}{111}\right) = \left(\frac{2}{111}\right) = 1$ since $111 \equiv 7 \pmod 4$.

d. Since $1009 \equiv 1 \pmod 4$ by the reciprocity law for Jacobi symbols we have $\left(\frac{1009}{2307}\right) = \left(\frac{2307}{1009}\right) = \left(\frac{289}{1009}\right) = \left(\frac{17}{1009}\right)^2 = 1$.

e. Since $2663 \equiv 3299 \equiv 3 \pmod 4$ by the reciprocity law for Jacobi symbols we have $\left(\frac{2663}{3299}\right) = \left(\frac{3299}{2663}\right) = \left(\frac{636}{2663}\right) = \left(\frac{4}{2663}\right)\left(\frac{159}{2663}\right) = leg1592663$ since $3229 \equiv 636 \pmod{2663}$. Since $159 \equiv 2663 \equiv 3 \pmod 4$, by the reciprocity law for Jacobi symbols we have $\left(\frac{159}{2663}\right) = -\left(\frac{2663}{159}\right) = -\left(\frac{85}{159}\right)$ since $2663 \equiv 85 \pmod{159}$. Since $85 \equiv 1 \pmod 4$, the reciprocity law for Jacobi symbols shows that $\left(\frac{85}{159}\right) \equiv \left(\frac{159}{85}\right) = \left(\frac{-11}{85}\right) = \left(\frac{-1}{85}\right)\left(\frac{11}{85}\right) = \left(\frac{11}{85}\right)$, since $85 \equiv 1 \pmod 4$. By the reciprocity law for Jacobi symbols we have $\left(\frac{11}{85}\right) = \left(\frac{85}{11}\right) = \left(\frac{8}{11}\right) = \left(\frac{2}{11}\right)^3 = (-1)^3 = -1$ since $11 \equiv 3 \pmod 8$. It follows that $\left(\frac{2663}{3299}\right) = -1$.

f. Since $10001 = 1 \pmod 4$ the reciprocity law for Jacobi symbols shows that $\left(\frac{10001}{20003}\right) = \left(\frac{20003}{10001}\right) = \left(\frac{1}{10001}\right) = 1$, where we have used the periodicity of the Jacobi symbol and the congruence $20003 \equiv 1 \pmod{10001}$.

11.3.2. By the reciprocity law for Jacobi symbols it follows that if $(15, n) = 1$ then $\left(\frac{15}{n}\right) = \left(\frac{n}{15}\right)$ if $n \equiv 1 \pmod 4$ and $\left(\frac{15}{n}\right) = -\left(\frac{n}{15}\right)$ if $n \equiv 3 \pmod 4$. Note that $\left(\frac{n}{15}\right) = \left(\frac{n}{3}\right)\left(\frac{n}{5}\right)$. We have $\left(\frac{n}{3}\right) = 1$ when $n \equiv 1 \pmod 3$ and $\left(\frac{n}{3}\right) = -1$ when $n \equiv 2 \pmod 3$. We have $\left(\frac{n}{5}\right) = 1$ when $n \equiv 1$ or $4 \pmod 5$ and $\left(\frac{n}{5}\right) = -1$ when $n \equiv 2$ or $3 \pmod 5$. It follows that $\left(\frac{n}{15}\right) = 1$ when $n \equiv 1, 2, 4$, or $8 \pmod{15}$ and $\left(\frac{n}{15}\right) = -1$ when $n \equiv 7, 1, , 13$, or $14 \pmod{15}$. It follows by the Chinese remainder theorem that $\left(\frac{15}{n}\right) = 1$ if and only if $n \equiv 1, 7, 11, 17, 43, 49, 53$, or $59 \pmod{60}$.

11.3.3. We have $\left(\frac{30}{n}\right) = \left(\frac{2}{n}\right)\left(\frac{15}{n}\right)$. By Theorem 11.10(iv) $\left(\frac{2}{n}\right) = 1$ when $n \equiv \pm 1 \pmod 8$. From Exercise 2, $\left(\frac{15}{n}\right) = 1$ when $n \equiv 1, 7, 11, 17, 43, 49, 53$, or $59 \pmod{60}$. By the Chinese remainder theorem, $\left(\frac{2}{n}\right) = \left(\frac{15}{n}\right) = 1$, when $n \equiv 1, 7, 17, 49, 61, 67, 77$, or $109 \pmod{120}$ and $\left(\frac{2}{n}\right) = \left(\frac{15}{n}\right) = -1$ when $n \equiv 13, 19, 29, 37, 71, 83, 91, 101, 103, 107, 113$, or $119 \pmod{120}$.

11.3.4. Since $\left(\frac{a}{n}\right) = \left(\frac{a}{p}\right)\left(\frac{a}{q}\right) = 1$, we know that $\left(\frac{a}{p}\right)$ and $\left(\frac{a}{q}\right)$ have the same sign. If they are both equal to 1, then $a \equiv x^2 \pmod p$ and $a \equiv y^2 \pmod q$ for some integers x and y. We can solve the system of congruences $b \equiv x \pmod p, b \equiv y \pmod q$. Then $b^2 \equiv a \pmod p$ and $b^2 \equiv a \pmod q$, so by the Chinese remainder theorem, $b^2 \equiv a \pmod{pq}$, which shows that a is a quadratic residue of $pq = n$. This contradiction shows that $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right) = -1$.

11.3.5. We have $21 = 3 \cdot 7$. The only quadratic nonresidue of 3 is 2. The quadratic nonresidues of 7 are 3, 5, and 6. From Exercise 4, we need to solve each of the systems of congruences $x \equiv a \pmod 3, x \equiv b \pmod 7$ where a is a quadratic nonresidue of 3 and b is a quadratic nonresidue of 7. For $x \equiv 2 \pmod 3, x \equiv 3 \pmod 7$, we have $x = 17$. For $x \equiv 2 \pmod 3, x \equiv 5 \pmod 7$, we have $x = 5$. And for $x \equiv 2 \pmod 3, x \equiv 6 \pmod 7$, we have $x = 20$. So the pseudo-squares modulo 21 are 5, 17 and 20.

11.3.6. We follow the strategy in the solution to Exercise 5. We have $35 = 5 \cdot 7$. The set of quadratic nonresidues modulo 5 is $S = \{2, 3\}$ and the set of quadratic nonresidues modulo 7 is $T = \{3, 5, 6\}$. We form the 6 systems of congruence $x \equiv a \pmod 5, x \equiv b \pmod 7$, where $a \in S$ and $b \in T$, and solve them. We

find that the pseudo-squares modulo 35 are: 3, 12, 13, 17, 27, and 33.

11.3.7. We follow the strategy in the solution to Exercise 5. We have $143 = 11 \cdot 13$. The set of quadratic nonresidues modulo 11 is $S = \{2, 6, 7, 8, 10\}$ and the set of quadratic nonresidues modulo 13 is $T = \{2, 5, 6, 7, 8, 11\}$. We form the 30 systems of congruence $x \equiv a \pmod{11}, x \equiv b \pmod{13}$, where $a \in S$ and $b \in T$, and solve them. We find that the pseudo-squares modulo 143 are: 1, 3, 4, 9, 12, 14, 16, 23, 25, 27, 36, 38, 42, 48, 49, 53, 56, 64, 69, 75, 81, 82, 92, 100, 103, 108, 113, 114, 126, and 133.

11.3.8. Suppose that $(a, b) = 1, b$ is odd and positive, and $a = (-1)^s 2^t q$ where q is odd. It follows that $\left(\frac{a}{b}\right) = \left(\frac{(-1)^s 2^t q}{b}\right) = \left(\frac{-1}{b}\right)^s \left(\frac{2}{b}\right)^t \left(\frac{q}{b}\right) = (-1)^{(b-1)/2}(-1)^{(b^2-1)/8}\left(\frac{q}{b}\right) = (-1)^{(b-1)/2+(b^2-1)/8}\left(\frac{q}{b}\right)$.

11.3.9. Since n is odd and square-free, n has prime factorization $n = p_1 p_2 \cdots p_r$. Let b be one of the $(p-1)/2$ quadratic nonresidues of p_1, so that $\left(\frac{b}{p_1}\right) = -1$. By the Chinese Remainder Theorem, let a be a solution to the system of linear congruences:

$$\begin{aligned} x &\equiv b \pmod{p_1} \\ x &\equiv 1 \pmod{p_2} \\ &\vdots \\ x &\equiv 1 \pmod{p_r} \end{aligned}$$

Then $\left(\frac{a}{p_1}\right) = \left(\frac{b}{p_1}\right) = -1, \left(\frac{1}{p_2}\right) = 1, \ldots \left(\frac{a}{p_r}\right) = \left(\frac{1}{p_r}\right) = 1$. Therefore $\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right)\cdots\left(\frac{a}{p_r}\right) = (-1) \cdot 1 \cdots 1 = -1$.

11.3.10. a. Let a be the integer given in Exercise 9. If $\{k_1, k_2, \ldots, k_{\phi(n)}\}$ is a reduced residue system modulo n, then so is $\{ak_1, ak_2, \ldots, ak_{\phi(n)}\}$ by Theorem 6.13. Then we have

$$\sum_{i=1}^{\phi(n)}\left(\frac{k_i}{n}\right) = \sum_{i=1}^{\phi(n)}\left(\frac{ak_i}{n}\right) = \sum_{i=1}^{\phi(n)}\left(\frac{a}{n}\right)\left(\frac{k_i}{n}\right) = \sum_{i=1}^{\phi(n)} -\left(\frac{k_i}{n}\right) = -\sum_{i=1}^{\phi(n)}\left(\frac{k_i}{n}\right).$$

Therefore $\sum_{i=1}^{\phi(n)}\left(\frac{k_i}{n}\right) = 0$.

b. Let $R = \sum\left(\frac{k}{n}\right)$ where the sum is taken over all K in a reduced residue system with $\left(\frac{k}{n}\right) = 1$. Let $N = \sum\left(\frac{k}{n}\right)$ where the sum is taken over all K in a reduced residue system with $\left(\frac{k}{n}\right) = -1$. By part (a), $R - N = \sum\left(\frac{k}{n}\right) = 0$. Therefore $R = N$.

11.3.11. a. Note that $(a, b) = (b, r_1) = (r_1, r_2) = \cdots = (r_{n-1}, r_n) = 1$ and since the q_i are even, the r_i are odd. Since $r_0 = b$ and $a \equiv \epsilon_1 r_1 \pmod{b}$ we have $\left(\frac{a}{b}\right) = \left(\frac{\epsilon_1 r_1}{r_0}\right) = \left(\frac{\epsilon_1}{r_0}\right)\left(\frac{r_1}{r_0}\right) = \left(\frac{\epsilon_1}{r_0}\right)\left(\frac{r_0}{r_1}\right)(-1)^{(r_0-1)/2\cdot(r_1-1)/2}$ by Theorem 11.11. If $\epsilon_1 = 1$, then $\left(\frac{a}{b}\right) = (-1)^{(r_0-1)/2\cdot(\epsilon_1 r_1-1)/2}\left(\frac{r_0}{r_1}\right)$ If $\epsilon_1 = -1$, then $\left(\frac{\epsilon_1}{r_0}\right) = (-1)^{(r_0-1)/2}$ and we have $\left(\frac{a}{b}\right) = (-1)^{(r_0-1)/2\cdot(r_1+1)/2}\left(\frac{r_0}{r_1}\right) = -1^{(r_0-1)/2\cdot(-r_1-1)/2}\left(\frac{r_0}{r_1}\right) = (-1)^{(r_0-1)/2\cdot(\epsilon_1 r_1-1)/2}\left(\frac{r_0}{r_1}\right)$ since $(r_1+1)/2$ and $(-r_1-1)/2$ have the same parity. Similarly, $\left(\frac{r_0}{r_1}\right) = (-1)^{(r_1-1)/2\cdot(\epsilon_2 r_2-1)/2}\left(\frac{r_1}{r_2}\right)$, so $\left(\frac{a}{b}\right) = (-1)^{(r_0-1)/2\cdot(\epsilon_1 r_1-1)/2+(r_1-1)/2\cdot(\epsilon_2 r_2-1)/2}\left(\frac{r_1}{r_2}\right)$. Proceed inductively until the last step, when $\left(\frac{r_n}{r_{n-1}}\right) = \left(\frac{1}{r_{n-1}}\right) = 1$.

b. If either $r_{i-1} \equiv 1 \pmod 4$ or $\epsilon_i r_i \equiv 1 \pmod 4$, then $(r_{i-1}-1)/2 \cdot (\epsilon_1 r_i - 1)/2$ is even. Otherwise, that is, if $r_{i-1} \equiv \epsilon_i r_i \equiv 3 \pmod 4$, then $(r_{i-1}-1)/2 \cdot (\epsilon_i r_i - 1)/2$ is odd. Then $(r_{n-1}-1)/2 \cdot (\epsilon_n r_n - 1)/2$ the exponent in part (a) is even or odd as T is even or odd.

11.3.12. If $a > 0$ and $b > 0$, then Theorem 11.11 says $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{(a-1)/2(b-1)/2}$. If $a < 0$ and $b > 0$, then $|a| = -a > 0$ and $|b| = b$. Then $\left(\frac{a}{|b|}\right)\left(\frac{b}{|a|}\right) = \left(\frac{-1}{b}\right)\left(\frac{-a}{b}\right)\left(\frac{b}{-a}\right) = (-1)^{(b-1)/2}(-1)^{(-a-1)/2\cdot(b-1)/2}$ by Theorem 11.10(iii) and 11.11. The total exponent on -1 on the right hand side is $(b-1)/2 + (-a-1)/2 \cdot (b-1)/2 = (1 + (-a-1)/2(b-1)/2) = (-a+1)/2 \cdot (b-1)/2 = (-1)^{(a-1)/2\cdot(b-1)/2}$. Similarly if $a > 0$ and $b < 0$. If $a < 0$ and $b < 0$, we apply Theorems 11.10 and 11.11 to get $\left(\frac{a}{|b|}\right)\left(\frac{b}{|a|}\right) =\leq -1-b\left(\frac{-a}{-b}\right)\left(\frac{-1}{-a}\right)\left(\frac{-b}{-a}\right) = (-1)^{(-b-1)/2}(-1)^{(-a-1)/2}\left(\frac{-a}{-b}\right)\left(\frac{-b}{-a}\right) = (-1)^{(-b-1)/2+(-a-1)/2}(-1)^{(-b-1)/2\cdot(-a-1)/2}$. The total exponent of -1 is $(-b-1)/2 + (-a-1)/2 + (-b-1)/2 \cdot (-a-1)/2 = (-a-1)/2\,((-b-1)/2+1) + (-b-1)/2 = (-a-1)/2\,((-b+1)/2 + (-b+1)/2) - 1 = (-b+1)/2\,((-a-1)/2+1) - 1 = (-b+1)/2 \cdot (-a+1)/2 - 1 = (a-1)/2 \cdot (b-1)/2 - 1$. Then $\left(\frac{a}{|b|}\right)\left(\frac{b}{|a|}\right) = (-1)^{(a-1)/2\cdot(b-1)/2-1} = -(-1)^{(a-1)/2\cdot(b-1)/2}$.

11.3.13. a. We have $\left(\frac{5}{12}\right) = \left(\frac{5}{2}\right)^2\left(\frac{5}{3}\right) = (-1)^2 \cdot \left(\frac{2}{3}\right) = 1 \cdot (-1) = -1$.

b. We have $\left(\frac{13}{20}\right) = \left(\frac{13}{2}\right)^2\left(\frac{13}{5}\right) = (-1)^2 \cdot \left(\frac{3}{5}\right) = 1 \cdot (-1) = -1$.

c. We have $\left(\frac{101}{200}\right) = \left(\frac{101}{2}\right)^3\left(\frac{101}{5}\right)^2 = (-1)^3 \cdot 1^2 = -1$.

11.3.14. If $a > 0$, then $\left(\frac{2}{|a|}\right) = \left(\frac{2}{a}\right) = (-1)^{(a^2-1)/8}$. If $a < 0$, then $\left(\frac{2}{|a|}\right) = \left(\frac{2}{-a}\right) = (-1)^{((-a)^2-1)/8} = (-1)^{(a^2-1)/8}$. But a is odd $a \equiv 1 \pmod{}4)$ so $a \equiv 1$ or $5 \pmod 8$. $(-1)^{(a^2-1)/8} = 1$ if $a \equiv 1 \pmod 8$ and $(-1)^{(a^2-1)/8} = -1$ if $a \equiv 5$. Hence $\left(\frac{2}{|a|}\right) = \left(\frac{a}{2}\right)$.

11.3.15. Let $n_1 = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ and $n_2 = q_1^{b_1}q_2^{b_2}\cdots q_s^{b_s}$ be the prime factorizations of n_1 and n_2. Then by the definition of the Kronecker symbol, we have $\left(\frac{a}{n_1n_2}\right) = \left(\frac{a}{p_1}\right)^{a_1}\cdots\left(\frac{a}{p_r}\right)^{a_r}\left(\frac{a}{q_1}\right)^{b_1}\cdots\left(\frac{a}{q_s}\right)^{b_s} = \left(\frac{a}{n_1}\right)\left(\frac{a}{n_2}\right)$.

11.3.16. Write $n = 2^km$, where m is odd. Then if a is odd, we have $\left(\frac{a}{n}\right) = \left(\frac{a}{2^km}\right) = \left(\frac{a}{2}\right)^k\left(\frac{a}{m}\right)$ by the definition. From Exercises 12 and 14 $\left(\frac{a}{n}\right) = \left(\frac{2}{|a|}\right)^k\left(\frac{a}{m}\right) = \left(\frac{2}{|a|}\right)^k\left(\frac{m}{|a|}\right) = \left(\frac{2^km}{|a|}\right) = \left(\frac{n}{|a|}\right)$. If a is even, then n is odd, since $(a,n) = 1$. Write $a = 2^st$ with t odd. Then $\left(\frac{a}{n}\right) = \left(\frac{2^st}{n}\right) = \left(\frac{2}{n}\right)^s\left(\frac{t}{n}\right)$. By Exercise 12, $\left(\frac{a}{n}\right) = \left(\frac{2}{n}\right)^s(-1)^{(n-1)/2\cdot(t-1)/2}\left(\frac{n}{|t|}\right)$, as desired.

11.3.17. If a is odd, then by Exercise 16, we have $\left(\frac{a}{n_1}\right) = \left(\frac{n_1}{|a|}\right)$. By Theorem 11.10($i$), we have $\left(\frac{n_1}{|a|}\right) = \left(\frac{n_2}{|a|}\right) = \left(\frac{a}{n_2}\right)$, using Exercise 16 again. If a is even, say $a = 2^st$ with t odd, Exercise 16 gives $\left(\frac{a}{n_1}\right) = \left(\frac{2}{n_1}\right)^s(-1)^{(t-1)/2\cdot(n_1-1)/2}\left(\frac{n_1}{|t|}\right)$ and $\left(\frac{a}{n_2}\right) = \left(\frac{2}{n_2}\right)^s(-1)^{(t-1)/2\cdot(n_2-1)/2}\left(\frac{n_2}{|t|}\right)$. Since $n_1 \equiv n_2 \pmod{\mid t \mid}$, we have $\left(\frac{n_1}{|t|}\right) = \left(\frac{n_2}{|t|}\right)$, and since $4 \mid a, m_1 \equiv m_2 \pmod 4$ and so $(-1)^{(t-1)/2\cdot(n_1-1)/2} = (-1)^{(t-1)/2\cdot(n_2-1)/2}$. Now $a \equiv 0 \pmod 4$, so $s \geq 2$. If s is 2, then certainly $\left(\frac{2}{n_1}\right)^2 = \left(\frac{2}{n_2}\right)^2$. If $s > 2$, then $8 \mid a$ and $m_1 \equiv m_2 \pmod 8$, so $\left(\frac{2}{n_1}\right) = (-1)^{(n_1^2-1)/8} = (-1)^{(n_2^2-1)/8} = \left(\frac{2}{n_2}\right)$. Therefore $\left(\frac{a}{n_1}\right) = \left(\frac{a}{n_2}\right)$.

11.3.18. If a is odd, then $|a|$ is not a square. Then a has an odd prime divisor p which appears to an odd power $k, a = p^kr$, with r odd. Let m be a nonresidue modulo p. By the Chinese Remainder Theorem, let n be a solution to the system $n \equiv m \pmod p, n \equiv 1 \pmod r$. Then $\left(\frac{a}{n}\right) = \left(\frac{n}{|a|}\right) = \left(\frac{n}{p}\right)^k\left(\frac{n}{r}\right) = \left(\frac{s}{p}\right)^k\left(\frac{1}{r}\right) = (-1)^k = -1$. If a is even, $a = 2^st$ with t odd and $s \geq 2$. If s is odd, then $s \geq 3$. Let n be a solution to $n \equiv 5 \pmod 8, n \equiv 1 \pmod{\mid t \mid}$. Then $\left(\frac{a}{n}\right) = \left(\frac{2}{n}\right)^s(-1)^{(t-1)/2\cdot(n-1)/2}\left(\frac{m}{|t|}\right) = \left(\frac{2}{n}\right)\cdot\mid\cdot\left(\frac{1}{|t|}\right) = -1$. If s is even, then t can't be a square since a isn't. If $t \equiv 3 \pmod 4$, let n be a solution to $n \equiv -1 \pmod 4, n \equiv 1 \pmod{\mid t \mid}$. Then $\left(\frac{a}{n}\right) = (-1)^{(t-1)/2(n-1)/2}\left(\frac{n}{|t|}\right) = -1 \cdot 1 = -1$. If $t \equiv \pmod 4, |t|$ is not a square and $|t| = p^kr$ with k odd for some prime p. Let m be a nonresidue of p and let n be a solution to $n \equiv$

$m \pmod p, n \equiv 1 \pmod r, n \equiv 1 \pmod 2$. Then $\left(\frac{a}{n}\right) = \left(\frac{n}{p^k r}\right) = \left(\frac{n}{p^k}\right)\left(\frac{n}{r}\right) = \left(\frac{m}{p}\right)^k \left(\frac{1}{r}\right) = (-1)^k = -1$.

11.3.19. If $a \equiv 1 \pmod 4$, then $|a| \equiv 1 \pmod 4$ if $a > 0$ and $|a| \equiv -1 \pmod 4$ if $a < 0$, so by Exercise 16 we have $\left(\frac{a}{|a|-1}\right) = \left(\frac{|a|-1}{|a|}\right) = \left(\frac{-1}{|a|}\right) = (-1)^{(|a|-1)/2} = 1$ if $a > 0$ and $= -1$ if $a < 0$. If $a \equiv 0 \pmod 4$, $a = 2^s t$ with t odd and $t \geq 2$, then by Exercise 16 $\left(\frac{a}{|a|-1}\right) = \left(\frac{2}{|a|-1}\right)^s (-1)^{(t-1)/2}\left(\frac{|a|-1}{|t|}\right)$. Since $s \geq 2$, check that $\left(\frac{2}{|a|-1}\right)^s = 1, (\mid a \mid -1 \equiv 7 \pmod 8$ if $s > 2)$. Also $(-1)^{(t-1)/2}\left(\frac{|a|-1}{|t|}\right) = (-1)^{(t-1)/2}\left(\frac{-1}{|t|}\right) = (-1)^{(t-1)/2+(|t|-1)/2} = 1$ if $t > 0$ and $= -1$ if $t < 0$.

11.3.20. The essential operation at each step is division, so this computation is equivalent to the division algorithm in complexity. Therefore, by Lame's theorem, the Jacobi symbol $\left(\frac{a}{b}\right)$ can be evaluated in $O((\log_2 b)^2)$ bit operations.

11.4. Euler Pseudoprimes

11.4.1. We find that $2^{(561-1)/2} = 2^{280} = \left(2^{10}\right)^{28} \equiv (-98)^{28} \equiv \left(-98^2\right)^{14} \equiv 67^{14} \equiv \left(67^2\right)^7 \equiv 1^7 = 1 \pmod{561}$. Furthermore, we see that $\left(\frac{2}{561}\right) = 1$ since $561 \equiv 1 \pmod 8$.

11.4.2. Note that $2^{45} \equiv 1 \pmod{15841}$, so $2^{(15841-1)/2} \equiv 2^{7920} \equiv 2^{45 \cdot 176} \equiv 1 \pmod{15841}$. Since $15841 \equiv 1 \pmod 8$ we have $\left(\frac{2}{15841}\right) = 1$, so 15841 is an Euler pseudoprime. Now $15841 - 1 = 2^4 \cdot 990$ and $2^{990} \equiv 2^{45 \cdot 22} \equiv 1^{22} \equiv 1 \pmod{15841}$, so 15841 passes Miller's test for the base 2. Therefore 15841 is a strong pseudoprime to the base 2. Next, $15841 = 7 \cdot 31 \cdot 73$, then $7 - 1 = 6 \mid 15840, 31 - 1 = 30 \mid 15840$ and $73 - 1 = 72 \mid 15840$, so by Theorem 6.7, 15841 is a Carmichael number.

11.4.3. Suppose that n is an Euler pseudoprime to both the bases a and b. Then $a^{(n-1)/2} \equiv \left(\frac{a}{n}\right)$ and $b^{(n-1)/2} \equiv \left(\frac{b}{n}\right)$. It follows that $(ab)^{(n-1)/2} \equiv \left(\frac{a}{n}\right)\left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right)$. Hence n is an Euler pseudoprime to the base ab.

11.4.4. Since n is an Euler pseudoprime to the base b, we have $b^{(n-1)/2} \equiv \left(\frac{b}{n}\right) \pmod n$. Then $(n-b)^{(n-1)/2} \equiv (-b)^{(n-1)/2} \equiv (-1)^{(n-1)/2}(b)^{(n-1)/2} \equiv \left(\frac{-1}{n}\right)\left(\frac{b}{n}\right) \equiv \left(\frac{-b}{n}\right) \equiv \left(\frac{n-b}{n}\right) \pmod n$. So n is an Euler pseudoprime to the base $n - b$.

11.4.5. Suppose that $n \equiv 5 \pmod 8$ and n is an Euler pseudoprime to the base 2. Since $n \equiv 5 \pmod 8$ we have $\left(\frac{2}{n}\right) = -1$. Since n is an Euler pseudoprime to the base 2, we have $2^{(n-1)/2} \equiv \left(\frac{2}{n}\right) = -1 \pmod n$. Write $n - 1 = 2^2 t$ where t is odd. Since $2^{((n-1))/2} \equiv 2^{2t} \equiv -1 \pmod n$, n is a strong pseudoprime to the base 2.

11.4.6. Write $n = 12k+5$. Then $n-1 = 12k+4 = 2^2(3k+1) = 2^2 t$ with t odd. Since n is an Euler pseudoprime to the base 3, we have $3^{(n-1)/2} \equiv 3^{6k+2} \equiv 3^{2t} \equiv \left(\frac{3}{n}\right) \pmod n$. If $\left(\frac{3}{n}\right) = -1$, then n passes Miller's test for the base 3 since $3^{2t} \equiv -1 \pmod n$. If $\left(\frac{3}{n}\right) = 1$, then $3^{2t} \equiv 1 \pmod n$ and so $3^t \equiv \pm 1 \pmod n$, and again n passes Miller's test for the base 3. Therefore n is a strong pseudoprime to the base 3.

11.4.7. Let n be an Euler pseudoprime to the base 5 such that $n \equiv 5 \pmod{20}$. Then $n = 20k + 5$ and $n - 1 = 20k + 4 = 2^2(5k + 1) = 2^2 t$, and $5^{(n-1)/2} \equiv 5^{2(5k+1)} \equiv 5^{2t} \equiv \left(\frac{5}{n}\right) \pmod n$. If $\left(\frac{5}{n}\right) = -1$, then n satisfies Miller's test to the base 5. If $\left(\frac{5}{n}\right) = 1$, then $5^{2t} \equiv 1 \pmod n$ and so $5^t \equiv -1$ and n satisfies Miller's test to the base 5. Therefore n is a strong pseudoprime to the base 5.

11.4.8. We sketch a proof. First note that the number of solutions to $x^{(n-1)/2} \equiv 1 \pmod{p_j^{a_j}}$ is the same as the number of solutions to $((n-1)/2)\text{ind}_r x \equiv 0 \pmod{\phi(p_j^{a_j})}$, which is $((n-1)/2, p_j^{a_j-1}(p_j - 1)) =$

$((n-1)/2, p_j - 1)$, since $p_j \nmid (n-1)$. It suffices, then, to consider only the case where all the $a_j = 1$. By the Chinese remainder theorem, there are $N = \prod_{j=1}^{m}((n-1)/2, p_j - 1)$ solutions to $x^{(n-1)/2} \equiv 1 \pmod{n}$. Now suppose $b^{(n-1)/2} \equiv 1 \pmod{n}$. Prove the following two facts: If $k_j \geq k$, then $\left(\frac{b}{p_j}\right) = 1$ for all solutions b, and if $k_j < k$, then $\left(\frac{b}{p_j}\right) = 1$ for exactly $1/2$ of the solutions b and $\left(\frac{b}{p_j}\right) = -1$ for the other $1/2$. (Use an argument similar to that in the proof of Theorem 11.17.) This shows that if $k_1 = k$, then every solution b of $x^{(n-1)/2} \equiv 1 \pmod{(n-1)/2}$ is a base for which n is an Euler pseudoprime, and if $k_j < k$, then exactly $N/2$ of the solutions b are bases for which n is an Euler pseudoprimes. Next, we count the number of solutions b to $x^{(n-1)/2} \equiv -1 \pmod{n}$ for which $\left(\frac{b}{n}\right) = -1$. Prove that if $k_1 = k$ then there are N such b, and that if $k_1 \neq k$ then there are no such b. (Do the cases $k_1 < k$ and $k_1 > k$ separately.) Putting all these cases together yields the result.

11.4.9. Using Exercise 8, we compute $561 = 3 \cdot 11 \cdot 17$. Then $561 = 1 + 2^4 35$, $3 = 1 + 2$, $11 = 1 + 2 \cdot 5$, and $17 = 1 + 2^4$, so $k = 4, k_1 = 1, k_2 = 1$, and $k_3 = 4$. Since $a_1 = 1$ is odd and $k_1 = 1 < k = 4$, we see that $\delta_n = 1/2$. Then the number we seek is $(1/2)((561-1)/2, 3-1)((561-1)/2, 11-1)((561-1)/2, 17-1) = (1/2)(280, 2)(280, 10)(280, 16) = (1/2)2 \cdot 10 \cdot 16 = 80$. So there are 80 different values for b.

11.4.10. Using Exercise 8, we compute $1729 = 7 \cdot 13 \cdot 19$, and we have $7 = 1+2 \cdot 3, 19 = 1+2 \cdot 9$, and $13 = 1+2^2 3$. Therefore, $k_1 = 1, k_2 = 1, k_3 = 2, q_1 = 3, q_2 = 9$, and $q_3 = 3$. Also $n = 1729 = 1 + 2^6 27$, so $k = 6$ and $q = 27$. We compute $\delta_n = 1/2$ since $k_i < k$ and all a_i's are odd. Then the number of integers b for which 1729 is an Euler pseudoprime to the base b is given by $(1/2)(869, 6)(869, 12)(869, 18) = (1/2) \cdot 6 \cdot 12 \cdot 18 = 648$.

11.5. Zero-Knowledge Proofs

11.5.1. We check that both 47 and 67 are congruent to 3 modulo 4. If p is a prime congruent to 3 modulo 4, then $(\pm x^4)^{(p+1)/4} \equiv x^{p+1} \equiv x^2 \pmod{p}$, by Fermat's little theorem. In this case, we have $x^2 \equiv \pm 2070^{(47+1)/4} \equiv \pm 2^{12} \equiv 7 \pmod{47}$, and $x^2 \equiv \pm 2070^{(67+1)/4} \equiv \pm 7^{17} \equiv \pm 23 \pmod{67}$. Next, since x^2 is a quadratic residue modulo 3149, it must be a quadratic residue modulo each of the factors of 3149. We compute $\left(\frac{7}{47}\right) = 1$, $\left(\frac{-7}{47}\right) = -1$, $\left(\frac{23}{67}\right) = 1$, and $\left(\frac{-23}{67}\right) = -1$. Therefore we solve the system $x^2 \equiv 7 \pmod{47}, x^2 \equiv 23 \pmod{67}$, to find $x^2 = 1229$.

11.5.2. Note that $103 \cdot 107 = 11021$, and we take $x^4 \equiv 1686 \pmod{11021}$. As in Exercise 1, we compute $x^2 \equiv \pm 1686^{(103+1)/4} \equiv \pm 1686^{26} \equiv \pm 55 \pmod{103}$. and $x^2 \equiv \pm 1686^{(107+1)/4} \equiv \pm 1686^{27} \equiv \pm 9 \pmod{107}$. Then we compute $\left(\frac{55}{103}\right) = 1$, $\left(\frac{-55}{103}\right) = -1$, $\left(\frac{9}{107}\right) = 1$, and $\left(\frac{-9}{107}\right) = -1$. So we solve the system $x^2 \equiv 55 \pmod{103}, x^2 \equiv 9 \pmod{107}$ to find $x^2 = 6750$.

11.5.3. Since $p, q \equiv 3 \pmod{4}$, -1 is not a quadratic residue modulo p or q. If the four square roots are found using the method in Example 9.19, then only one of each possibility for choosing $+$ or $-$ can yield a quadratic residue in each congruence, so there is only one system which results in a square.

11.5.4. Paula sends $x \equiv 1101^2 \equiv 303 \pmod{1961}$. since $1863^2 \equiv 1760 \pmod{1961}$, we have $v = 1760$, so she sends $y \equiv v\overline{x} \equiv 1760 \cdot 1385 \equiv 77 \pmod{1961}$. Vince checks that $xy \equiv 1760 \pmod{1961}$ and sends 1 as his random bit. Paula then sends $s \equiv u\overline{r} \equiv 1863 \cdot 1188 \equiv 1236 \pmod{1961}$ Vince checks that $s^2 \equiv 1236^2 \equiv 77 \pmod{1961}$.

11.5.5. Paula sends $x = 1226, y = 625$. After receiving a 1, she sends $u\overline{r} = 689$.

11.5.6. Paula sends $x = 888^2 = 788544 \equiv 1388 \pmod{2491}$. Vince chooses and sends the subset $\{2, 3, 5\}$. Paula sends $y \equiv rv_2v_3v_5 \equiv 888 \cdot 877 \cdot 2001 \cdot 101 \equiv 2101 \pmod{2491}$. Vince computes $y^2 s_2 s_3 s_5 \equiv 2102^2 \cdot 2453 \cdot 1553 \cdot 494 \equiv 1388 \equiv x \pmod{2491}$.

11.5.7. The prover sends $x = 1403^2 = 1968409 \equiv 519 \pmod{2491}$. The verifier sends $\{1, 5\}$. The prover sends $y = 1425$. The verifier computes $y^2 z = 1425^2 \cdot 197 \cdot 494 \equiv 519 \equiv x \pmod{2491}$

11.5.8. a. First we find inverses modulo 2491 of the six numbers, getting $1688, 1741, 201, 1789, 161$, and 214, respectively. Next we square and reduce these numbers modulo 2491 to get $s_1 = 2131, s_2 = 2025, s_3 = 545, s_4 = 2077, s_5 = 1011$, and $s_6 = 958$.

b. Paula sends $y \equiv 1091 \cdot 1199 \cdot 2144 \cdot 557 \cdot 2200 \equiv 1474 \pmod{2491}$.

c. Vince computes $1474^2 \cdot 2025 \cdot 545 \cdot 1011 \cdot 958 \equiv 2074 \pmod{2491}$, and then checks that $1091^2 \equiv 2074 \pmod{2491}$ also.

11.5.9. a. First we find inverses modulo 3953 of the six numbers, getting $3333, 753, 411, 1319, 705$, and 1811, respectively. Next we square and reduce these numbers modulo 3953 to get $s_1 = 959, s_2 = 1730, s_3 = 2895, s_4 = 441, s_5 = 2900$, and $s_6 = 2684$.

b. Paula sends $y \equiv 403 \cdot 1001 \cdot 21 \cdot 989 \cdot 1039 \equiv 1074 \pmod{3953}$.

c. Vince checks that $1074^2 \cdot 959 \cdot 1730 \cdot 441 \cdot 2684 \equiv 336 \equiv 403^2 \pmod{3953}$.

11.5.10. If an integer a is a quadratic residue modulo n then it is also a quadratic residue modulo p and q, and so there are two square roots modulo each of p and q. The Chinese remainder theorem shows that there are, therefore, 4 square roots of a modulo n. For our algorithm, we choose x and note that x^2 is necessarily a quadratic residue modulo n. Then x^2 has 4 square roots modulo n, two of which are $\pm x$, so if we extract a square root b of x^2, the probability is $2/4 = 1/2$ that b is different from $\pm x$. If so, then $b^2 \equiv x^2 \pmod{n}$ and so $(b - x)(b + x) \equiv 0 \pmod{n}$. So we expect that either $(b - x, n)$ or $(b + x, n)$ yields a non-trivial factor for n. If the square root is not different from $\pm x$, we select a new integer and repeat the process. The probability that we fail to find a square root different from $\pm x$ after k tries is $1/2^k$. Therefore the probability that we succeed in factoring n is $1 - 1/2^k$ which approaches 1.

11.5.11. If Paula sends back a to Vince then $a^2 \equiv w^2 \pmod{n}$, with $a \not\equiv w \pmod{n}$. Then $a^2 - w^2 = (a-w)(a+w) \equiv 0 \pmod{n}$. By computing $(a-w, n)$ and $(a+w, n)$ Vince will likely produce a nontrivial factor of n.

CHAPTER 12

Decimal Fractions and Continued Fractions

12.1. Decimal Fractions

12.1.1. a. Using the recursive formulae from Theorem 12.1. Let $\gamma_0 = 2/5$. Then $c_1 = [10 \cdot (2/5)] = 4$, and $\gamma_1 = 10 \cdot (2/5) - 4 = 0$, so we're done, and the decimal expansion is 0.4.

b. Let $\gamma_0 = 5/12$. Then $c_1 = [10 \cdot (5/12)] = 4$ and $\gamma_1 = 10 \cdot (5/12) - 4 = 1/6$. Then $c_2 = [10 \cdot (1/6)] = 1$, and $\gamma_2 = 10 \cdot (1/6) - 1 = 2/3$. Then $c_3 = [10 \cdot (2/3)] = 6$, and $\gamma_3 = 10 \cdot (2/3) - 6 = 2/3 = \gamma_2$, so the sequence repeats and the decimal expansion is $0.41\overline{6}$.

c. Let $\gamma_0 = 12/13$. Then $c_1 = [10 \cdot (12/13)] = 9$, and $\gamma_1 = 10 \cdot (12/13) - 9 = 3/13$. Then $c_2 = [10 \cdot (3/13)] = 2$, and $\gamma_2 = 10 \cdot (3/13) - 2 = 4/13$. Then $c_3 = [10 \cdot (4/13)] = 3$, and $\gamma_3 = 10 \cdot (4/13) - 3 = 1/13$. Then $c_4 = [10 \cdot (1/13)] = 0$, and $\gamma_4 = 10 \cdot (1/13) - 0 = 10/13$. Then $c_5 = [10 \cdot (10/13)] = 7$, and $\gamma_5 = 10 \cdot (10/13) - 7 = 9/13$. Then $c_6 = [10 \cdot (9/13)] = 6$, and $\gamma_6 = 10 \cdot (9/13) - 6 = 12/13 = \gamma_0$. So the decimal expansion is $.\overline{923076}$.

d. Let $\gamma_0 = 8/15$. Then $c_1 = [10 \cdot (8/15)] = 5$, and $\gamma_1 = 10 \cdot (8/15) - 5 = 5/15$. Then $c_2 = [10 \cdot (5/15)] = 3$, and $\gamma_2 = 10 \cdot (5/15) - 3 = 5/15 = \gamma_1$. So the decimal expansion is $0.5\overline{3}$.

e. Let $\gamma_0 = 1/111$. Then $c_1 = [10 \cdot (1/111)] = 0$, and $\gamma_1 = 10 \cdot (1/111) - 0 = 10/111$. Then $c_2 = [10 \cdot (10/111)] = 0$, and $\gamma_2 = 10 \cdot (10/111) - 0 = 100/111$. Then $c_3 = [10 \cdot (100/111)] = 9$, and $\gamma_3 = 10 \cdot (100/111) - 9 = 1/111 = \gamma_1$. So the decimal expansion is $0.\overline{009}$.

f. Let $\gamma_0 = 1/1001$. Then $c_1 = [10 \cdot (1/1001)] = 0$, and $\gamma_1 = 10 \cdot (1/1001) - 0 = 10/1001$. Then $c_2 = [10 \cdot (10/1001)] = 0$, and $\gamma_2 = 10 \cdot (10/1001) - 0 = 100/1001$. Then $c_3 = [10 \cdot (100/1001)] = 0$, and $\gamma_3 = 10 \cdot (100/1001) - 0 = 1000/1001$. Then $c_4 = [10 \cdot (1000/1001)] = 9$, and $\gamma_4 = 10 \cdot (1000/1001) - 9 = 991/1001$. Then $c_5 = [10 \cdot (991/1001)] = 9$, and $\gamma_5 = 10 \cdot (991/1001) - 9 = 901/1001$. Then $c_6 = [10 \cdot (901/1001)] = 9$, and $\gamma_6 = 10 \cdot (901/1001) - 9 = 1/1001 = \gamma_0$. So the decimal expansion is $0.\overline{000999}$.

12.1.2. a. Let $\gamma_0 = 1/3$. Then $c_1 = [8 \cdot (1/3)] = 2$, and $\gamma_1 = 8 \cdot (1/3) - 2 = 2/3$. Then $c_2 = [8 \cdot (2/3)] = 5$, and $\gamma_2 = 8 \cdot (2/3) - 5 = 1/3 = \gamma_0$. So the base 8 expansion is $(.\overline{25})_8$.

b. Let $\gamma_0 = 1/4$. Then $c_1 = [8 \cdot (1/4)] = 2$, and $\gamma_1 = 8 \cdot (1/4) - 2 = 0$. So the base 8 expansion is $(.2)_8$.

c. Let $\gamma_0 = 1/5$. Then $c_1 = [8 \cdot (1/5)] = 1$, and $\gamma_1 = 8 \cdot (1/5) - 1 = 3/5$. Then $c_2 = [8 \cdot (3/5)] = 4$, and $\gamma_2 = 8 \cdot (3/5) - 4 = 4/5$. Then $c_3 = [8 \cdot (4/5)] = 6$, and $\gamma_3 = 8 \cdot (4/5) - 6 = 2/5$. Then $c_4 = [8 \cdot (2/5)] = 3$, and $\gamma_4 = 8 \cdot (2/5) - 3 = 1/5 = \gamma_0$. So the base 8 expansion is $(.\overline{1463})_8$.

d. Let $\gamma_0 = 1/6$. Then $c_1 = [8 \cdot (1/6)] = 1$, and $\gamma_1 = 8 \cdot (1/6) - 1 = 1/3$. Then $c_2 = [8 \cdot (1/3)] = 2$, and $\gamma_2 = 8 \cdot (1/3) - 2 = 2/3$ Then $c_3 = [8 \cdot (2/3)] = 5$, and $\gamma_3 = 8 \cdot (2/3) - 5 = 1/3 = \gamma_1$. So the base 8 expansion is $(.1\overline{25})_8$.

e. Let $\gamma_0 = 1/12$. Then $c_1 = [8 \cdot (1/12)] = 0$, and $\gamma_1 = 8 \cdot (1/12) - 0 = 2/3$. Then $c_2 = [8 \cdot (2/3)] = 5$, and $\gamma_2 = 8 \cdot (2/3) - 5 = 1/3$ Then $c_3 = [8 \cdot (1/3)] = 2$, and $\gamma_3 = 8 \cdot (1/3) - 2 = 2/3 = \gamma_1$. So the base 8 expansion is $(.0\overline{52})_8$.

f. Let $\gamma_0 = 1/22$. Then $c_1 = [8 \cdot (1/22)] = 0$, and $\gamma_1 = 8 \cdot (1/22) - 0 = 4/11$. Then $c_2 = [8 \cdot (4/11)] = 2$, and $\gamma_2 = 8 \cdot (4/11) - 2 = 10/11$. Then $c_3 = [8 \cdot (10/11)] = 7$, and $\gamma_3 = 8 \cdot (10/11) - 7 = 3/11$. Then $c_4 =$

$[8\cdot(3/11)]=2$, and $\gamma_4=8\cdot(3/11)-2=2/11$. Then $c_5=[8\cdot(2/11)]=1$, and $\gamma_5=8\cdot(2/11)-1=5/11$. Then $c_6=[8\cdot(5/11)]=3$, and $\gamma_6=8\cdot(5/11)-3=7/11$. Then $c_7=[8\cdot(7/11)]=5$, and $\gamma_7=8\cdot(7/11)-5=1/11$. Then $c_8=[8\cdot(1/11)]=0$, and $\gamma_8=8\cdot(1/11)-0=8/11$. Then $c_9=[8\cdot(8/11)]=5$, and $\gamma_9=8\cdot(8/11)-5=9/11$. Then $c_{10}=[8\cdot(9/11)]=6$, and $\gamma_{10}=8\cdot(9/11)-2=6/11$. Then $c_{11}=[8\cdot(6/11)]=4$, and $\gamma_{11}=8\cdot(6/11)-4=4/11=\gamma_1$. So the base 8 expansion is $(.0\overline{2721350564})_8$.

12.1.3. a. We reduce $12/100$ to get $3/25$.

b. Note that $.1\overline{2}=.1+(2/100)\sum_{n=1}^{\infty}1/10^n=(1/10)+(2/100)(1/(1-1/10))=11/90$.

c. Let $\alpha=.\overline{12}$. Then $100\alpha=12.\overline{12}$, so that $99\alpha=12$. Therefore $\alpha=12/99=4/33$.

12.1.4. a. We have $(.123)_7=1/7+2/7^2+3/7^3=66/343$.

b. Let $\alpha=(.0\overline{13})_6$. Then $6^2\alpha=(1.3\overline{13})_6$, so that $(6^2-1)\alpha=(1.3)_6=1+3/6=3/2$. Then $\alpha=3/(2\cdot35)=3/70$.

c. We have $(\overline{.17})_{11}=\sum_{n=1}^{\infty}(17)_{11}/(100)_{11}^n=(18/121)(1/(1-1/121))=3/20$.

d. Let $\alpha=(.\overline{ABC})_{16}$. Then $16^3\alpha=(ABC.\overline{ABC})_{16}$, so that $(16^3-1)\alpha=(ABC)_16=10\cdot16^2+11\cdot16+12=2748$. Then $\alpha=2748/(16^3-1)=916/1365$.

12.1.5. All prime divisors of $210=2\cdot3\cdot5\cdot7$ must divide b, so $b=2^r3^s5^t7^u$, with r,s,t, and u nonnegative integers.

12.1.6. a. Since $12=2^23$, we have $T=2^2|10^2$, and $U=3$. So the pre-period length is 2, and $\text{ord}_310=1$, so the period length is 1.

b. Since $30=20\cdot3$, we have $T=10|10^1$, and $U=3$. So the pre-period length is 1, and $\text{ord}_310=1$, so the period length is 1.

c. Since $75=5^23$, we have $T=5^2|10^2$, and $U=3$. So the pre-period length is 2, and $\text{ord}_310=1$, so the period length is 1.

d. Since $23=1\cdot23$, we have $T=1|10^0$, and $U=23$. So the pre-period length is 0, and $\text{ord}_{23}10=22$, so the period length is 22.

e. Since $56=2^37$, we have $T=2^3|10^3$, and $U=7$. So the pre-period length is 3, and $\text{ord}_710=6$, so the period length is 6.

f. Since $61=1\cdot61$, we have $T=1|10^0$, and $U=61$. So the pre-period length is 0, and $\text{ord}_{61}10=60$, so the period length is 60.

12.1.7. a. Since $4=2^2$, we have $T=2^2|12^1$, and $U=1$. So the pre-period length is 1, and $\text{ord}_112=0$, so the period length is 0.

b. Since $8=2^3$, we have $T=2^3|12^2$, and $U=1$. So the pre-period length is 2, and $\text{ord}_112=0$, so the period length is 0.

c. Since $10=2\cdot5$, we have $T=2|12^1$, and $U=5$. So the pre-period length is 1, and $\text{ord}_512=4$, so the period length is 4.

d. Since $24=2^33$, we have $T=2^23|12^2$, and $U=1$. So the pre-period length is 2, and $\text{ord}_112=0$, so the period length is 0.

e. Since $132 = 12 \cdot 11$, we have $T = 12|12^1$, and $U = 11$. So the pre-period length is 1, and $\text{ord}_{11}12 = 1$, so the period length is 1.

f. Since $360 = 2^3 3^2 5$, we have $T = 2^3 3^2 | 12^2$, and $U = 5$. So the pre-period length is 2, and $\text{ord}_5 12 = 4$, so the period length is 4.

12.1.8. If m is prime and b is a primitive root modulo m, then $\text{ord}_m b = m - 1$, so the period length of $1/m$ is $m - 1$. Conversely, if the period length is $m - 1$, then $\text{ord}_m b = m - 1$, but $\text{ord}_m b | m$ so $(m-1)|\phi(m)$, which implies m is prime.

12.1.9. If $p = 2$ or 5, the period length is 0. Otherwise, $\text{ord}_p b = n$ is the period length. Now, $\text{ord}_p b = n$ for exactly those primes dividing $10^n - 1$, but not dividing $10^m - 1$ for any $m < n$. Then, (a) $10 - 1 = 3^2, p = 3$ (b) $10^2 - 1 = 3^2 11, p = 11$ (c) $10^3 - 1 = 3 \cdot 11 \cdot 37, p = 37$ (d) $p = 101$ (e) $p = 41$ and 271 (f) $p = 7$ and 13.

12.1.10. a. We have $1/(b-1) = (1/b)\,(1/(1-1/b)) = (1/b)\sum_{j=0}^{\infty}(1/b)^j = (.\overline{1})_b$.

b. We have $1/(b+1) = (b-1)/(b^2-1) = (b-1)/b^2 \cdot 1/(1-1/b^2) = (b-1)/b^2 \sum_{j=0}^{\infty}(1/b^2)^j = (b-1)/b^2(1.\overline{01})_b = (b-1)(.\overline{01})_b = (.\overline{0b-1})_b$.

12.1.11. Using the construction from Theorem 12.2 and Example 12.1, we use induction to show that $c_k = k-1$ and $\gamma_k = (kb-k+1)/(b-1)^2$. The induction step is as follows: $c_{k+1} = [b\gamma_k] = \left[(kb^2 - bk + b)/(b-1)^2\right] = \left[(k(b-1)^2 + b(k+1) - k)/(b-1)^2\right] = \left[k + (b(k+1) - k)/(b-1)^2\right] = k$, and $\gamma_{k+1} = (k+1)b - k$, if $k \neq b-2$. If $k = b-2$, we have $c_{b-2} = b$, so we have determined $b-1$ consecutive digits of the expansion. From the binomial theorem, $(x+1)^a \equiv ax+1 \pmod{x^2}$, so $\text{ord}_{(b-1)^2} b = b-1$, which is the period length. Therefore we have determined the entire expansion.

12.1.12. By Theorem 12.4, a non-repeating expansion represents an irrational number. To see that $(.0123\ldots(b-1)10111213\ldots)_b$ is non-repeating, notice that the sequence of digits contains arbitrarily long strings of zeros.

12.1.13. The base b expansion is $(.100100001\ldots)_b$ which is non-repeating and therefore by Theorem 12.4 represents an irrational number.

12.1.14. Use the construction from Theorem 12.1 and Example 12.1, but replace b by b_n at the nth step: $c_n = [b_n\gamma_{n-1}]$ and $\gamma_n = b_n\gamma_{n-1} - c_n$. Then $0 \leq c_n < b_n$.

12.1.15. Let γ be a real number. Set $c_0 = [\gamma]$ and and $\gamma_1 = \gamma - c_0$. Then $0 \leq \gamma_1 < 1$ and $\gamma = c_0 + \gamma_1$. From the condition that $c_k < k$ for $k = 1, 2, 3, \ldots$, we must have $c_1 = 0$. Let $c_2 = [2\gamma_1]$ and $\gamma_2 = 2\gamma_1 - c_2$. Then $\gamma_1 = (c_2 + \gamma_2)/2$, so $\gamma = c_0 + c_1/1! + c_2/2! + \gamma_2/2!$ Now let $c_3 = [3\gamma_2]$ and $\gamma_3 = 3\gamma_2 - c_3$. Then $\gamma_2 = (c_3 + \gamma_3)/3$ and so $\gamma = c_0 + c_1/1! + c_2/2! + c_3/3! + \gamma_3/3!$. Continuing in this fashion, for each $k = 2, 3, \ldots$, define $c_k = [k\gamma_{k-1}]$ and $\gamma_k = k\gamma_{k-1} - c_k$. Then $\gamma = c_0 + c_1/1! + c_2/2! + c_3/3! + \cdots + c_k/k! + \gamma_k/k!$. Since each $\gamma_k < 1$, we know that $\lim_{k\to\infty} \gamma_k/k! = 0$, so we conclude that $\gamma = c_0 + c_1/1! + c_2/2! + c_3/3! + \cdots + c_k/k! + \cdots$.

12.1.16. Let $r < s$ be integers. Then the rational number $\frac{r}{s} - \left(\frac{c_1}{1!} + \frac{c_2}{2!} + \cdots + \frac{c_{s-1}}{s-1!}\right)$, where the c_i are given by Exercise 15, has common denominator $s!$, so let c_s be the corresponding numerator. Check that $0 \leq c_s < 1$.

12.1.17. In the proof of Theorem 12.1, the numbers $p\gamma_n$ are the remainders of b^n upon division by p. The process recurs as soon as some γ_i repeats a value. Since $1/p = (.\overline{c_1 c_2 \ldots c_{p-1}})$ has period length $p-1$, we have by Theorem 12.4 that $\text{ord}_p b = p-1$, so there is an integer k such that $b^k \equiv m \pmod{p}$. So the remainders of mb^n upon division by p are the same as the remainders of $b^k b^n$ upon division by p. Hence the nth digit of the expansion of m/p is determined by the remainder of b^{k+n} upon division by p. Therefore, it will be the same as the $(k+n)$th digit of $1/p$.

12.1.18. First note that $b^t \equiv -1 \pmod{p}$, since $\text{ord}_p b = 2t$. Now $p\gamma_j$ is the remainder of b^j upon division by p, and $p\gamma_{j+t}$ is the remainder of b^{j+t} upon division by p, which must be the same as the remainder of $-b^j$

upon division by p. By the division algorithm, $b^j = kp + r$, so $-b^j = -kp - r = -(k-1)p + (p-r)$. Hence, $c_j = [br/p]$, and $c_{j+t} = [b(p-r)/p]$. Let $br/p = a + x$ where a is an integer and $0 < x < 1$. Then $c_j + c_{j+t} = [br/p] + [b(p-r)/p] = [a+x] + [b-(a+x)] = a + b - a - 1 = b - 1$, as desired.

12.1.19. Suppose $n = TU$, with $T = 2^k$ and U odd. Then the period length of the binary expansion of $1/n$ is $\text{ord}_U 2$. If $\text{ord}_U 2 = n - 1$, then $U = n$. So n is prime, and 2 is a primitive root of n.

12.1.20. By Theorem 12.4, $n = TU$ with $(U, 10) = 1$ and every prime factor of T divides b. Then the length of the period of the decimal expansion of $1/n$ is $\text{ord}_U 10$ which can be no larger than $U - 1$, which occurs if and only if U is prime and 10 is a primitive root modulo U. Thus $T = 1$ and n is a prime with primitive root 10.

12.1.21. Suppose $e = h/k$. Then $k!(e - 1 - 1/1! - 1/2! - \cdots - 1/k!)$ is an integer. But this is equal to $k!(1/(k+1)! + 1/(k+2)! + \cdots) = 1/(k+1) + 1/(k+1)(k+2) + \cdots < 1/(k+1) + 1/(k+1)^2 + \cdots = 1/k < 1$. But $k!(e - 1 - 1/1! - 1/2! - \cdots - 1/k!)$ is positive, and therefore cannot be an integer, a contradiction.

12.1.22. We have $b = 14$ and $\gamma = 1/6$ in the formula from Exercise 21. So the jth digit in the base 14 expansion is given by $c_j = [14^j 1/6] - 14[14^{j-1}1/6]$. The possible values for U in Theorem 12.4 are 1, 2, 3, and 6. Since ϕ of each of these numbers is less than or equal to 2, the expansion for $1/6$ must have period 1 or 2. Computing, we have: $c_1 = [14 \cdot 1/6] - 14[1/6] = 2$; $c_2 = [14^2 1/6] - 14[14 \cdot 1/6] = 4$; and $c_3 = [14^3 1/6] - 14[14^2 1/6] = 9$. Therefore we have $1/6 = (.2494949...)_{14}$.

12.1.23. Let $\alpha = \displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{a_i}}{10^{i!}}$, and $\dfrac{p_k}{q_k} = \displaystyle\sum_{i=1}^{k} \frac{(-1)^{a_i}}{10^{i!}}$. Then $\left|\alpha - \dfrac{p_k}{q_k}\right| = \left|\displaystyle\sum_{i=k+1}^{\infty} \frac{(-1)^{a_i}}{10^{i!}}\right| \leq \displaystyle\sum_{i=k+1}^{\infty} \frac{1}{10^{i!}}$. As in the proof of Corollary 12.5.1, it follows that $\left|\alpha - \dfrac{p_k}{q_k}\right| < \dfrac{2}{10^{(k+1)!}}$, which shows that there can be no real number C as in Theorem 12.5. Hence, α must be transcendental.

12.1.24. We mimic the proof of Theorem 12.6. The only changes are that each d_{ij} is either 0 or 1. Form a new number $r = 0.d_1 d_2 d_3 d_4 \ldots$, by $d_i = 0$ if $d_{ii} = 1$ and $d_i = 1$ if $d_{ii} = 0$. Then r is different from every number in the listing and so is not in the listing. Therefore, the listing, no matter what it was, could not contain all the real numbers with decimal expansions consisting of only 0s and 1s.

12.1.25. Suppose $e = h/k$. Then $k!(e - 1 - 1/1! - 1/2! - \cdots 1/k!)$ is an integer. But this is equal to $k!(1/(k+1)! + 1/(k+2)! + \cdots) = 1/(k+1) + 1/(k+1)(k+2) + \cdots < 1/(k+1) + 1/(k+1)^2 + \cdots = 1/k < 1$. But $k!(1/(k+1)! + 1/(k+2)! + \cdots)$ is positive, and therefore can not be an integer, a contradiction.

12.1.26. a. We find $1/19 = (.024024024\ldots)_7$, so starting at the 7th position we have $0, 2, 4, 0, 2, 4, 0, 2, 4, 0$.

b. We find $1/21 = (.030303030303\ldots)_8$, so starting at the 6th position we have $3, 0, 3, 0, 3, 0, 3, 0, 3, 0$.

12.2. Finite Continued Fractions

12.2.1. a. We have $[2; 7] = 2 + 1/7 = 15/7$.

b. We have $[1;, 2, 3] = 1 + \dfrac{1}{2 + 1/3} = 1 + 3/7 = 10/7$.

c. We have $[0'5, 6] = \dfrac{1}{5 + (1/6)} = 6/31$.

d. We have $[3; 7, 15, 1] = 3 + \dfrac{1}{7 + \dfrac{1}{15 + 1}} = 3 + \dfrac{1}{1 + 1/16} = 355/113$. Note that this is a very good approximation for π.

e. We have $[1;1] = 1 + (1/1) = 2$.

f. We have $[1;1,1] = 1 + \dfrac{1}{1+1} = 3/2$.

g. We have $[1;1,1,1] = 1 + \dfrac{1}{1+\dfrac{1}{1+1}} = 5/3$

h. We have $[1;1,1,1,1] = 1 + \dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}} = 8/5$. Note that the numerators and denominators in these last four exercises are Fibonacci numbers.

12.2.2. a. We have $[10;3] = 10 + 1/3 = 31/3$.

b. We have $[3;2,1] = 3 + \dfrac{1}{2+1/1} = 3 + 1/3 = 10/3$.

c. We have $[0;1,2,3] = 0 + \dfrac{1}{1+\dfrac{1}{2+1/3}} = 3/10$. Compare this with part (b).

d. We have $[2;1,2,1] = 2 + \dfrac{1}{1+\dfrac{1}{2+1/1}} = 11/4$.

e. We have $[2;1,2,1,1,4] = 2 + \dfrac{1}{1+\dfrac{1}{2+\dfrac{1}{1+\dfrac{1}{1+1/4}}}} = 87/32$.

f. We have $[1;2,1,2] = 1 + \dfrac{1}{2+\dfrac{1}{1+1/2}} = 11/8$.

g. We have $[1;2,1,2,1] = 1 + \dfrac{1}{2+\dfrac{1}{1+\dfrac{1}{2+1/1}}} = 15/11$.

h. We have $[1;2,1,2,1,2] = 1 + \dfrac{1}{2+\dfrac{1}{1+\dfrac{1}{2+\dfrac{1}{1+1/2}}}} = 41/30$.

12.2.3. a. Using the construction in the proof of Theorem 12.8, we let $r_0 = 18$ and $r_1 = 13$. Then $18 = 1 \cdot 13 + 5, 13 = 2 \cdot 5 + 3, 5 = 1 \cdot 3 + 2, 3 = 1 \cdot 2 + 1$, and $2 = 2 \cdot 1$. The sequence of quotient gives us the continued fraction $[1;2,1,1,2]$.

b. We perform the Euclidean algorithm on 32 and 17 to get $32 = 1 \cdot 17 + 15, 17 = 1 \cdot 15 + 2, 15 = 7 \cdot 2 + 1, 2 = 2 \cdot 1$. The sequence of quotients gives us $[1;1,7,2]$.

c. We perform the Euclidean algorithm on 19 and 9 to get $19 = 2 \cdot 9 + 1, 9 = 9 \cdot 1$. The sequence of quotients yields $[2;9]$.

d. We perform the Euclidean algorithm on 310 and 99 to get $310 = 3 \cdot 99 + 13, 99 = 7 \cdot 13 + 8, 13 = 1 \cdot 8 + 5, 8 = 1 \cdot 5 + 3, 5 = 1 \cdot 3 + 2, 3 = 1 \cdot 2 + 1, 2 = 2 \cdot 1$. The sequence of quotients yields $[3; 7, 1, 1, 1, 1, 2]$.

e. We perform the Euclidean algorithm on -931 and 1005 to get $-931 = -1 \cdot 1005 + 74, 1005 = 13 \cdot 74 + 43, 74 = 1 \cdot 43 + 31, 43 = 1 \cdot 31 + 12, 31 = 2 \cdot 12 + 7, 12 = 1 \cdot 7 + 5, 7 = 1 \cdot 5 + 25 = 2 \cdot 2 + 1, 2 = 2 \cdot 1$. The sequence of quotients yields $[-1; 13, 1, 1, 2, 1, 1, 2, 2]$.

f. We perform the Euclidean algorithm on 831 and 8110 to get $831 = 0 \cdot 8110 + 831, 8110 = 9 \cdot 831 + 631, 831 = 1 \cdot 631 + 200, 631 = 3 \cdot 200 + 31, 200 = 6 \cdot 31 + 14, 31 = 2 \cdot 14 + 3, 14 = 4 \cdot 3 + 2, 3 = 1 \cdot 2 + 1, 2 = 2 \cdot 1$. The sequence of quotients gives us $[0; 9, 1, 3, 6, 2, 4, 1, 2]$.

12.2.4. a. We have $6/5 = 1 + 1/5$, so the continued fraction expansion is $[1; 5]$.

b. We have $22/7 = 3 + 1/7$, so the continued fraction expansion is $[3; 7]$.

c. The Euclidean algorithm gives: $19 = 0(29) + 19; 29 = 1(19) + 10; 19 = 1(10) + 9; 10 = 1(9) + 1; 9 = 9(1)$. The quotients give the continued fraction expansion $[0; 1, 1, 1, 9]$.

d. The Euclidean algorithm gives: $5 = 0(999) + 5; 999 = 199(5) + 4; 5 = 1(4) + 1; 4 = 4(1)$, so the continued fraction expansion is $[0; 199, 1, 4]$.

e. The Euclidean algorithm gives: $-943 = -1 \cdot 1001 + 58, 1001 = 17 \cdot 58 + 15, 58 = 3 \cdot 15 + 13, 15 = 1 \cdot 13 + 2, 13 = 6 \cdot 2 + 1, 2 = 2 \cdot 1$, so the continued fraction expansion is $[-1, 17, 3, 1, 6, 2]$.

f. The Euclidian algorithm gives: $873 = 0 \cdot 4867 + 873, 4867 = 5 \cdot 873 + 502, 873 = 1 \cdot 502 + 371, 502 = 1 \cdot 371 + 131, 371 = 2 \cdot 131 + 109, 131 = 1 \cdot 109 + 22, 109 = 8 \cdot 22 + 21, 22 = 1 \cdot 21 + 1, 21 = 21 \cdot 1$, so the continued fraction expansion is $[0; 5, 1, 1, 2, 1, 4, 1, 21]$.

12.2.5. a. We compute $p_0 = 1, p_1 = 1 \cdot 2 + 1 = 3, p_2 = 1 \cdot 3 + 1 = 4, p_3 = 1 \cdot 4 + 3 = 7, p_4 = 2 \cdot 7 + 4 = 18$, and $q_0 = 1, q_1 = 2, q_2 = 1 \cdot 2 + 1 = 3, q_3 = 1 \cdot 3 + 2 = 5, q_4 = 2 \cdot 5 + 3 = 13$. Then the convergents are $C_0 = 1/1 = 1, C_1 = 3/2, C_2 = 4/3, C_3 = 7/5, C_4 = 18/13$.

b. . We compute $p_0 = 1, p_1 = 1 \cdot 1 + 1 = 2, p_2 = 7 \cdot 2 + 1 = 15, p_3 = 2 \cdot 15 + 2 = 32$, and $q_0 = 1, q_1 = 1, q_2 = 7 \cdot 1 + 1 = 8, q_3 = 2 \cdot 8 + 1 = 17$. Then the convergents are $C_0 = 1, C_1 = 2, C_2 = 15/8, C_3 = 32/17$.

c. We compute $p_0 = 2, p_1 = 2 \cdot 9 + 1 = 19$, and $q_0 = 1, q_1 = 9$. Then the convergents are $C_0 = 2, C_1 = 19/9$.

d. We compute the sequence of p_i to be 3, 22, 25, 47, 72, 119, 310, and the sequence of q_i to be 1, 7, 8, 15, 23, 38, 99, so the convergents are 3, $22/7$, $25/8$, $47/15$, $72/23$, $119/38$, $310/99$.

e. We compute the sequence of p_i to be $-1, -12, -13, -25, -63, -88, -151, -390, -931$, and the sequence of q_i to be $1, 13, 14, 27, 68, 95, 163, 421, 1005$, so the convergents are -1, $-12/13$, $-13/14$, $-25/27$, $-63/68$, $-88/95$, $-151/163$, $-390/421$, $-931/1005$.

f. We compute the sequence of p_i to be 0, 1, 1, 4, 25, 54, 241, 295, 831, and the sequence of q_i to be 1, 9, 10, 39, 244, 527, 2352, 2879, 8110, so the convergents are 0, $1/9$, $1/10$, $4/39$, $25/244$, $54/527$, $241/2352$, $295/2879$, $831/8110$.

12.2.6. a. The convergents are $C_0 = 1, C_1 = 1 + 1/5 = 6/5$.

b. The convergents are $C_0 = 3, C_1 = 3 + 1/7 = 22/7$.

c. We compute the sequence of p_i to be $0, 1, 1, 2, 19$, and the sequence of q_i to be $1, 1, 2, 3, 29$, so the convergents are $0, 1, 1/2, 2/3, 19/29$.

d. We compute the sequence of p_i to be $0, 1, 1, 5$, and the sequence of q_i to be $1, 199, 200, 999$, so the convergents are $0, 1/199, 1/200, 5/999$.

e. We compute the sequence of p_i to be $-1, -16, -49, -65, -439, -943$, and the sequence of q_i to be $1, 17, 52, 69, 466, 1001$, so the convergents are $-1, -16/17, -49/52, -65/69, -439/466, -943/1001$.

f. We compute the sequence of p_i to be 0, 1, 1, 2, 5, 7, 33, 40, 873, and the sequence of q_i to be 1, 5, 6, 11, 28, 39, 184, 223, 4867, so the convergents are 0, 1/5, 1/6, 2/11, 5/28, 7/39, 33/184, 40/223, 873/4867.

12.2.7. For Exercise 5: (a) $3/2 > 7/5$ and $1 < 4/3 < 18/13$ (b) $2 > 32/17$ and $1 < 15/8$ (c) vacuously true (d) $22/7 > 47/15 > 119/38$ and $3 < 25/8 < 72/23, < 310/99$ (e) $-12/13 > -25/27 > -88/95 > -390/421$ and $-1 < -13/14 < -63/68, < -151/163 < -931/1005$ (f) $1/9 > 4/39 > 54/527 > 295/2879$ and $0 < 1/10 < 25/244 < 241/2352 < 831/8110$.

12.2.8. The recursion formula for the Fibonacci sequence tells us that the Euclidean algorithm for f_{k+1}/f_k gives the following sequence of equations: $f_{k+1} = 1(f_k) + f_{k-1}; f_k = 1(f_{k-1}) + f_{k-2}; \ldots; f_2 = 1(f_1)$. So $f_{k+1}/f_k = [1; 1, 1, \ldots, 1]$ (k-times).

12.2.9. Let $\alpha = r/s$. The Euclidean Algorithm for $1/\alpha = s/r < 1$ gives $s = 0(r) + s; r = a_0(s) + a_1$, and continues just like for r/s.

12.2.10. The recursion formula for the p_i tells us that the Euclidean algorithm for p_k/p_{k-1} gives the following sequence of equations: $p_k = a_k p_{k-1} + p_{k-2}; \ldots; p_1 = a_1 p_0 + 1; p_0 = a_0(1)$. So $p_k/p_{k-1} = [a_k; a_{k-1}, \ldots, a_0]$. Similarly for the q_k.

12.2.11. Proceed by induction. Assume $q_j \geq f_j$ for $j < k$. Then $q_k = a_k q_{k-1} + q_{k-2} \geq a_k f_{k-1} + f_{k-2} \geq f_{k-1} + f_{k-2} = f_k$, as desired.

12.2.12. Rewrite the last step of the Euclidean Algorithm: $r_{n-1} = q_n r_n = (q_n - 1)r_n + r_n; r_n = q_{n+1} r_n$, so $q_{n+1} = 1$ and we have $[a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots, a_n - 1, 1]$.

12.2.13. By Exercise 10, we have $p_n/p_{n-1} = [a_n; a_{n-1}, \ldots, a_0] = [a_0; a_1, \ldots, a_n] = p_n/q_n = r/s$ if the continued fraction is symmetric. Then, $q_n = p_{n-1} = s$ and $p_n = r$, so by Theorem 12.10 we have $p_n q_{n-1} - q_n p_{n-1} = r q_{n-1} - s^2 = (-1)^{n-1}$. Then $r q_{n-1} = s^2 + (-1)^{n-1}$ and so $r | s^2 - (-1)^n$. Conversely, if $r | s^2 + (-1)^{n-1}$, then $(-1)n - 1 = p_n q_{n-1} - q_n p_{n-1} = r q_{n-1} - p_{n-1} s$. So $r | p_{n-1} s + (-1)^{n-1}$ and hence $r | (s^2 + (-1)^{n-1}) - (p_{n-1} s + (-1)^{n-1}) = s(s - p_{n-1})$. Since $s, p_{n-1} < r$ and $(r, s) = 1$, we have $s = p_{n-1}$. Then $[a_n; a_{n-1}, \ldots, a_0] = p_n/p_{n-1} = r/s = [a_0; a_1, \ldots, a_n]$.

12.2.14. If a, b are integers, then Section 1.5, Exercise 16 gives $a = q_0 b + r_1; b = q_1 r_1 + r_2; r_1 = q_2 r_2 + r_3; \ldots; r_{n-1} = q_n r_n$, with $-b/2 < r_1 \leq b/2$ and $r_{j-1}/2 < r_j \leq r_{j-1}/2$ for each j. Then $a/b = [q_0; q_1, \ldots, q_n]$ following the construction in Theorem 12.6

12.2.15. Note that the notation $[a_0; a_1, \ldots, a_n]$ makes sense, even if the a_j are not integers. Use induction. Assume the statement is true for k odd and prove it for $k + 2$. Define $a'_k = [a_k; a_{k+1}, a_{k+2}]$ and check that $a'_k < [a_k; a_{k+1}, a_{k+2} + x] = a'_k + x'$. Then $[a_0; a_1, \ldots, a_{k+2}] = [a_0; a_1, \ldots, a'_k] > [a_0; a_1, \ldots, a'_k + x'] = [a_0; a_1, \ldots, a_{k+2} + x]$. Proceed similarly for k even.

12.2.16. a. By Exercise 8 we have $13 = 8 + 5$, all Fibonacci numbers, which gives $8/5 = [1; 1, 1, 1, 1]$.

b. We have $17 = 12 + 5$ and $12/5 = [2; 2, 2]$.

c. We have $19 = 12 + 7$ and $12/7 = [1; 1, 2, 2]$.

d. We check the continued fraction for each of $(23 - j)/j$ for $j = 1, 2, \ldots, 11$ and find no solution.

e. We have $27 = 18 + 9$ and $18/9 = 2$.

f. We have $29 = 21 + 8$ and $21/8 = [2, 1, 1, 1, 2]$.

12.3. Infinite Continued Fractions

12.3.1. a. We compute $a_0 = [\sqrt{2}] = 1, \alpha_1 = 1/(\sqrt{2}-1) = \sqrt{2}+1, a_1 = [\alpha_1] = 2, \alpha_2 = \frac{1}{(\sqrt{2}+1)-2} = \sqrt{2}+1 = \alpha_1$. Therefore the sequence repeats, and we have $\sqrt{2} = [1; 2, 2, \ldots]$.

b. We compute $a_0 = [\sqrt{3}] = 1, \alpha_1 = 1/(\sqrt{3}-1) = (\sqrt{3}+1)/2, a_1 = [\alpha_1] = 1, \alpha_2 = \frac{1}{(\sqrt{3}+1)/2-1} = \sqrt{3}+1, a_2 = [\alpha_2] = 2, \alpha_3 = \frac{1}{(\sqrt{3}+1)-2} = (\sqrt{3}+1)/2 = \alpha_1$. Therefore the sequence repeats, and we have $\sqrt{3} = [1; 1, 2, 1, 2, \ldots]$.

c. We compute $a_0 = [\sqrt{5}] = 2, \alpha_1 = 1/(\sqrt{5}-2) = \sqrt{5}+2, a_1 = [\alpha_1] = 4, \alpha_2 = 1/((\sqrt{5}+2)-4) = \sqrt{5}+2 = \alpha_1$. Therefore the sequence repeats, and we have $\sqrt{5} = [2; 4, 4, \ldots]$.

d. We compute $a_0 = [(\sqrt{5}+1)/2 = 1, \alpha_1 = \frac{1}{(\sqrt{5}+1)/2)-1} = (\sqrt{5}+1)/2 = \alpha_0$. This gives $(\sqrt{5}+1)/2 = [1; 1, 1, \ldots]$.

12.3.2. a. We compute $a_0 = [\sqrt[3]{2}] = 1$, $\alpha_1 = 1/(\sqrt[3]{2}-1)$,$a_1 = [\alpha_1] = 3$, $\alpha_2 = \frac{1}{1/(\sqrt[3]{2}-1)-3} = (\sqrt[3]{2}-1)/(3\sqrt[3]{2}-4)$, $a_2 = [\alpha_2] = 1$, $\alpha_3 = \frac{1}{(1/\alpha_2)-a_2} = (3\sqrt[3]{2}-4)/(4\sqrt[3]{2}-5)$, $a_3 = [\alpha_3] = 5$, $\alpha_4 = \frac{1}{(1/\alpha_3)-a_3} = (4\sqrt[3]{2}-5)/(23\sqrt[3]{2}-29)$, $a_4 = [\alpha_4] = 1$, so the first five partial quotients are $1, 3, 1, 5, 1$.

b. We compute $a_0 = [2\pi] = 6$, $\alpha_1 = 1/(2\pi-6)$, $a_1 = [\alpha_1] = 3$, $\alpha_2 = \frac{1}{1/(2\pi-6)-3} = (-2\pi+6)/(6\pi-19)$, $a_2 = [\alpha_2] = 1$, $\alpha_3 = \frac{1}{(1/\alpha_2)-a_2} = (-6\pi+19)/(8\pi-25)$, $a_3 = [\alpha_3] = 1$, $\alpha_4 = \frac{1}{(1/\alpha_3)-a_3} = (-14\pi+44)/(106\pi-333)$, $a_4 = [\alpha_4] = 1$, so the first five partial quotients are $6, 3, 1, 1, 7$.

c. We compute $a_0 = [(e-1)/(e+1)] = 0$, $\alpha_1 = 1/((e-1)/(e+1)) = (e+1)/(e-1)$, $a_1 = [\alpha_1] = 2$, $\alpha_2 = \frac{1}{1/(\alpha_1)-2} = -(e-1)/(e-3)$, $a_2 = [\alpha_2] = 6$, $\alpha_3 = \frac{1}{(1/\alpha_2)-a_2} = -(e-3)/(7e-19)$, $a_3 = [\alpha_3] = 10$, $\alpha_4 = \frac{1}{(1/\alpha_3)-a_3} = -(7e-19)/(71e-193)$, $a_4 = [\alpha_4] = 14$, so the first five partial quotients are $0, 2, 6, 10, 14$.

d. We compute $a_0 = [(e^2-1)/(e^2+1)] = 0$, $\alpha_1 = 1/((e^2-1)/(e^2+1)) = (e^2+1)/(e^2-1)$, $a_1 = [\alpha_1] = 1$, $\alpha_2 = \frac{1}{1/(\alpha_1)-1} = -(e^2-1)/2$, $a_2 = [\alpha_2] = 3$, $\alpha_3 = \frac{1}{(1/\alpha_2)-a_2} = 1/(e^2-7)$, $a_3 = [\alpha_3] = 5$, $\alpha_4 = \frac{1}{(1/\alpha_3)-a_3} = -(e^2-7)/(5e^2-37)$, $a_4 = [\alpha_4] = 7$, so the first five partial quotients are $0, 1, 3, 5, 7$.

12.3.3. From Example 12.11, we have $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots]$. We compute the convergents until we have a denominator greater than 100000: 3, 22/7, 333/106, 355/113, 103933/33102, 104348/33215, 208341/66317, 312689/99532, 833719/265381, ... Therefore, the best approximation with denominator less than 100000 is 312689/99532.

12.3.4. a. Using Theorem 12.9, we compute the sequence of p_k and q_k to get the following convergents: $2, 3, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}$.

b. The ninth convergent is $\frac{1264}{465}$. The tenth is $\frac{1457}{536}$, so the tenth is the best.

12.3.5. If $a_1 > 1$, let $A = [a_2; a_3, \ldots]$. Then $[a_0; a_1, \ldots] + [-a_0 - 1; 1, a_1 - 1, a_2, a_3, \ldots] = a_0 + \dfrac{1}{a_1 + (1/A)} + \left(-a_0 - 1 + \dfrac{1}{1 + \dfrac{1}{a_1 - 1 + (1/A)}} \right) = 0$. Similarly if $a_1 = 1$.

12.3.6. Without loss of generality, k is odd. Theorem 12.11 says that the odd convergents decrease to α, and the even convergents increase to α, so $p_{k+1}/q_{k+1} < \alpha < p_k/q_k$. The hint follows after Theorem 12.10. Notice that $(q_{k+1} - q_k)^2 > 0$, so $q_{k+1}^2 + q_k^2 > 2q_{k+1}q_k$. Then dividing by $2q_{k+1}^2 q_k^2$ gives $|\alpha - (p_k/q_k)| + |\alpha - (p_{k+1}/q_{k+1})| = 1/(q_k q_{k+1}) < 1/(2q_k^2) + 1/(2q_{k+1}^2)$, and the proposition follows.

12.3.7. If $\alpha = [a_0; a_1, a_2, \ldots]$, then $1/\alpha = 1/[a_0; a_1, a_2, \ldots] = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cdots}} = [0; a_0, a_1, a_2, \ldots]$. Then the kth convergent of $1/\alpha$ is $[0; a_0, a_1, a_2, \ldots, a_{k-1}] = 1/[a_0; a_1, a_2, \ldots, a_{k-1}]$, which is the reciprocal of the $(k-1)$st convergent of α.

12.3.8. Suppose $|\alpha - (p_k/q_k)| \geq 1/(\sqrt{5}q_k^2)$, for $k = j-1, j, j+1$. Note that $x + 1/x \geq \sqrt{5}$ implies $(\sqrt{5}-1)/2 < x < (\sqrt{5}+1)/2$. Then as in the hint to Exercise 6, $|\alpha - (p_{j-1}/q_{j-1})| + |\alpha - (p_j/q_j)| = 1/(q_{j-1}q_j) > 1/(\sqrt{5}q_{j-1}^2) + 1/(\sqrt{5}q_j^2)$, where the inequality is strict since the left side is rational. Then $\sqrt{5} > q_j/q_{j-1} + q_{j-1}/q_j$, so by the note, $(\sqrt{5}-1)/2 < q_j/q_{j-1}) < (\sqrt{5}+1)/2$. Similarly, $(\sqrt{5}-1)/2 < q_{j+1}/q_j < (\sqrt{5}+1)/2$. Then using $q_{j+1} = a_j q_j + q_{j-1}$ we have $(\sqrt{5}+1)/2 > q_{j+1}/q_j = a_j + (q_{j-1}/q_j) > 1 + (\sqrt{5}-1)/2 = (\sqrt{5}+1)/2$, which is a contradiction.

12.3.9. By Theorem 12.17, such a p/q is a convergent of α. We have $(\sqrt{5}+1)/2 = [1; 1, 1, \ldots]$, so $q_n = f_n$ (Fibonacci) and $p_n = q_{n+1}$. Then $\lim_{n\to\infty} q_{n-1}/q_n = \lim_{n\to\infty} q_{n-1}/p_{n-1} = 2/(\sqrt{5}+1) = (\sqrt{5}-1)/2$. So $\lim_{n\to\infty}\left((\sqrt{5}+1)/2 + (q_{n-1}/q_n)\right) = (\sqrt{5}+1)/2 + (\sqrt{5}-1)/2 = \sqrt{5}$. So $(\sqrt{5}+1)/2 + (q_{n-1}/q_n) > c$ only finitely often. Whence, $1/\left((\sqrt{5}+1)/2 + (q_{n-1}/q_n)\right) q_n^2 < 1/(cq_n^2)$. The following identity finishes the proof. Note that $\alpha_n = \alpha$ for all n. Then $|\alpha - (p_n/q_n)| = |(\alpha_{n+1}p_n + p_{n-1})/(\alpha_{n+1}q_n + q_{n-1}) - (p_n/q_n)| = |(-(p_n q_{n-1} - p_{n-1}q_n))/(q_n(\alpha q_n + q_{n-1}))| = 1/(q_n^2(\alpha + (q_{n-1}/q_n))$.

12.3.10. Note that $\alpha = (1\cdot\alpha + 0)/(0\cdot\alpha + 1)$.

12.3.11. If β is equivalent to α, then $\beta = (a\alpha + b)/(c\alpha + d)$. Solving for α gives $\alpha = (-d\beta + b)/(c\beta - a)$, so α is equivalent to β.

12.3.12. Say $\beta = (a\alpha + b)/(c\alpha + d)$ and $\gamma = (e\beta + f)/(g\beta + h)$. Then $\gamma = (e(a\alpha + b)/(c\alpha + d) + f)/(g(a\alpha + b)/(c\alpha + d) + h) = ((ea + fc)\alpha + (eb + df))/((ga + ch)\alpha + (gb + dh))$, so γ and α are equivalent.

12.3.13. If $a \neq 0$, then $r/s = ((rb)a + 0)/((sa)b + 0)$, so r/s and a/b are equivalent. If $a = 0$ then $r/s = (1\cdot a + r)/(0\cdot b + s)$.

12.3.14. First note that with α_j defined in the usual way, $\alpha_j = 1/(a_{j+1} + \alpha_{j+1})$, so α_j is equivalent to α_{j+1}. From the transitivity of Exercise 12, we have α equivalent to α_j for all j. The solution then follows easily.

12.3.15. Note that $p_{k,t}q_{k-1} - q_{k,t}p_{k-1} = t(p_{k-1}q_{k-1} - q_{k-1}p_{k-1}) + (p_{k-2}q_{k-1} - p_{k-1}q_{k-2}) = \pm 1$. Thus $p_{k,t}$ and $q_{k,t}$ are relatively prime.

12.3.16. Consider the function $f(t) = (at + b)/(ct + d)$, where $a/b > c/d$. Then $f'(t) = (ad - bc)/(ct + d)^2 > 0$ for all t. Therefore, $f(x)$ is a strictly increasing function. Now as t goes from 0 to a_k, we see that $g(t) = p_{k,t}/q_{k,t}$ goes from C_{k-2} to C_k. Now if k is even, we have $C_{k-2} < C_k$, and $g(t)$ is a function of the same form as $f(t)$. Therefore, as t increases, so must the pseudoconvergents. If k is odd, the argument is similar.

12.3.17. See, for example, the classic work by O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig, Teubner (1929).

12.3.18. We have $\pi = [3; 7, 15, \ldots]$ for which the first convergents are $3/1, 33/7, 333/106, \ldots$. Then $p_{2,t}/q_{2,t} = (tp_1 + p_0)/(tq_1 + q_0) = (t22 + 3)/(t7 + 1)$ for $t = 1, 2, \ldots, 14$, so the pseudoconvergents are: $25/8$, $47/15$, $69/22$, $91/29$, $113/36$, $135/43$, $157/50$, $179/57$, $201/64$, $223/71$, $245/78$, $267/85$, $289/92$, and $311/99$.

12.3.19. Using Exercise 17, we test each of the pseudoconvergents in Exercise 18 and find that $|\pi - 179/57| < |\pi - 22/7|$.

12.3.20. The smallest denominator of a pseudoconvergent greater than 71 is $39 + 71 > 100$, so the 8th convergent $193/71$ is the best approximation.

12.3.21. (Proof by Rob Johnson.) Note first that if $b < d$, then $|a/b - c/d| < 1/2d^2$ implies that $|ad - bc| < b/2d < 1/2$, but since $b \neq d$, $|ad - bc|$ is a positive integer, and so is greater than $1/2$. Thus $b \geq d$. Now assume that c/d is not a convergent of the continued fraction for a/b. Since the denominators of the convergents increase to b, there must be two successive convergents p_n/q_n and p_{n+1}/q_{n+1} such that $q_n < d < q_{n+1}$. Next, by the triangle inequality we have $1/2d^2 > \left|\frac{a}{b} - \frac{c}{d}\right| = \left|\frac{c}{d} - \frac{p_n}{q_n}\right| - \left|\frac{a}{b} - \frac{p_n}{q_n}\right| \geq \left|\frac{c}{d} - \frac{p_n}{q_n}\right| - \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right|$, since the $n+1$st convergent is on the other side of a/b from the nth convergent. Since the numerator of the first difference is a nonzero integer, and applying Corollary 12.3 to the second difference, we have the last expression greater than or equal to $1/dq_n - 1/q_{n+1}q_n$. If we multiply through by d^2 we get $\frac{1}{2} > \frac{d}{q_n}\left(1 - \frac{d}{q_{n+1}}\right) > 1 - \frac{d}{q_{n+1}}$ since $d/q_n > 1$. From which we deduce that $1/2 < d/q_{n+1}$.

Now the convergents p_n/q_n and p_{n+1}/q_{n+1} divide the line into three regions. As c/d could be in any of these, there are three cases. Case 1: If c/d is between the convergents, then $\frac{1}{dq_n} \leq \left|\frac{c}{d} - \frac{p_n}{q_n}\right|$ since the numerator of the fraction is a positive integer and the denominators on both sides of the inequality are the same. This last is less than or equal to $\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_{n+1}q_n}$ since the $n+1$st convergent is farther from the nth convergent than c/d and where we have applied Corollary 12.3. But this implies that $d \geq q_{n+1}$, a contradiction. Case 2. If c/d is closer to p_n/q_n, then again $\frac{1}{dq_n} \leq \left|\frac{c}{d} - \frac{p_n}{q_n}\right| \leq \left|\frac{a}{b} - \frac{c}{d}\right|$ since a/b is on the other side of the nth convergent from c/d. But this last is less than $1/2d^2$ and if we multiply through by d we have $1/q_n < 1/2d$, which implies that $q_n > d$, a contradiction. Case 3. If c/d is closer to p_{n+1}/q_{n+1}, then with the same reasoning as in Case 2, we have $\frac{1}{dq_{n+1}} \leq \left|\frac{c}{d} - \frac{p_{n+1}}{q_{n+1}}\right| < \left|\frac{a}{b} - \frac{c}{d}\right| < 1/2d^2$. But this implies that $d/q_{n+1} < 1/2$ contradicting the inequality established above. Having exhausted all the cases, we must conclude that c/d must be a convergent of the continued fraction for a/b.

12.3.22. From the proof of Theorem 12.8, we see that finding the convergents of a rational number involves exactly the same calculations as finding the greatest common divisor of the numerator and denominator. By Corollary 3.13.1 (to Lame's Theorem) this takes $O((\log_2 n)^3)$ operations.

12.4. Periodic Continued Fractions

12.4.1. a. Using Theorem 12.20, we have $\alpha_0 = \sqrt{7}, a_0 = 2, P_0 = 0, Q_0 = 1, P_1 = 2 \cdot 1 - 0 = 2, Q_1 = \frac{7-2^2}{1} = 3, \alpha_1 = \frac{2+\sqrt{7}}{3}, a_1 = 1, P_2 = 1 \cdot 3 - 2 = 1, Q_2 = \frac{7-1^2}{3} = 2, \alpha_2 = \frac{1+\sqrt{7}}{2}, a_2 = 1, P_3 = 1 \cdot 2 - 1 = 1, Q_3 = \frac{7-1^2}{2} = 3, \alpha_3 = \frac{1+\sqrt{7}}{3}, a_3 = 1, P_4 = 1 \cdot 3 - 1 = 2, Q_4 = \frac{7-2^2}{3} = 1, \alpha_4 = \frac{2+\sqrt{7}}{1}, a_4 = 4, P_5 = 4 \cdot 1 - 2 = 2, Q_5 = \frac{7-2^2}{1} = 3, \alpha_5 = \alpha_1$, so $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$.

b. As in part (a), we find $\sqrt{11} = [3; \overline{3, 6}]$.

c. As in part (a), we find $\sqrt{23} = [4; \overline{1, 3, 1, 8}]$.

d. As in part (a), we find $\sqrt{47} = [6; \overline{1, 5, 1, 12}]$.

e. As in part (a), we find $\sqrt{59} = [7; \overline{1, 2, 7, 2, 1, 14}]$.

f. As in part (a), we find $\sqrt{94} = [9; \overline{1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18}]$.

12.4.2. a. As in Exercise 1, we find $\sqrt{101} = [10; \overline{20}]$.

b. As in Exercise 1, we find $\sqrt{103} = [10; \overline{6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6, 20}]$.

c. As in Exercise 1, we find $\sqrt{107} = [10; \overline{2, 1, 9, 1, 2, 20}]$.

d. As in Exercise 1, we find $\sqrt{201} = [14; \overline{5, 1, 1, 1, 2, 1, 8, 1, 2, 1, 1, 1, 5, 28}]$.

e. As in Exercise 1, we find $\sqrt{203} = [14; \overline{4, 28}]$.

f. As in Exercise 1, we find $\sqrt{209} = [14; \overline{2, 5, 3, 2, 3, 5, 2, 28}]$.

12.4.3. a. As in Exercise 1, we find $1 + \sqrt{101} = [2; \overline{2}]$.

b. As in Exercise 1, we find $(2 + \sqrt{5}/3 = [1; \overline{2, 2, 2, 1, 12, 1}]$.

c. . As in Exercise 1, we find $(5 - \sqrt{7})/4 = [0; 1, 1, \overline{2, 3, 10, 3}]$.

12.4.4. a. $\alpha_0 = (1+\sqrt{3})/2, a_0 = 1, P_0 = 1, Q_0 = 2, P_1 = 1\cdot 2-1 = 1, Q_1 = (3-1^2)/2 = 1, \alpha_1 = (1+\sqrt{3})/1, a_1 = 2, P_2 = 2\cdot 1 - 1 = 1, Q_2 = (3-1^2)/1 = 2, \alpha_2 = \alpha_0$, so $\alpha = [\overline{1; 2}]$.

b. $\alpha_0 = (14 + \sqrt{37})/3, a_0 = 6, P_0 = 14, Q_0 = 3, P_1 = 6\cdot 3 - 14 = 4, Q_1 = (37 - 4^2)/3 = 7, \alpha_1 = (4 + \sqrt{37})/7, a_1 = 1, P_2 = 1\cdot 7 - 4 = 3, Q_2 = (37 - 3^2)/7 = 4, \alpha_2 = (3 + \sqrt{37})/4, a_2 = 2, P_3 = 2\cdot 4 - 3 = 5, Q_3 = (37 - 5^2)/4 = 3, \alpha_3 = (5 + \sqrt{37})/3, a_3 = 3, P_4 = 3\cdot 3 - 5 = 4, Q_4 = (37 - 4^2)/3 = 7, \alpha_4 = (4 + \sqrt{37})/7 = \alpha_1$, so $\alpha = [6; \overline{1, 2, 3}]$.

c. $7 \nmid 2 - 13^2$, so apply Lemma 10.5 to get $\alpha = (-13 + \sqrt{2})/-7 = (-91 + \sqrt{98})/-49$. Then $a_0 = 1, P_0 = -91, Q_0 = -49$, etc. Then $\alpha = [1; 1, 1, \overline{1, 8, 1, 18}]$.

12.4.5. a. Let $x = [2; 1, \overline{5}]$. Then $x = [2; 1, y]$, where $y = [5; \overline{5}]$. Since $y = [5; y]$, we have $y = 5 + 1/y$, so $y^2 - 5y - 1 = 0$, and since y is positive, $y = (5+\sqrt{29})/2$. Then $x = 2 + \dfrac{1}{5 + (1/y)} = (3y+2)/(y+1) = (23 + \sqrt{29})/10$.

b. Let $x = [2; \overline{1, 5}]$, then $x = [2; y]$, where $y = [\overline{1; 5}]$. Then $y = [1; 5, y] = 1+1/(1+1/y)$, so $5y^2 - 5y - 1 = 0$, and y is positive, so $y = (5 + 3\sqrt{5})/10$. Then $x = 2 + 1/y = (-1 + 3\sqrt{5})/2$.

c. Let $x = [\overline{2; 1, 5}]$. Then $x = [2; 1, 5, x] = 2 + \dfrac{1}{1 + 1/(5 + (1/x))} = (17x+3)/(6x+1)$, so $6x^2 - 16x - 3 = 0$. Noting that x is positive gives $x = (8 + \sqrt{82})/6$.

12.4.6. a. Let $y = [3; \overline{3}]$, so that $y = [3, y] = 3 + 1/y$, which simplifies to $y^2 - 3y - 1$, which has one solution greater than 3, namely $y = (3 + \sqrt{13})/2$. Then $[1; 2, \overline{3}] = [1; 2, y] = 1 + 1/(2 + (1/y)) = (5 + \sqrt{13})/6$.

b. Let $y = [2; \overline{3, 2}]$, so that $y = [2; 3, y] = 2 + 1/(3 + (1/y)) = (7y + 2)/(3y + 1)$, which implies that $3y^2 - 6y - 2 = 0$, which has positive solution $y = (3 + \sqrt{15})/3$. Then $[1; \overline{2, 3}] = [1; y] = 1 + 1/y = (-1 + \sqrt{15})/2$.

c. Let $y = [\overline{1; 2, 3}] = 1 + \dfrac{1}{2 + 1/(3 + (1/y))} = (10y + 3)/(7y + 2)$, so that $7y^2 - 8y - 3 = 0$, which has positive solution $y = (4 + \sqrt{37})/7$.

12.4.7. a. From Exercise 8, we have $[3; \overline{6}] = \sqrt{3^2 + 1} = \sqrt{10}$.

b. From Exercise 8, we have $[4; \overline{8}] = \sqrt{4^2 + 1} = \sqrt{17}$.

c. From Exercise 8, we have $[5;\overline{10}] = \sqrt{5^2+1} = \sqrt{26}$.

d. From Exercise 8, we have $[6;\overline{12}] = \sqrt{6^2+1} = \sqrt{37}$.

12.4.8. a. We have $\alpha_0 = \sqrt{d^2+1}, a_0 = [\sqrt{d^2+1}] = d, P_0 = 0, Q_0 = 1, P_1 = d, Q_1 = ((d^2-1)-d_2)/1 = 1, \alpha_1 = d+\sqrt{d^2+1}, a_1 = 2d, P_2 = 2d-d = d, Q_2 = d^2+1-d2 = 1, \alpha_2 = \alpha_1$, so $a_1 = a_2 = \cdots = 2d$. Thus, $\sqrt{d^2+1} = [d;\overline{2d}]$.

b. From part (a), we have $\sqrt{101} = \sqrt{10^2+1} = [10;\overline{20}], \sqrt{290} = \sqrt{17^2+1} = [17;\overline{34}], \sqrt{2210} = \sqrt{47^2+1} = [47;\overline{94}]$.

12.4.9. a. $\alpha_0 = \sqrt{d^2-1}, a_0 = d-1, P_0 = 0, Q_0 = 1, P_1 = (d-1)(1)-0 = d-1, Q_1 = ((d^2-1)-(d-1)^2)/1 = 2d-2, \alpha_1 = (d-1+\sqrt{d^2-1})/(2(d-1)) = 1/2+1/2\sqrt{(d+1)/(d-1)}, a_1 = 1, P_2 = 1(2d-2)-(d-1) = d-1, Q_2 = (d^2-1-(d-1)^2)/(2d-2) = 1, \alpha_2 = (d-1+\sqrt{d^2-1})/1, a_2 = 2d-2, P_3 = 2(d-1)(1)-(d-1) = d-1 = P_1, Q_3 = ((d^2-1)-(d-1)^2)/1 = 2d-2 = Q_1$, so $\alpha = [d-1;\overline{1, 2(d-1)}]$.

b. $\alpha_0 = \sqrt{d^2-d}, a_0 = [\sqrt{d^2-d}] = d-1$, since $(d-1)^2 < d^2-d < d^2$. Then $P_0 = 0, Q_0 = 1, P_1 = d-1, Q_1 = d-1, \alpha_1 = ((d-1)+\sqrt{d^2-d})/(d-1) = 1+\sqrt{d/(d-1)}, a_1 = 2, P_2 = d-1, Q_2 = 1, \alpha_2 = ((d-1)+\sqrt{d^2-d})/1, a_2 = 2(d-1), P_3 = P_1, Q_3 = Q_1$. Therefore,$\sqrt{d^2-d} = [d-1;\overline{2, 2(d-1)}]$.

c. Applying parts(a) and (b) we compute $\sqrt{99} = \sqrt{10^2-1} = [9;\overline{1, 18}], \sqrt{110} = \sqrt{11^2-11} = [10;\overline{2, 20}], \sqrt{272} = \sqrt{17^2-17} = [16;\overline{2, 32}]$, and $\sqrt{600} = \sqrt{25^2-25} = [24;\overline{2, 48}]$.

12.4.10. a. Note that $d-1 < \sqrt{d^2-2} < d$. We compute $\alpha_0 = \sqrt{d^2-2}, a_0 = d-1, P_0 = 0, Q_0 = 1, P_1 = d-1, Q_1 = 2d-3, \alpha_1 = ((d-1)+\sqrt{d^2-2})/(2d-3), ((d-1)+(d-1))/(2d-3) < \alpha_1 < ((d-1)+d)/(2d-3)$, so $a_1 = 1, P_2 = d-2, Q_2 = 2, \alpha_2 = (d-2+\sqrt{d^2-2})/2, a_2 = d-2, P_3 = d-2, Q_3 = 2d-3, \alpha_3 = ((d-2)+\sqrt{d^2-2})/(2d-3), a_3 = 1, P_4 = d-1, Q_4 = 1, \alpha_4 = ((d-1)+\sqrt{d^2-2})/1, a_4 = 2d-2, P_5 = d-1 = P_1, Q_5 = 2d-3 = Q_1$. So $\alpha = [d-1;\overline{1, d-2, 1, 2d-2}]$.

b. Note that $d < \sqrt{d^2+2} < d+1$. We compute $\alpha_0 = \sqrt{d^2+2}, a_0 = d, P_0 = 0, Q_0 = 1, P_1 = d, Q_1 = 2, \alpha_1 = (d+\sqrt{d^2+2})/2, (d+d)/2 < \alpha_1 < (d+d+1)/2, a_0 = d, P_2 = d, Q_2 = 1, \alpha_2 = (d+\sqrt{d^2+2})/1, a_2 = 2d, P_3 = d = P_1, Q_3 = 2 = Q_1$. So $\alpha = [d;\overline{d, 2d}]$.

c. Using parts (a) and (b), we compute $\sqrt{47} = \sqrt{7^2-2} = [6;\overline{1, 5, 1, 12}], \sqrt{51} = \sqrt{7^2+2} = [7;\overline{7, 14}]$, $\sqrt{287} = \sqrt{17^2-2} = [16;\overline{1, 15, 1, 32}]$.

12.4.11. a. Note that $d < \sqrt{d^2+4} < d+1$. We compute $\alpha_0 = \sqrt{d^2+4}, a_0 = d, P_0 = 0, Q_0 = 1, P_1 = d, Q_1 = 4, \alpha_1 = (d+\sqrt{d^2+4})/4, a_1 = [2d/4] = (d-1)/2$, since d is odd. Then, $P_2 = d-2, Q_2 = d, \alpha_2 = (d-2+\sqrt{d^2+4})/d, ((d-2)+d)/d < \alpha_2 < (d-2+d+1)/d$, so $a_2 = 1, P_3 = 2, Q_3 = d, \alpha_3 = (2+\sqrt{d^2+4})/d, a_3 = 1, P_4 = d-2, Q_4 = 4, \alpha_4 = (d-2+\sqrt{d^2+4})/4, (d-2+d)/4 = (d-1)/2 < \alpha_4 < (d-2+d+1)/4$, so $a_4 = (d-1)/2, P_5 = d, Q_5 = 1, \alpha_5 = (d+\sqrt{d^2+4})/1, a_5 = 2d, P_6 = d = P_1, Q_6 = 4 = Q_1$, so $\alpha = [d;\overline{(d-1)/2, 1, 1, (d-1)/2, 2d}]$.

b. Note that $d-1 < \sqrt{d^2-4} < d$. We compute $\alpha_0 = \sqrt{d^2-4}, a_0 = d-1, P_0 = 0, Q_0 = 1, P_1 = d-1, Q_1 = 2d-5, \alpha_1 = (d-1+\sqrt{d^2-4})/(2d-5), (d-1+d-1)/(2d-5) < \alpha_0 < (d-1+d)/(2d-5)$ and $d > 3$ so $a_1 = 1, P_2 = d-4, Q_2 = 4, a_2 = (d-4+\sqrt{d^2-4})/4, a_2 = (d-3)/2, P_3 = d-2, Q_3 = d-2, \alpha_3 = (d-2+\sqrt{d^2-4})/(d-2), a_3 = 2, P_4 = d-2, Q_4 = 4, \alpha_4 = (d-2+\sqrt{d^2-4})/4, a_4 = (d-3)/2, P_5 = d-4, Q_5 = 2d-5, \alpha_5 = (d-4+\sqrt{d^2-4})/(2d-5), a_5 = 1, P_6 = d-1, Q_6 = 1, \alpha_6 = (d-1+\sqrt{d^2-4})/1, a_6 = 2d-2, P_7 = d-1 = P_1, Q_7 = 2d-5 = Q_1$, so $\alpha = [d-1;\overline{1, (d-3)/2, 2, (d-3)/2, 1, 2d-2}]$.

12.4.12. Let $\alpha = \sqrt{a^2+1}$. Then by Exercise 4 part (a), we have $\alpha = [a;\overline{2a}]$, which has period length one. Conversely, suppose the period length of the continued fraction for $\sqrt{d}$ is one, say $\sqrt{d} = [a;\overline{2a}]$, the form required for the square root of an integer. Then $[a;\overline{2a}] = [a;x]$, where $x = [2a;\overline{2a}]$. Then $x = [2a;x] =$

$2a + (1/x)$, and so $x^2 - 2ax - 1 = 0$. Since x is positive, we have $x = a + \sqrt{a^2+1}$. Then $\sqrt{d} = [a; x] = a + (1/x) = \sqrt{a^2+1}$. So $d = a^2 + 1$.

12.4.13. Suppose $\sqrt{d}$ has period length 2. Then $\sqrt{d} = [a; \overline{c, 2a}]$ from the discussion preceding Example 12.16. Then $\sqrt{d} = [a; y]$ with $y = [\overline{c; 2a}] = [c; 2a, y] = c + 1/(2a + (1/y)) = (2acy + c + y)/(2ay + 1)$. Then $2ay^2 - 2acy - c = 0$, and since y is positive, we have $y = (2ac + \sqrt{(2ac)^2 + 4(2a)c})/(4a) = (ac + \sqrt{(ac)^2 + 2ac})/(2a)$. Then $\sqrt{d} = [a; y] = a + (1/y) = a + 2a/(ac + \sqrt{(ac)^2 + 2ac}) = \sqrt{a^2 + 2a/c}$, so $d = a^2 + 2a/c$, and $b = 2a/c$ is an integral divisor of $2a$. Conversely, let $\alpha = \sqrt{a^2 + b}$ and $b|2a$, say $kb = 2a$. Then $a_0 = [\sqrt{a^2 + b}] = a$, since $(a^2 < a^2 + b < (a + 1)^2$. Then $P_0 = 0, Q_0 = 1, P_1 = a, Q_1 = b, \alpha_1 = (a + \sqrt{a^2 + b})/b, a_1 = 4k, P_2 = a, Q_2 = 1, \alpha_2 = (a + \sqrt{a^2 + b})/1, a_2 = 2a, P_3 = a = P_1, Q_3 = b = Q_1$, so $\alpha = [a; \overline{4k, 2a}]$, which has period length 2.

12.4.14. a. We have $(\alpha_1+\alpha_2)' = ((a_1+b_1\sqrt{d})/c_1+(a_2+b_2\sqrt{d})/c_2)' = (((a_1c_2+a_2c_1)+(b_1c_2+b_2c_1)\sqrt{d})/c_1c_2)' = ((a_1c_2 + a_2c_1) - (b_1c_2 + b_2c_1)\sqrt{d})/c_1c_2 = (a_1 - b_1\sqrt{d})/c_1 + (a_2 - b_2\sqrt{d})/c_2 = \alpha_1' + \alpha_2'$.

b. We have $(\alpha_1-\alpha_2)' = ((a_1+b_1\sqrt{d})/c_1-(a_2+b_2\sqrt{d})/c_2)' = (((a_1c_2-a_2c_1)+(b_1c_2-b_2c_1)\sqrt{d})/c_1c_2)' = ((a_1c_2 - a_2c_1) - (b_1c_2 - b_2c_1)\sqrt{d})/c_1c_2 = (a_1 - b_1\sqrt{d})/c_1 - (a_2 - b_2\sqrt{d})/c_2 = \alpha_1' - \alpha_2'$.

c. $(\alpha_1\alpha 2)' = ((a_1+b_1\sqrt{d})/c_1 \cdot (a_2+b_2\sqrt{d})/c_2)' = (((a_1a_2+b_1b_2d)+(a_1b_2+a_2b_1)\sqrt{d})/c_1c_2)' = ((a_1a_2 + b_1b_2d) - (a_1b_2 + a_2b_1)\sqrt{d})/c_1c_2 = (a_1(a_2 - b_2\sqrt{d}) - b_1\sqrt{d}(a_2 - b_2\sqrt{d}))/c_1c_2 = (a_1 - b_1\sqrt{d})/c_1 \cdot (a_2 - b_2\sqrt{d})/c_2 = \alpha_1'\alpha_2'$.

12.4.15. a. We have $1 + \sqrt{5} > 1$, but $(1 + \sqrt{5})' = 1 - \sqrt{5} < -1$. Hence, by Theorem 12.21, the continued fraction of $1 + \sqrt{5}$ is not purely periodic.

b. We have $2 + \sqrt{8} > 1$ and $-1 < (2 + \sqrt{8})' = 2 - \sqrt{8} < 0$, so by Theorem 12.21 the continued fraction expansion of $2 + \sqrt{8}$ is purely periodic.

c. We have $4 + \sqrt{17} > 1$ and $-1 < (4 + \sqrt{17})' = 4 - \sqrt{17} < 0$, so by Theorem 12.21 the continued fraction expansion of $4 + \sqrt{17}$ is purely periodic.

d. We have $(11 - \sqrt{10})/9 < 1$, so by Theorem 12.21, the continued fraction expansion of $(11 - \sqrt{10})/9$ is not purely periodic.

e. We have $(3 + \sqrt{23})/2 > 1$ and $-1 < ((3 + \sqrt{23})/2)' = (3 - \sqrt{23})/2 < 0$, so by Theorem 12.21 the continued fraction expansion of $(3 + \sqrt{23})/2$ is purely periodic.

f. We have $(17 + \sqrt{188})/3 > 1$ but $((17 + \sqrt{188})/3)' = (17 - \sqrt{188})/3 > 0$, so by Theorem 12.21 the continued fraction expansion of $(17 + \sqrt{188})/3$ is not purely periodic.

12.4.16. If $\alpha = (a + \sqrt{b})/c$ is reduced, then $1 < (a + \sqrt{b})/c$ and $-1 < (a - \sqrt{b})/c < 0$. Adding the first two inequalities gives $0 < 2a/c$, so a and c have the same sign. If they were both negative, then $(a - \sqrt{b})/c$ would be positive, contrary to assumption, so a and c are both positive. Then $1 < (a + \sqrt{b})/c$ implies $c < a + \sqrt{b}$. Also $-1 < (a - \sqrt{b})/c < 0$ implies $c > \sqrt{b} - a > 0$. This gives us all the desired inequalities. The converse is proved by reversing these steps.

12.4.17. Let $\alpha = (a+\sqrt{b})/c$. Then $-1/\alpha' = -(c)/(a-\sqrt{b}) = (ca+\sqrt{bc^2})/(b-a^2) = (A+\sqrt{B})/C$, say. By Exercise 16, $0 < a < \sqrt{b}$ and $\sqrt{b} - a < c < \sqrt{b} + a < 2\sqrt{b}$. Multiplying by c gives $0 < ca < \sqrt{bc^2}$ and $\sqrt{bc^2} - ca < c^2 < \sqrt{bc^2} + ca < 2\sqrt{bc^2}$. That is, $0 < A < \sqrt{B}$ and $\sqrt{B} - A < c^2 < \sqrt{B} + A < 2\sqrt{B}$. Multiply $\sqrt{b} - a < c$ by $\sqrt{b} + a$ to get $C = b - a^2 < \sqrt{bc^2} + ca = A + \sqrt{B}$. Multiply $c < \sqrt{b} + a$ by $\sqrt{b} - a$ to get $\sqrt{B} - A = \sqrt{bc^2} - ac < b - a^2 = C$. So, $-1/\alpha'$ satisfies all the inequalities in Exercise 16, and therefore is reduced.

12.4.18. If $y = [2A; 2, \ldots, 2, y]$ with k 2's (A an integer > 1), then the simple continued fraction for y has period $k + 1$. Now prove by induction that $[0; 2, \ldots, 2, y] = (a_{k-1}y + a_{k-2})/(a_k y + a_{k-1})$. For the basis steps,

$k = 1$ and 2, take $a_0 = 1$ and $a_{-1} = 0$. Thus y satisfies the equation $y - 2A = (a_{k-1}y + a_{k-2})/(a_k y + a_{k-1})$ which simplifies to $a_k y^2 - 2Aa_k y = 2Aa_{k-1} + a_{k-2}$. Define B by $2Aa_{k-1} + a_{k-2} = Ba_k$. Then $y^2 - 2Ay = B$ or $(y - A)^2 = A^2 + B$. Thus if B is a positive integer, $D = B + A^2$, and we have $\sqrt{D} = y - A = [A; 2, ..., 2, y]$ with a simple continued fraction expansion of period $k + 1$. Now using $a_{k-2} = a_k - 2a_{k-1}$, the equation above becomes $2(A - 1)a_{k-1} = (B - 1)a_k$. This must be an integer divisible by $2a_k$ and a_{k-1}, so let it be $2ta_{k-1}a_k$, where t is a positive integer. Then we have a solution with $B = 1 + 2ta_{k-1}$, $A = 1 + ta_k$, and $D = (1 + ta_k)^2 + 1 + 2ta_{k-1}$. This completes the proof.

12.4.19. Start with $\alpha_0 = \sqrt{D_k} + 3^k + 1$ (this will have the same period since it differs from $\sqrt{D_k}$ by an integer) and use induction. Apply the continued fraction algorithm to show $\alpha_{3i} = \sqrt{D_k} + 3^k - 2 \cdot 3^{k-i} + 2/(2 \cdot 3^{k-i})$, $i = 1, 2, \ldots, k$, but $\alpha_{3k+3i} = \sqrt{D_k} + 3^k - 2/(2 \cdot 3^i)$, $i = 1, 2, \ldots, k - 1$, and $\alpha_{6k} = \sqrt{D_k} + 3^k + 1 = \alpha_0$. Since $\alpha_i \neq \alpha_0$ for $i < 6k$ we see that the period is $6k$.

12.5. Factoring Using Continued Fractions

12.5.1. We have $19^2 - 2^2 = (19 - 2)(19 + 2) \equiv 0 \pmod{119}$. Then $(19 - 2, 119) = (17, 119) = 17$ and $(19 + 2, 119) = (21, 119) = 7$ are factors of 119.

12.5.2. In expanding the continued fraction of $\sqrt{1537}$, we have $P_0 = 0, Q_0 = 1, a_0 = 39, P_1 = 39, Q_1 = 16, a_1 = 4, P_2 = 25, Q_2 = 57, a_2 = 1, P_3 = 32, Q_3 = 9, a_3 = 7$, and $P_4 = 31, Q_4 = 64, a_4 = 1$. Since $Q_4 = 8^2$ is a square and has even index, we examine the congruence $p_3^2 \equiv Q_4 \pmod{1537}$. The third convergent of $\sqrt{1537}$ is $\frac{1529}{39}$, so $p_3 = 1529$ and the congruence is $1529^2 \equiv 8^2 \pmod{1537}$. This implies that $(1529^2 - 8^2) = (1529 - 8)(1529 + 8) \equiv 0 \pmod{1537}$, which does not lead to a factor, since $1529 + 8 = 1537$, so we continue: $P_5 = 33, Q_5 = 7, a_5 = 10, P_6 = 37, Q_6 = 24, a_6 = 3, P_7 = 35, Q_7 = 13, a_7 = 5, P_8 = 30, Q_8 = 49, a_8 = 7$. Since $Q_8 = 7^2$ is a square and has even index, we examine the congruence$p_7^2 \equiv Q_8 \pmod{1537}$. Since $p_7 = 309089$, we have $309089^2 \equiv 7^2 \pmod{1537}$, which implies that $(309089^2 - 7^2) = (309089 - 7)(309089 + 7) \equiv 0 \pmod{1537}$. Then we find that $(309089 - 7, 1537) = 309082, 1537) = 29$ and $(309089 + 7, 1537) = (309096, 1537) = 53$ are factors of 1537.

12.5.3. Using a computer to generate lists $[k, \alpha_k, a_k, P_k, Q_k, \sqrt{Q_k}]$, we have $[1, \sqrt{13290059}, 3645, 0, 1, 1]$, $[2, (3645 + \sqrt{13290059})/4034, 1, 3645, 4034, \sqrt{4034}]$, $[3, (389 + \sqrt{13290059})/3257, 1, 389, 3257, \sqrt{3257}]$, $[4, (2868 + \sqrt{13290059})/1555, 4, 2868, 1555, \sqrt{1555}]$, $[5, (3352 + \sqrt{13290059})/1321, 5, 3352, 1321, \sqrt{1321}]$, $[6, (3253 + \sqrt{13290059})/2050, 3, 3253, 2050, 5\sqrt{82}]$, $[7, (2897 + \sqrt{13290059})/2389, 2, 2897, 2389, \sqrt{2389}]$, $[8, (1881 + \sqrt{13290059})/4082, 1, 1881, 4082, \sqrt{4082}]$, $[9, (2201 + \sqrt{13290059})/2069, 2, 2201, 2069, \sqrt{2069}]$, $[10, (1937 + \sqrt{13290059})/4610, 1, 1937, 4610, \sqrt{4610}]$, $[11, (2673 + \sqrt{13290059})/1333, 4, 2673, 1333, \sqrt{1333}]$, $[12, (2659 + \sqrt{13290059})/4666, 1, 2659, 4666, \sqrt{4666}]$, $[13, (2007 + \sqrt{13290059})/1985, 2, 2007, 1985, \sqrt{1985}]$, $[14, (1963 + \sqrt{13290059})/4754, 1, 1963, 4754, \sqrt{4754}]$, $[15, (2791 + \sqrt{13290059})/1157, 5, 2791, 1157, \sqrt{1157}]$, $[16, (2994 + \sqrt{13290059})/3739, 1, 2994, 3739, \sqrt{3739}]$, $[17, (745 + \sqrt{13290059})/3406, 1, 745, 3406, \sqrt{3406}]$, $[18, (2661 + \sqrt{13290059})/1823, 3, 2661, 1823, \sqrt{1823}]$, $[19, (2808 + \sqrt{13290059})/2965, 2, 2808, 2965, \sqrt{2965}]$, $[20, (3122 + \sqrt{13290059})/1195, 5, 3122, 1195, \sqrt{1195}]$, $[21, (2853 + \sqrt{13290059})/4310, 1, 2853, 4310, \sqrt{4310}]$, $[22, (1457 + \sqrt{13290059})/2591, 1, 1457, 2591, \sqrt{2591}]$, $[23, (1134 + \sqrt{13290059})/4633, 1, 1134, 4633, \sqrt{4633}]$, $[24, (3499 + \sqrt{13290059})/226, 31, 3499, 226, \sqrt{226}]$, $[25, (3507 + \sqrt{13290059})/4385, 1, 3507, 4385, \sqrt{4385}]$, $[26, (878 + \sqrt{13290059})/2855, 1, 878, 2855, \sqrt{2855}]$, $[27, (1977 + \sqrt{13290059})/3286, 1, 1977, 3286, \sqrt{3286}]$, $[28, (1309 + \sqrt{13290059})/3523, 1, 1309, 3523, \sqrt{3523}]$, $[29, (2214 + \sqrt{13290059})/2381, 2, 2214, 2381, \sqrt{2381}]$, $[30, (2548 + \sqrt{13290059})/2855, 2, 2548, 2855, \sqrt{2855}]$, $[31, (3162 + \sqrt{13290059})/1153, 5, 3162, 1153, \sqrt{1153}]$, $[32, (2603 + \sqrt{13290059})/5650, 1, 2603, 5650, 5\sqrt{226}]$, $[33, (3047 + \sqrt{13290059})/709, 9, 3047, 709, \sqrt{709}]$, $[34, (3334 + \sqrt{13290059})/3067, 2, 3334, 3067, \sqrt{3067}]$, $[35, (2800 + \sqrt{13290059})/1777, 3, 2800, 1777, \sqrt{1777}]$, $[36, (2531 + \sqrt{13290059})/3874, 1, 2531, 3874, \sqrt{3874}]$, $[37, (1343 + \sqrt{13290059})/2965, 1, 1343, 2965, \sqrt{2965}]$, $[38, (1622 + \sqrt{13290059})/3595, 1, 1622, 3595, \sqrt{3595}]$, $[39, (1973 + \sqrt{13290059})/2614, 2, 1973, 2614, \sqrt{2614}]$, $[40, (3255 + \sqrt{13290059})/1031, 6, 3255, 1031, \sqrt{1031}]$, $[41, (2931 + \sqrt{13290059})/4558, 1, 2931, 4558, \sqrt{4558}]$, $[42, (1627 + \sqrt{13290059})/2335, 2, 1627, 2335, \sqrt{2335}]$, $[43, (3043 + \sqrt{13290059})/1726, 3, 3043, 1726, \sqrt{1726}]$, $[44, (2135 + \sqrt{13290059})/5059, 1, 2135, 5059, \sqrt{5059}]$, $[45, (2924 + \sqrt{13290059})/937, 7, 2924, 937, \sqrt{937}]$, $[46, (3635 + \sqrt{13290059})/82, 88, 3635, 82, \sqrt{82}]$, $[47, (3581 + \sqrt{13290059})/5689, 1, 3581, 5689, \sqrt{5689}]$, $[48, (2108 + \sqrt{13290059})/1555, 3, 2108, 1555, \sqrt{1555}]$, $[49, (2557 + \sqrt{13290059})/4342, 1, 2557, 4342, \sqrt{4342}]$,

$[50, (1785+\sqrt{13290059})/2327, 2, 1785, 2327, \sqrt{2327}]$, $[51, (2869+\sqrt{13290059})/2174, 2, 2869, 2174, \sqrt{2174}]$, $[52, (1479+\sqrt{13290059})/5107, 1, 1479, 5107, \sqrt{5107}]$, $[53, (3628+\sqrt{13290059})/25, 290, 3628, 25, 5]$. So we have $Q_{53} = 5^2$. Using a computer again, we find that $p_{52} = 352701086822481292500 2106 \equiv 2467124 \pmod{13290059}$. Then $(13290059, 2467124-5) = 4261$ and $(13290059, 2467124+5) = 3119$, so we have $13290059 = 3119 \cdot 4261$.

12.5.4. First, $x^2 = \prod_{j=1}^{r} x_j^2 \equiv \prod_{j=1}^{r} \left((-1)^{e_{0j}} \prod_{k=1}^{m} p_k^{e_{kj}}\right) \equiv (-1)^{e_{01}+e_{02}+\cdots+e_{0r}} \prod_{k=1}^{m} p_k^{e_{k1}+e_{k2}+\cdots+e_{kr}} \equiv (-1)^{2e_0} \prod_{k=1}^{m} p_k^{2e_k} \equiv y^2 \pmod{n}$. Once $x_1, x_2, \ldots, x_r$ have been found satisfying the r congruences and the m equations, we can form a solution to $x^2 \equiv y^2 \pmod{n}$ and finish the factorization process as in the continued fraction method.

12.5.5. We have $17^2 = 289 \equiv 3 \pmod{143}$ and $19^2 = 361 \equiv 3 \cdot 5^2 \pmod{143}$. Combining these, we have $(17 \cdot 19)^2 \equiv 3^2 5^2 \pmod{143}$. Hence, $323^2 \equiv 15^2 \pmod{143}$. It follows that $323^2 - 15^2 = (323-15)(323+15) \equiv 0 \pmod{143}$. This produces the two factors $(323-15, 143) = (308, 143) = 11$ and $(323+15, 143) = (338, 143) = 13$ of 143.

12.5.6. We have a notational problem, since p_i is used for two things. Let p_1 etc. stand for the primes. Let π_k stand for the "p_k's" in the continued fraction development, as in Theorem 12.22. Then from Theorem 12.22, we have $\pi_k^2 \equiv (-1)^{k-1} Q_{k+1} \pmod{n}$, for all k. Then $\pi_{k_i-1} \equiv (-1)^{k_i-2} \prod_{j=1}^{r} p_j^{k_{ij}} \pmod{n}$. Then $\left(\prod_{i=1}^{t} \pi_{k_i-1}\right)^2 \equiv \prod_{i=1}^{t} \pi_{k_i-1}^2 \equiv \prod_{i=1}^{t} \left((-1)^{k_i-2} \prod_{j=1}^{r} p_j^{k_{ij}}\right) \equiv (-1)^{\sum_{i=1}^{t} k_i = 2} \prod_{i=1}^{t} \prod_{j=1}^{r} p_j^{k_{ij}} \pmod{n}$ Since $\sum_{i=1}^{t} k_i = 2$ is even we have the last term congruent to $\prod_{i=1}^{t} \prod_{j=1}^{r} p_j^{k_{ij}} \equiv \prod_{j=1}^{r} p_j^{\sum_{i=1}^{t} k_{ij}} \equiv \prod_{j=1}^{r} p_j^{w} \pmod{n}$ where w is even. Therefore, this last term is a perfect square, say m^2, and the very first expression is also a square, say P^2. Then we have $P^2 \equiv m^2 \pmod{n}$ and now we may proceed as in Example 12.17.

12.5.7. We use a computer to find $p_0 \equiv 3465, p_{11} \equiv 1211442, p_{27} \equiv 6764708, p_{33} \equiv 6363593$, and $p_{40} \equiv 8464787 \pmod{12007001}$. The product of these reduces to $P \equiv 9815310 \pmod{12007001}$. Then $Q = \sqrt{Q_1 Q_{12} Q_{28} Q_{34} Q_{40}} = 1247455$. Then the factors of 12007001 are $(12007001, P-Q) = 3001$ and $(12007001, P+Q) = 4001$.

12.5.8. We compute Q_i until we have a subset which has only prime factors of 2,3, and 5, each occurring an even number of times, in total. We find $Q_4 = 720 = 2^4 3^2 5$ and $Q_{10} = 405 = 3^4 5$. Further $p_4 = 750943$ and $p_{10} = 3143053051$. Then, following Exercise 6, $(750943^2 \cdot 3143053051^2 \equiv 720 \cdot 405 \equiv 540^2 \pmod{197209})$. Then $(750943 \cdot 3143053051 - 540, 197209) = 199$, and $(750943 \cdot 3143053051 + 540, 197209) = 991$, which gives us $197209 = 199 \cdot 991$.

CHAPTER 13

Some Nonlinear Diophantine Equations

13.1. Pythagorean Triples

13.1.1. a. Since $z = m^2 + n^2 \leq 40$, we have $m \leq 6$. The triples we seek are those in Table 13.1 with $z \leq 40$: (3,4,5), (5,12,13), (15,8,17), (7,24,25), (21,20,29), and (35,12,37).

b. These would be triples which are multiples of the primitive triples. In addition to those in part (a), we have (6,8,10), (9,12,15), (12,16,20), (15,20,25), (18,24,30), (21,28,35), (24,32,40), (10,24,26), (15,36,39), and (30,16,34).

13.1.2. If $3 \nmid x$ or y, then $x^2 \equiv y^2 \equiv 1 \pmod 3$. But then $z^2 \equiv 1 + 1 \equiv 2 \pmod 3$ which is impossible.

13.1.3. By Lemma 13.1, 5 divides at most one of x, y, and z. If $5 \nmid x$ or y, then $x^2 \equiv \pm 1 \pmod 5$ and $y^2 \equiv \pm 1 \pmod 5$. Then, $z^2 \equiv 0, 2$, or $-2 \pmod 5$. But ± 2 is not a quadratic residue modulo 5, so $z^2 \equiv 0 \pmod 5$, whence $5 \mid z$.

13.1.4. From Theorem 13.1, one of m and n must be even, so $2 \mid mn$. Therefore, $4 \mid 2mn = y$.

13.1.5. Let k be an integer ≥ 3. If $k = 2n + 1$, let $m = n + 1$. Then m and n have opposite parity, $m > n$ and $m^2 - n^2 = 2n + 1 = k$, so m and n define the desired triple. If k has an odd divisor $d > 1$, then use the construction above for d and multiply the result by k/d. If k has no odd divisors, then $k = 2^j$ for some integer $j > 1$. Let $m = 2^{j-1}$ and $n = 1$. Then $k = 2mn$, $m > n$, and m and n have opposite parity, so m and n define the desired triple.

13.1.6. Proceed by induction. The basis step is $x_1^2 = y_1^2 = 3^2 + 4^2 = 5^2 = z_1^2$. Assume that x_n, y_n, z_n is a Pythagorean triple. Then

$$\begin{aligned} x_{n+1}^2 + y_{n+1}^2 &= (3x_n + 2z_n + 1)^2 + (3x_n + 2z_n + 2)^2 \\ &= 18x_n^2 + 8z_n^2 + 5 + 24x_n y_n + 18x_n + 12z_n \\ &= (16x_n^2 + 9z_n^2 + 24x_n y_n + 16x_n + 12z_n + 4) \\ &\quad +(2x_n^2 - z_n^2 + 2x_n + 1) \\ &= z_{n+1}^2 + (x_n^2 + 2x_n + (x_n^2 - z_n^2)) \\ &= z_{n+1}^2 + (x_n + 1)^2 - y_n^2 \\ &= z_{n+1}^2 \end{aligned}$$

which completes the induction step.

13.1.7. Substituting $y = x+1$ into the Pythagorean equation gives us $2x^2+2x+1 = z^2$, which is equivalent to $m^2 - 2z^2 = -1$ where $m = 2x + 1$. Dividing by z^2 yields $m^2/z^2 - 2 = -1/z^2$. Note that $m/z \geq 1$, $1/z^2 = 2 - m^2/z^2 = (\sqrt{2} + m/z)(\sqrt{2} - m/z) < 2(\sqrt{2} - m/z)$. So by Theorem 12.18, m/z must be a convergent of the continued fraction expansion of $\sqrt{2}$. Further, by the proof of Theorem 12.13, it must be one of the even-subscripted convergents. Therefore each solution is given by the recurrence $m_{n+1} = 3m_n + 2z_n$, $z_{n+1} = 2m_n + 3m_n$. (See, e.g., Theorem 13.11.) Substituting x back in yields the recurrences of Exercise 6.

13.1.8. $2y^2 = z^2 - x^2 = (z - x)(z + x)$. x and y have the same parity, so $(z - x)/2$ and $(z + x)/2$ are integers. It suffices to assume $(x, z) = 1$. Then either $((z - x)/2, z + x) = 1$, and then $y^2 = ((z - x)/2)(z + x) = m^2n^2$, and solving $(z - x)/2 = m^2$ and $z + x = n^2$ for x and y gives $x = (m^2 - 2n^2)/2, y = mn, z =$

$(m^2+2n^2)/2$. Or $((z+x)/2, z-x) = 1$ which gives $x = (2m^2-n^2)/2, y = mn, z = (2m^2+n^2)/2$.

13.1.9. See Exercise 15 with $p = 3$.

13.1.10. All primitive solutions are given as follows: Let r, s, t be arbitrary integers, with $(r, s, t) = 1$. Then let $x_0 = 2rt, y_0 = 2st, z_0 = t^2 - r^2 - s^2$, and $w_0 = t^2 + r^2 + s^2$. Let $d = (x_0, y_0, z_0, w_0)$. Then $x = x_0/d, y = y_0/d, z = z_0/d, w = w_0/d$ is a primitive solution.

13.1.11. We must find all primitive triples containing a divisor of 12: 2, 3, 4, 6, or 12. Such a triple must have $x = m^2 - n^2, y = 2mn, z = m^2 + n^2$, and $(m, n) = 1$. So only y is even. If $y = 2mn = 2$, then $m = n = 1$, and $x = 0$, which is not allowed. If $y = 2mn = 4$, then $m = 2$, and $n = 1$, so $x = 3$ and $z = 5$. If $y = 6 = 2mn$ then $m = 3, n = 1$, which are not of opposite parity. If $y = 12 = 2mn$, then either $m = 6, n = 1, x = 35$, and $z = 37$; or $m = 3, n = 2, x = 5$, and $z = 13$. Now $z \neq 3$ since 9 is not the sum of two squares. If $x = 3 = m^2 - n^2 = (m+n)(m-n)$, then $m = 2, n = 1, y = 4$, and $z = 5$. Multiples of these triples containing 12 are (9,12,15), (35,12,37), (5,12,13), and (12,16,20).

13.1.12. Let m be odd. Then all solutions are given by $x = m, y = (x^2-1)/2, z = (x^2+1)/2$.

13.1.13. If m is positive, then all solutions are given by $x = 2m, y = m^2 - 1, z = m^2 + 1$.

13.1.14. Suppose x is odd and has prime factorization $x = p_1^{a_1} \cdots p_r^{a_r}$. If x is part a Pythagorean triple, then it can be factored as $x = def$ where f is the greatest common divisor of x, y, and $z, d = m - n$, and $e = m + n$, so that $de = m^2 - n^2$ where m and n are given by Theorem 13.1. We need to count the number of such factorizations. Since $(d, e) = 1$, a prime factor p_i of x can only divide one of d and e. Thus, there are $2a_i + 1$ ways that the a_i factors of p_i can be distributed among d, e, and f, namely, either $0, 1, 2, \ldots$, or r of them divide d and the rest divide f or $0, 1, 2, \ldots$, or r of them divide e and the rest divide f. This gives $(2a_1 + 1)(2a_2 + 1) \cdots (2a_r + 1) = \tau(x^2)$ ways, except, we can not have $f = x$, and if $d > e$ then d and e reverse roles, so we have $(\tau(x^2) - 1)/2$ different ways. The argument for x even is similar.

13.1.15. Check that if $m > \sqrt{p}n$ then $x = (m^2 - pn^2)/2, y = mn, z = (m^2 + pn^2)/2$ is a solution. Conversely, if x, y, z is a primitive solution, then $y^2 = (z^2 - x^2)/p$, so $p \mid (z \pm x)$. Take $m^2 = z \mp x$ and $n^2 = (z \pm x)/p$.

13.1.16. Rewrite the equation as $x^2 + y^2 = (xy/z)^2$. Then xy/z must be an integer and from Theorem 13.1, we have $x = m^2 - n^2, y = 2mn$, and $xy/z = m^2 + n^2$, for some integers m and n. Then $z = 2mn(m^2 - n^2)/(m^2 + n^2)$, but to ensure that z in an integer, we multiply x, y, and z by $(m^2 + n^2)$ and get $x = (m^2 - n^2)(m^2 + n^2) = (m^4 - n^4), y = 2mn(m^2 + n^2)$, and $z = 2mn(m^2 - n^2)$. This is the form of every solution.

13.1.17. Substituting $f_n = f_{n+2} - f_{n+1}$ and $f_{n+3} = f_{n+2} + f_{n+1}$ into $(f_n f_{n+3})^2 + (2f_{n+1}f_{n+2})^2$ yields $(f_{n+2} - f_{n+1})^2(f_{n+2}+f_{n+1})^2+4f_{n+1}^2f_{n+2}^2 = (f_{n+2}^2-f_{n+1}^2)^2+4f_{n+1}^2f_{n+2}^2 = f_{n+2}^4-2f_{n+1}^2f_{n+2}^2+f_{n+1}^4+4f_{n+1}^2f_{n+2}^2 = f_{n+2}^4 + 2f_{n+1}^2f_{n+2}^2 + f_{n+1}^4 = (f_{n+2}^2 + f_{n+1}^2)^2$, which proves the result.

13.1.18. Let x, y, and z be the sides of such a triangle. Then (x, y, z) is a Pythagorean triple and there must be integers (m, n) such that $x = m^2 - n^2$, $y = 2mn$ and $z = m^2 + n^2$. Since the triangle is a right triangle with legs x and y, its area is $xy/2$. If the area equals the perimeter, we have $x + y + z = xy/2$. Substituting the above relations gives us $(m^2 - n^2) + 2mn + (m^2 + n^2) = (m^2 - n^2)2mn/2$. Simplifying and dividing through by m gives us $2m + 2n = (m^2 - n^2)n$. We factor both sides and divide by $m + n$ to get $2 = (m - n)n$, which tells us that $n = 1$ or 2. If $n = 1$, then $m - n = 2$ and so $m = 3$, which implies that $(x, y, z) = (8, 6, 10)$. If $n = 2$, then $m - n = 1$ and so $m = 3$, which implies that $(x, y, z) = (5, 12, 13)$ and these are the only solutions.

13.2. Fermat's Last Theorem

13.2.1. Assume without loss of generality that $x < y$. Then $x^n + y^n = x^2x^{n-2} + y^2y^{n-2} < (x^2 + y^2)y^{n-2} = z^2y^{n-2} < z^2z^{n-2} = z^n$.

13.2.2. Let $n \geq 3$ be an integer and x, y, z be a solution to $x^n + y^n = z^n$. If n has an odd prime factor p, say $n = pk$, then we have $(x^k)^p + (y^k)^p = (z^k)^p$, so x^k, y^k, z^k is a solution to $x^p + y^p = x^p$, a contradiction. If n has no odd factor, then n is a power of 2. Since $n > 2, 4 \mid n$, say $n = 4k$. Now 4 plays the role of p above to give a solution to $x^4 + y^4 = z^4$, also a contradiction.

13.2.3. a. If $p \mid x, y$, or z, then certainly $p \mid xyz$. If not, then by Fermat's Little Theorem, $x^{p-1} \equiv y^{p-1} \equiv z^{p-1} \equiv 1 \pmod{p}$. Hence, $1 + 1 \equiv 1 \pmod{p}$, which is impossible.

b. We know $a^p \equiv a \pmod{p}$ for every integer a. Then $x^p + y^p \equiv z^p \pmod{p}$ implies $x + y \equiv z \pmod{p}$, so $p \mid x + y - z$.

13.2.4. We have $z^2 + (y^2)^2 = (x^2)^2$. If $y^2 = 2mn$, then $m = u^2$, and $n = 2v^2$. Then $z^2 = m^2 - n^2 = u^4 - v^4$, and $u^4 = m^2 < m^2 + n^2 < x^2 < x^4$, so we have a smaller positive solution. If $y^2 = m^2 - n^2$, $z = 2mn$, and $x^2 = m^2 + n^2$, then $x^2y^2 = m^4 - n^4$ which is a smaller solution, since $m^2 < x^2$.

13.2.5. Let x and y be the lengths of the legs and z be the hypotenuse. Then $x^2 + y^2 = z^2$. If the area is a perfect square, we have $A = \frac{1}{2}xy = r^2$. Then, if $x = m^2 - n^2$, and $y = 2mn$, we have $r^2 = mn(m^2 - n^2)$. All of these factors are relatively prime, so $m = a^2$, $n = b^2$, and $m^2 - n^2 = c^2$, say. Then, $a^4 - b^4 = c^2$, which contradicts Exercise 4.

13.2.6. It suffices to take x and z odd. We have $(x^2)^2 + (2y^2)^2 = z^2$. Then $x^2 = m^2 - n^2$ and $2y^2 = 2mn$, so $m = r^2$ and $n = s^2$. Then, $x^2 = r^4 - s^4$ which contradicts Exercise 4.

13.2.7. We use the method of infinite descent. Assume there is a nonzero solution with where $|x|$ is minimal. Then $(x, y) = 1$. Also x and z cannot both be even, because then y would be odd and then $z^2 \equiv 8 \pmod{18}$, but 8 is not a quadratic residue modulo 16. Therefore x and z are both odd, since $8y^4$ is even. From here it is easy to check that $(x, z) = 1$. We may also assume (by negating if necessary) that $x \equiv 1 \pmod{4}$ and $z \equiv 3 \pmod{4}$. Clearly $x^2 > |z|$. We have $8y^4 = x^4 - z^2 = (x^2 - z)(x^2 + z)$. Since $z \equiv 3 \pmod{4}$, we have $x^2 - z \equiv 2 \pmod{4}$, so $m = (x^2 - z)/2$ is odd, and $n = (x^2 + z)/4$ is an integer. Since no odd prime can divide both m and n, we have $(m, n) = 1$, $m, n > 0$ and $mn = y^4$, whence $m = r^4$ and $n = s^4$, with $(r, s) = 1$. So now $r^4 + 2s^4 = m + 2n = x^2$. This implies $(x, r) = 1$, since no odd prime divides r and x but not s, and r and x are both odd. Also, $|x| > r^2 > 0$. Now consider $2s^4 = (x^2 - r^4) = (x - r^2)(x + r^2)$. Then, s must be even since a difference of squares is not congruent to 2 (mod 4), so $s = 2t$ and $32t^4 = (x - r^2)(x + r^2)$. Recalling $x \equiv 1 \pmod{4}$ and r is odd, we have $U = (x + r^2)/2$ is odd and $V = (x - r^2)/16$ is an integer. Again $(U, V) = 1$ and $UV = t^4$, but we don't know the sign of x. So $U = \pm u^4$ and $V = \pm v^4$, depending on the sign of x. Now $r^2 = \pm(u^4 - 8v^4)$. But since u is odd, the sign can't be $-$ (or else $r^2 \equiv 7 \pmod{8}$.) So the sign is $+$ (hence x is positive), and we have $u^4 - 8v^4 = r^2$. Finally, $|v| > 0$ because $|x + r^2| > 0$. So we haven't reduced to a trivial case. Then, $u^4 = U < |x + r^2|/2 < x$, so $|u| < x$, and so $|x|$ was not minimal. This contradiction shows that there are no nontrivial solutions.

13.2.8. For the basis step, note that $f_2/f_1 = 1$. Suppose that $f_k/f_{k-1} = [1; 1, 1, \ldots, 1]$, where there are $k - 1$ 1s in the continued fraction. Then we have $f_{k+1}/f_k = (f_k + f_{k-1})/f_k = 1 + 1/\frac{f_k}{f_{k-1}}$. Using Exercise 7 of Section 10.3, we have that $f_{k+1}/f_k = [1; 1, 1, \ldots, 1]$, where there are k 1s in the continued fraction.

13.2.9. (Solution by John R. Ramsden.) First we mimic the construction of the solutions to the Pythagorean equation to solve the 2nd-order diophantine equation $x^2 + 3y^2 = z^2$ and find that all solutions are given by $\pm 2^e x = m^2 - 3n^2$, $2^e y = 2mn$, $2^e = m^2 + 3n^2$ for relatively prime integers m and n, where (i) x and y are odd and z is even if and only if $e = 1$ and m and n are both odd, and (ii) x and z are odd and y is even if and only if $e = 0$ and m and n have opposite parity.

Now consider the diophantine equation $x^4 + 3y^4 = z^4$ modulo 8. If x and y are odd and z is even, we get a contradiction, so case (i) above doesn't happen for the 4th-order specialization. Assume we have a nontrivial solution with x, y and z pairwise relatively prime and z as small as possible. Then as above, since we must be in case (ii), $x^2 = \pm(m^2 - 3n^2)$, $y^2 = 2mn$ and $z^2 = m^2 + 3n^2$. If m were even and n odd, then m^2 would be divisible by 4 and $n^2 \equiv 1 \pmod{4}$, so that $z^2 = m^2 + 3n^2 \equiv 3 \pmod{4}$ a contradiction, therefore m is odd and n is even. Then if $x^2 = -(m^2 - 3n^2) \equiv -1 \pmod{4}$, also a contradiction, so we have $x^2 = m^2 - 3n^2$.

Now, since $(m,n)=1$ and n is even and $y^2=2mn$, we have $m=U^2$, $n=2V^2$ and so $y=2UV$, for some integers U and V. Then $x^2=U^4-12V^4$ and $z^2=U^4+12V^4$, both of which are diophantine equations which can be considered as 2nd-order of the type solved above. If we do, we find integers $(p,q)=(r,s)=1$, with p and q of opposite parity and r and s of opposite parity such that $U^2=p^2-3q^2=r^2+3s^2$ and $V^2=pq=rs$. This last equation shows that $p=P^2$, $q=Q^2$, $r=R^2$, and $s=S^2$ for some integers P,Q,R and S. Since $(p,q)=(r,s)=1$ and $pq=rs$, there must be integers $(a,d)=(b,c)=1$ such that $p=ab$, $q=cd$, $r=ac$, $s=bd$ (this is the so-called "Lucas Lemma".) Then since $p=P^2=ab$, we let $a=fA^2$, $b=fB^2$ where f is squarefree. Then since $r=R^2=ac=fA^2c$, we see that $c=fC^2$ for some C. But since $(b,c)=1$, we have that $f=1$, so $V^2=pq=rs$ and $(A,D)=(B,C)=1$. So we have $p=A^2B^2, q=C^2D^2, r=A^2C^2$, and $s=B^2D^2$. When we substitute these in for U^2 in the two solutions above we get $(A^4-3D^4)B^4=(A^4+3D^4)C^4$. Since $(B,C)=1$ we must have, for some integer E, $A^4-3D^4=EC^4$ and $A^4+3D^4=EB^4$. Now, since $U^2=r^2+3s^2$, it's easy to check that $(r,6)=1$ and hence $(A,6)=1$. If we add and subtract the last two equations we get $2A^4=E(B^4+C^4)$ and $6D^4(B^4-C^4)$. Since $(A,6)=1$, we see that E divides 2 and 6 and since it's positive, we conclude $E=1$ or 2. If $E=1$, then the second equation $A^4+3D^4=EB^4$ is an equation of the same type but with smaller positive value for z, completing the descent argument. If $E=2$, then we add the equations and divide by 2 to get $A^4=B^4+C^4$ and note that Fermat showed this equations has no nontrivial solutions.

13.2.10. We add 1 to both sides of the equation to get $y^2+1=x^3+8$ Reducing modulo 4 yields $y^2+1\equiv x^3 \pmod 4$. Since $y^2\equiv 0$ or $1 \pmod 4$, then $x^3\equiv 1$ or $2 \pmod 4$, but the only solution to these last congruences is $x\equiv 1 \pmod 4$. Now $x^3+8=(x+2)(x^2+2x+4)$ and $x^2+2x+4\equiv 3 \pmod 4$. Therefore x^2+2x+4 is divisible by a prime $p\equiv 3 \pmod 4$, since a product of primes congruent to 1 modulo 4 is again congruent to 1 modulo 4. But then $p\mid y^2+1$, which implies that -1 is a quadratic residue modulo p, which is not possible by Theorem 11.5. Therefore there are no solutions to the equation.

13.2.11. If x were even, the $y^2=x^3+23\equiv 3 \pmod 4$, which is impossible, so x must be odd, making y even, say $y=2v$. If $x\equiv 3 \pmod 4$, then $y^2\equiv 3^3+23\equiv 2 \pmod 4$ which is also impossible, so $x\equiv 1 \pmod 4$. Add 4 to both sides of the equation to get $y^2+4=4v^2+4=x^3+27=(x+3)(x^2-3x+9)$. Then $z=x^2-3x+9\equiv 1-3+9\equiv 3 \pmod 4$, so a prime $p\equiv 3 \pmod 4$ must divide z. Then $4v^2+4\equiv 0 \pmod p$ or $v^2\equiv -1 \pmod p$. But this shows that a prime congruent to 3 modulo 4 has -1 as a quadratic residue, which contradicts Theorem 11.5. Therefore, the equation has no solutions.

13.2.12. If x is even, then modulo 8 the equation becomes $y^2\equiv 5 \pmod 8$ which is impossible, since 5 is not a quadratic residue modulo 8. If $x\equiv 1 \pmod 4$ then $y^2\equiv 2 \pmod 4$, which is also impossible, since 2 is not a quadratic residue modulo 4. Therefore $x\equiv 3$ or $7 \pmod 8$. Suppose $x\equiv 3 \pmod 8$. Subtract $72=2\cdot 6^2$ from both sides of the equation to get $y-2\cdot 6^2=x^3-27=(x-3)(x^2+3x+9)$. First note that if $3\mid x$, then $3\mid y$ so that $x=3a$ and $y=3b$ for some integers a and b. Then the equation becomes $b^2=3a^3+5$, which implies $b^2\equiv 2 \pmod 3$, but 2 is not a quadratic residue modulo 3. Therefore $3\nmid x$. Now note that $x^2+3x+9\equiv 3 \pmod 8$. The product of integers congruent to 1 or 7 modulo 8 is again congruent to 1 or 7 modulo 8. Therefore, a prime p congruent to 3 or 5 modulo 8 must divide x^2+3x+9. Then the equation becomes $y^2\equiv 2\cdot 6^2 \pmod p$ (since $x-3\neq 0$) which implies that 2 is a quadratic residue modulo $p\equiv \pm 3 \pmod 8$, which is impossible. Therefore $x\not\equiv 3 \pmod 8$. Now suppose $x\equiv 7 \pmod 8$. Subtract $18=2\cdot 3^2$ from both sides of the equation to get $y=2\cdot 3^2=x^3+27=(x+3)(x^2-3x+9)$. Then $x^2-3x+9\equiv 5 \pmod 8$ and, as above, must be divisible by a prime p congruent to 3 or 5 modulo 8. Then we have $y^2\equiv 2\cdot 3^2$ which implies that 2 is a quadratic residue modulo $p\equiv \pm 3 \pmod 8$, which is impossible. Therefore, there are no solutions to the diophantine equation.

13.2.13. If there were two perfect squares in a Pythagorean triple, then we would have a solution of either the equation in Theorem 13.3 or the equation in Exercise 4, both of which have no nontrivial solutions.

13.2.14. We compute $x^2+y^2=(3k^2-1)^2+(k(k^2-3))^2=k^6+3k^4+3k^2+1=(k^2+1)^3=z^3$.

13.2.15. Assume $n\nmid xyz$, and $(x,y,z)=1$. Now $(-x)^n=y^n+z^n=(y+z)(y^{n-1}-y^{n-2}z+\cdots+z^{n-1})$, and these factors are relatively prime, so they are nth powers, say $y+z=a^n$, and $y^{n-1}-y^{n-2}z+\cdots+z^{n-1}=\alpha^n$, whence $x=a\alpha$. Similarly, $z+x=b^n$, and $(z^{n-1}-z^{n-2}x+\cdots+x^{n-1})=\beta^n$, $-y=b\beta$, $x+y=c^n$, and $(x^{n-1}-x^{n-2}y+\cdots+y^{n-1})=\gamma^n$, and $-z=c\gamma$. Since $x^n+y^n+z^n\equiv 0 \pmod p$, we have $p\mid$

xyz, say $p \mid x$. Then $\gamma^n = (x^{n-1} - x^{n-2}y + \cdots + y^{n-1}) \equiv y^{n-1} \pmod{p}$. Also $2x \equiv b^n + c^n + (-a)^n \equiv 0 \pmod{p}$, so by the condition on p, we have $p \mid abc$. If $p \mid b$ then $y = -b\beta \equiv 0 \pmod{p}$, but then $p \mid x$ and y, a contradiction. Similarly, p cannot divide c. Therefore, $p \mid a$, so $y \equiv -z \pmod{p}$, and so $\alpha^n \equiv (y^{n-1} - y^{n-2}z + \cdots + z^{n-1}) \equiv ny^{n-1} \equiv n\gamma^n \pmod{p}$. Let g be the inverse of $\gamma \pmod{p}$, then $(ag)^n \equiv n \pmod{p}$, which contradicts the condition that there is no solution to $w^n \equiv n \pmod{p}$.

13.2.16. Let k and z be any positive integers. Then substituting the suggested expressions gives us $w^3 + x^3 + y^3 = 729k^{12}z^3 + \left(1 - 9k^3\right)^3 z^3 + 27k^3\left(1 - 3k^3\right)^3 z^3 = 729k^{12}z^3 + z^3 - 27k^3z^3 + 243k^6z^3 - 729k^9z^3 + 27k^3z^3 - 243k^6z^3 + 729k^9z^3 - 729\,k^{12}z^3 = z^3$, as desired.

13.2.17. Note that $3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216 = 6^3$.

13.2.18. Let m and n be positive integers. Substituting the suggested expressions yields a large, 28th degree polynomial in m and n on each side of the equation. Inspection reveals both polynomials to be the same.

13.2.19. If $m \geq 3$ then modulo 8 we have $3^n \equiv -1 \pmod{8}$ which is impossible, so $m = 1$ or 2. If $m = 1$, then $3^n = 2 - 1 = 1$ which implies that $n = 0$ which is not a positive integer, so we have no solutions in this case. If $m = 2$, then $3^n = 2^2 - 1 = 3$, which implies that $n = 1$, and this is the only solution.

13.2.20. If $m \geq 3$, we have $3^n - 1 \equiv 0 \pmod{8}$, which implies that $n = 2k$ for some integer k. Then $3^{2k} - 1 = (3^k - 1)(3^k + 1) = 2^m$, so that $3^k - 1$ and $3^k + 1$ must be powers of 2 which differ by 2. Therefore $3^k + 1 = 4$ and $3^k - 1 = 2$ and hence $k = 1$, $n = 2$ and $m = 3$. If $m = 2$, then $3^n = 5$ has no solution. If $m = 1$, then $3^n = 2 + 1$, and so $n = 1$. So the only solutions are $m = 3, n = 2$ and $m = 1 = n$.

13.2.21. a. Substituting the expressions into the left-hand side of the equation yields $a^2 + b^2 + (3ab - c)^2 = a^2 + b^2 + 9a^2b^2 - 6abc + c^2 = (a^2 + b^2 + c^2) + 9a^2b^2 - 6abc$. Since (a, b, c) is a solution to Markoff's equation, we substitute $a^2 + b^2 + c^2 = 3abc$ to get the last expression equal to $3abc + 9a^2b^2 - 6abc = 9a^2b^2 - 3abc = 3ab(3ab - c)$, which is the right-hand side of Markoff's equation evaluated at these expressions.

b. Case 1: If $x = y = z$, then Markoff's equations becomes $3x^2 = 3xyz$ so that $1 = yz$. Then $y = z = 1$ and then $x = 1$ so the only solution in this case is $(1, 1, 1)$.

Case 2: If $x = y \neq z$, then $2x^2 + z^2 = 3x^2z$ which implies that $x^2|z^2$ or $x|z$, say $dx = z$. Then $2x^2 + d^2x^2 = 3dx^3$ or $2 + d^2 = 3dx$ or $2 = d(3x - d)$. So $d|2$, but since $x \neq z$, we must have $d = 2$. Then $3x - d = 1$ so that $x = 1 = y$ and $z = 2$, so the only solution in this case is $(1, 1, 2)$.

Case 3: Assume $x < y < z$. From $z^2 - 3xyz + x^2 + y^2 + z^2$ we apply the quadratic formula to get $2z = 3xy \pm \sqrt{9x^2y^2 - 4(x^2 + y^2)}$. Note that $8x^2y^2 - 4x^2 - 4y^2 = 4x^2(y^2 - 1) + 4y^2(x^2 - 1) > 0$ so in the "minus" case of the quadratic formula, we have $2z < 3xy - \sqrt{9x^2y^2 - 8x^2y^2} = 3xy - xy = 2xy$, or $z < xy$. But $3xyz = x^2 + y^2 + z^2 < 3z^2$ so that $xy < z$, a contradiction, therefore we must have the "plus" case in the quadratic formula and $2z = 3xy + \sqrt{9x^2y^2 - 4(x^2 + y^2)} > 3xy$, so that $z > 3xy - z$. This last expression is the formula for the generation of z in part (a). Therefore, by successive use of the formula in part (a), we will reduce the value of $x + y + z$ until it is one of the solutions in Case 1 or Case 2.

13.2.22. Assume $x^m + 1 = y^n$, with x, y, m, m positive integers and $m, n \geq 2$. Note that $\text{rad}(x^m \cdot 1 \cdot y^n) = \text{rad}(xy) \leq xy \leq \max(x^2, y^2)$. Then by the *abc* conjecture, we have $x^m < y^n = \max(x^n, 1, y^n) < K(\epsilon)\max(x^2, y^2)^{1+\epsilon}$. Therefore at least one of the inequalities $x^m < K(\epsilon)x^{2(1+\epsilon)}$ and $y^m < K(\epsilon)y^{2(1+\epsilon)}$ must hold. Suppose the first one holds. Assume $m \geq 3$ and set $\epsilon = 1/4$. Then $m - 2(1 + \epsilon) = m - 5/2 \geq m/6$. The inequality becomes $x^{m/6} \leq x^{m-2(1+\epsilon)} < K(1/4)$, so that $x^m < K(1/4)^6$. Therefore there can be only finitely many values of x^m and hence of $y^n = x^m + 1$. Similarly, if the other inequality holds, there are only finitely many solutions with $n \geq 3$. Therefore, we have shown that, assuming the *abc* conjecture, there can be only finitely many solutions to the Catalan equation with $m, n \geq 3$.

13.2.23. Let $\epsilon > 0$ be given then the abc Conjecture gives us $\max(|a|,|b|,|c|) \leq K(\epsilon)\text{rad}(abc)^{1+\epsilon}$ for integers $(a,b) = 1$ and $a+b = c$. Set $M = \log K(\epsilon)/\log 2+(3+3\epsilon)$. Suppose x, y, z, a, b, c are positive integers with $(x,y) = 1$ and $x^a+y^b = c^z$, so that we have a solution to Beal's equation. Assume $\min(a,b,c) > M$. From the abc Conjecture, and since $\text{rad}(x^a y^b y^c) = \text{rad}(xyz)$, we have $\max(x^a, y^b, y^c) \leq K(\epsilon)\text{rad}(xyz)^{1+\epsilon} \leq (xyz)^{1+\epsilon}$. If $\max(x,y,z) = x$, then we would have $x^a \leq K(\epsilon)x^{3(1+\epsilon)}$. Taking $\log$'s of both sides yields $a \leq \log K(\epsilon)/\log x + (3+3\epsilon) < \log K(\epsilon)/\log 2 + (3+3\epsilon) = M$, a contradiction. Similarly if the maximum is y or z. Therefore, if the abc Conjecture is true, there are no solutions to the Beal conjecture for sufficiently large exponents.

13.2.24. a. Suppose d is a congruent number. Then there is a right triangle with rational legs a, b and rational hypotenuse c, which has area d. Since the triangle is a right triangle, we have, by the Pythagorean theorem, that $a^2 + b^2 = c^2$. And since a and b are the legs of the triangle, its area is $ab/2 = d$. Hence $2d = ab$ as desired. Conversely, suppose there are rational integers a, b and c such that $a^2 + b^2 = c^2$ and $ab = 2d$. Then by the Pythagorean theorem, a, b, c are the lengths of the sides of a right triangle, with c as the hypotenuse. Its area is $ab/2 = d$, and so d is a congruent number.

b. By part (a), we note that $(3/2)^2 + (20/3)^2 = (41/6)^2$ and that $(3/2)(20/3) = 10 = 2 \cdot 5$, and so 5 is a congruent number. Note that $3^2 + 4^2 = 5^2$ and $3 \cdot 4 = 2 \cdot 6$, and so 6 is a congruent number. Note that $(35/12)^2 + (24/5)^2 = (337/60)^2$ and $(35/12)(24/5) = 2 \cdot 7$, and so 7 is a congruent number. Note that $6^2 + 8^2 = 10^2$ and $6 \cdot 8 = 2 \cdot 24$, and so 24 is a congruent number. Note that $5^2 + 12^2 = 13^2$ and $5 \cdot 12 = 2 \cdot 30$ and so 30 is a congruent number.

13.2.25. a. If 1 is a congruent number, then there exist rational numbers r, s and t such that $r^2 + s^2 = t^2$ and $rs/2 = 1$. Let $r = a/d$, $s = b/d$ and $t = c/d$, where a, b, c, and d are integers and d is the least common denominator of the rational numbers r, s and t. Then $a^2 + b^2 = (rd)^2 + (sd)^2 = d^2t^2 = c^2$, so (a,b,c) is a Pythagorean triple and represents a right triangle whose area is $ab/2 = (rd)(sd)/2 = (d^2)(rs/2) = d^2$, a perfect square. Conversely, if there is a right triangle whose area is a perfect square, d^2, then it is represented by a Pythagorean triple (a,b,c), and $a^2 + b^2 = c^2$. We can divide through by d^2 to get $(a/d)^2 + (b/d)^2 = (c/d)^2$ and so this represents a right triangle with sides $(a/d, b/d, c/d)$ and area $1/2(a/d)(b/d) = (ab/2)(1/d^2) = d^2/d^2 = 1$.

b. Suppose 1 is a congruent number. Then by part (a), there exist integers a, b, c and d, such that $a^2 + b^2 = c^2$ and $ab/2 = d^2$. If we add and subtract 4 times the second equation from the first we get $a^2 + 2ab + b^2 = (a+b)^2 = c^2 + (2d)^2$ and $a^2 - 2ab + b^2 = (a-b)^2 = c^2 - (2d)^2$. Since the right hand sides of both equations are squares, then so is their product, and we have $(c^2 + (2d)^2)(c^2 - (2d)^2) = c^4 - (2d)^4 = (a+b)^2(a-b)^2$, but this is a solution to $x^4 - y^4 = z^2$, which contradicts Exercise 4. Therefore 1 is not a congruent number.

13.2.26. a. If $d = 1$ or 2, then $d/a = 1$, so the equations in the preamble become $x^2 + 2ay^2 + 8z^2 = 1$ and $x^2 + 2ay^2 + 32z^2 = 1$. If either y or z were nonzero, then the left hand sides of both of these equations would be greater than 1. Therefore, $y = z = 0$ and we have $x^2 = 1$ in both equations. So the only solutions to both equations are $(\pm 1, 0, 0)$ and we have $n = m = 2$.

b. If $d = 3$ or 10, then $d/a = 3$ or 5, which forces $z = 0$ in both equations. If $d = 3$, both equations reduce to $x^2 + 2y^2 = 3$, which has solutions $(\pm 1, \pm 1, 0)$ and so $n = m = 4$. If $d = 10$, both equations reduce to $x^2 + 4y^2 = 5$ which has solutions $(\pm 1, \pm 1, 0)$ and so $n = m = 4$.

c. If $d = 11$, the first equation becomes $x^2 + 2y^2 + 8z^2 = 11$, so $|z| < 2$. If $z = 0$, we have $x^2 + 2y^2 = 11$, which has solutions $(\pm 3, \pm 1, 0)$, which is 4 solutions. If $z = \pm 1$, we have $x^2 + 2y^2 = 3$, which has solutions $(\pm 1, \pm 1, \pm 1)$ which gives us 8 more solutions for a total of $n = 12$. The second equation becomes $x^2 + 2y^2 + 32^2 = 11$, which forces $z = 0$, and we have $x^2 + 2y^2 = 11$, which has solutions $(\pm 3, \pm 1, 0)$ for a total of $m = 4$ solutions.

d. If $d = 34$, the first equation becomes $x^2 + 4y^2 + 8z^2 = 17$, so $|z| < 2$. If $z = 1$, we have $x^2 + 4y^2 = 9$ which gives us the 4 solutions $(\pm 3, 0, \pm 1)$. If $z = 0$, we have $x^2 + 4y^2 = 17$, which gives us the 4 solutions $(\pm 1, \pm 2, 0)$ for a total of $n = 8$ solutions. The second equation becomes $x^2 + 4y^2 + 32z^2 =$

17, which forces $z = 0$ and we have $x^2 + 4y^2 = 17$, giving us the $m = 4$ solutions $(\pm 1, \pm 2)$.

e. If $d \equiv 5$ or $7 \pmod 8$ both equations reduce to $x^2 + 2y^2 \equiv 5$ or $7 \pmod 8$. If y is even, this implies that 5 or 7 is a quadratic residue modulo 8, which is a contradiction. If y is odd, then $2y^2 \equiv 2 \pmod 8$ and so $x^2 \equiv 3$ or $5 \pmod 8$, which is also a contradiction. Therefore $n = m = 0$. If $d \equiv 6 \pmod 8$, both equations reduce to $x^2 + 4y^2 \equiv 3$ or $7 \pmod 8$. If y is even, this implies that 3 or 7 is a quadratic residue modulo 8, which is a contradiction. If y is odd then $x^2 \equiv 7$ or $3 \pmod 8$, also a contradiction. Therefore $n = m = 0$.

f. If $d = 1$ or 2, part (a) shows that $2m = 4 \neq 2 = n$, so 1 and 2 are not congruent numbers by Tunnell's theorem. If $d = 3$ or 10, part (b) shows that $2m = 8 \neq 4 = n$, so 3 and 10 are not congruent numbers. If $d = 11$, then part (c) shows that $2m = 8 \neq 12 = n$, so 11 is not a congruent number.

g. One solution is $(24, 17/6, 145/6)$. Another is $(15/2, 136/15, 353/30)$.

13.3. Sums of Squares

13.3.1. a. We compute $377 = 13 \cdot 29 = (3^2 + 2^2)(5^2 + 2^2) = (3 \cdot 5 + 2 \cdot 2)^2 + (3 \cdot 2 - 2 \cdot 5)^2 = 19^2 + 4^2$.

b. We compute $650 = 13 \cdot 50 = (3^2 + 2^2)(7^2 + 1^2) = (3 \cdot 7 + 2 \cdot 1)^2 + (3 \cdot 1 - 2 \cdot 7)^2 = 23^2 + 11^2$.

c. We compute $1450 = 29 \cdot 50 = (5^2 + 2^2)(7^2 + 1^2) = (5 \cdot 7 + 2 \cdot 1)^2 + (5 \cdot 1 - 2 \cdot 7)^2 = 37^2 + 9^2$.

d. We compute $18850 = 377 \cdot 50 = (19^2 + 4^2)(7^2 + 1^2) = (19 \cdot 7 + 4 \cdot 1)^2 + (19 \cdot 1 - 4 \cdot 7)^2 = 137^2 + 9^2$.

13.3.2. The integers in parts **a.**, **g.**, and **h.** all have primes $\equiv 3 \pmod 4$ appearing to an odd power in their factorizations, and therefore can not be written as the sum of two squares.

13.3.3. a. We compute $34 = 5^2 + 3^2$.

b. . We compute $90 = 3^2 10 = 3^2(3^2 + 1) = 9^2 + 3^2$.

c. We compute $100 = 10^2 + 0^2$.

d. We compute $490 = 7^2 10 = 7^2(3^2 + 1) = 21^2 + 7^2$.

e. We compute $21658 = 7^2 \cdot 2 \cdot 13 \cdot 17 = 7^2(1^2 + 1^2)(3^2 + 2^2)(4^2 + 1^2) = 7^2(1^2 + 1^2)((3 \cdot 4 + 2 \cdot 1)^2 + (3 \cdot 1 - 2 \cdot 4)^2) = 7^2(1^2 + 1^2)(14^2 + 5^2) = 7^2((1 \cdot 14 + 1 \cdot 5)^2 + (1 \cdot 5 - 1 \cdot 14)^2) = 7^2(19^2 + 9^2) = 133^2 + 63^2$.

f. We compute $324608 = 2^{10} \cdot 317 = 32^2(14^2 + 11^2) = 448^2 + 352^2$.

13.3.4. A square must be $\equiv 1$ or $0 \pmod 4$, so $x^2 - y^2 \equiv \pm 1$ or $0 \pmod 4$. Conversely, let $n = 4^m k$, with $4 \nmid k$. Then $n = 4^m \left((k+1)/2\right)^2 - 4^m \left((k-1)/2\right)^2$, which is the sum of two squares if $m \geq 1$ or if k is odd.

13.3.5. a. We have $3 = 1^2 + 1^2 + 1^2$.

b. We have $90 = 8^2 + 5^2 + 1^2$.

c. We have $11 = 3^2 + 1^2 + 1^2$.

d. We have $18 = 3^2 + 3^2 + 0^2$.

e. There are no solutions since $23 \equiv 7 \pmod 8$. See Exercise 6.

f. There are no solutions since $28 = 4 \cdot 7$. See Exercise 7.

13.3.6. Since $x^2 \equiv 0, 1$, or $4 \pmod 8$, we have $x^2 + y^2 + z^2 \equiv 0, 1, 2, 3, 4, 5$, or $6 \pmod 8$. So there are no solutions to $x^2 + y^2 + z^2 \equiv 7 \pmod 8$.

13.3.7. Let $n = x^2 + y^2 + z^2 = 4^m(8k + 7)$. If $m = 0$, see Exercise 6. If $m \geq 1$, then n is even, so 0 or 2 of x, y, z are odd. If 2 are odd, $x^2 + y^2 + z^2 \equiv 2$ or $6 \pmod 8$, but then $4 \nmid n$, a contradiction, so all of x, y, z are even. Then $4^{m-1}(8k + 7) = (\frac{x}{2})^2 + (\frac{y}{2})^2 + (\frac{z}{2})^2$ is the sum of 3 squares. Repeat until $m = 0$ and use Exercise 6 to get a contradiction.

13.3.8. For a counterexample, we have $4 = 2^2 + 0^2 + 0^2$, and $3 = 1^2 + 1^2 + 1^2$, but $3 + 4 = 7$, which is not the sum of 3 squares.

13.3.9. a. We compute $105 = 7 \cdot 15 = (2^2 + 1^2 + 1^2 + 1^2)(3^2 + 2^2 + 1^2 + 1^2) = (2 \cdot 3 + 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 1)^2 + (2 \cdot 2 - 1 \cdot 3 + 1 \cdot 1 - 1 \cdot 1)^2 + (2 \cdot 1 - 1 \cdot 1 - 1 \cdot 3 + 1 \cdot 2)^2 + (2 \cdot 1 + 1 \cdot 1 - 1 \cdot 2 - 1 \cdot 3)^2 = 10^2 + 1^2 + 0^2 + 2^2$.

b. We compute $510 = 15 \cdot 34 = (3^2 + 2^2 + 1^2 + 1^2)(4^2 + 4^2 + 1^2 + 1^2) = (3 \cdot 4 + 2 \cdot 4 + 1 \cdot 1 + 1 \cdot 1)^2 + (3 \cdot 4 - 2 \cdot 4 + 1 \cdot 1 - 1 \cdot 1)^2 + (3 \cdot 1 - 2 \cdot 1 - 1 \cdot 4 + 1 \cdot 4)^2 + (3 \cdot 1 + 1 \cdot 2 - 1 \cdot 4 - 1 \cdot 4)^2 = 22^2 + 4^2 + 1^2 + 3^2$.

c. We compute $238 = 7 \cdot 34 = (2^2 + 1^2 + 1^2 + 1^2)(4^2 + 4^2 + 1^2 + 1^2) = (2 \cdot 4 + 1 \cdot 4 + 1 \cdot 1 + 1 \cdot 1)^2 + (2 \cdot 4 - 1 \cdot 4 + 1 \cdot 1 - 1 \cdot 1)^2 + (2 \cdot 1 - 1 \cdot 1 - 1 \cdot 4 + 1 \cdot 4)^2 + (2 \cdot 1 + 1 \cdot 1 - 1 \cdot 4 - 1 \cdot 4)^2 = 14^2 + 4^2 + 1^2 + 5^2$.

d. We compute $3570 = 15 \cdot 238 = (3^2 + 2^2 + 1^2 + 1^2)(14^2 + 4^2 + 1^2 + 5^2) = (3 \cdot 14 + 2 \cdot 4 + 1 \cdot 1 + 1 \cdot 5)^2 + (3 \cdot 4 - 2 \cdot 14 + 1 \cdot 5 - 1 \cdot 1)^2 + (3 \cdot 1 - 5 \cdot 2 - 1 \cdot 14 + 1 \cdot 4)^2 + (3 \cdot 5 + 2 \cdot 1 - 1 \cdot 4 - 1 \cdot 14)^2 = 56^2 + 12^2 + 17^2 + 1^2$.

13.3.10. a. We have $6 = 2^2 + 1^2 + 1^2 + 0^2$.

b. We have $12 = 2^2 + 2^2 + 2^2 + 0^2 = 3^2 + 1^2 + 1^2 + 1^2$.

c. We have $21 = 4^2 + 2^2 + 1^2 + 0^2$.

d. We have $89 = 9^2 + 2^2 + 2^2 + 0^2$.

e. We have $99 = 9^2 + 4^2 + 1^2 + 1^2$.

f. We have $555 = 15 \cdot 37 = (3^2 + 2^2 + 1^2 + 1^2)(6^2 + 1^2 + 0^2 + 0^2) = (3 \cdot 6 + 2 \cdot 1 + 1 \cdot 0 + 1 \cdot 0)^2 + (3 \cdot 1 - 2 \cdot 6 + 1 \cdot 0 - 1 \cdot 0)^2 + (3 \cdot 0 - 2 \cdot 0 - 1 \cdot 6 + 1 \cdot 1)^2 + (3 \cdot 0 + 1 \cdot 0 - 1 \cdot 1 - 1 \cdot 6)^2 = 20^2 + 9^2 + 5^2 + 7^2$.

13.3.11. Let $m = n - 169$. Then m is the sum of four squares: $m = x^2 + y^2 + z^2 + w^2$. If, say, x, y, z are 0, then $n = w^2 + 169 = w^2 + 10^2 + 8^2 + 2^2 + 1^2$. If, say, x, y are 0, then $n = z^2 + w^2 + 169 = z^2 + w^2 + 12^2 + 4^2 + 3^2$. If, say, x is 0, then $n = y^2 + z^2 + w^2 + 169 = y^2 + z^2 + w^2 + 12^2 + 5^2$. If none are 0, then $n = x^2 + y^2 + z^2 + w^2 + 13^2$.

13.3.12. From Exercise 11, we need only check $n \leq 169$. Note that $50 = 7^2 + 1^2 = 5^2 + 4^2 + 3^2 = 4^2 + 4^2 + 3^2 + 3^2$, and $18 = 3^2 + 3^2 = 4^2 + 1^2 + 1^2 = 3^2 + 2^2 + 2^2 + 1^2$. So if $n - 50$ or $n - 18$ is the sum of 1,2, or 3 squares, then n is the sum of 5 squares. So we have eliminated the integers with $n - 50$ or $n - 18 = 4^m(8k + 7)$ (see Exercise 7). This leaves only the integers 1, 2, ...,18, 25, 33, 41, 26, 49, 57, 65, 73, 78, 81, 89, 97, 105, 110, 113, 121, 129, 137, 142, 145, 153, 161, 169, which can be checked separately.

13.3.13. If k is odd, then 2^k is not the sum of four positive squares. Suppose $k \geq 3$, and $2^k = x^2 + y^2 + z^2 + w^2$. Then either 0, 2 or 4 of the squares are odd. Modulo 8, we have $0 \equiv x^2 + y^2 + z^2 + w^2$, and since an odd square is congruent to 1 modulo 8, the only possibility is to have x, y, z, w all even. But then we can divide by 4 to get $2^{k-2} = (\frac{x}{2})^2 + (\frac{y}{2})^2 + (\frac{z}{2})^2 + (\frac{w}{2})^2$. Either $k - 2 \geq 3$ and we can repeat the argument, or $k - 2 = 1$, in which case we have 2 equal to the sum of four positive squares, a contradiction.

13.3.14. There are $[\sqrt{p} + 1]$ integers in the range $0 \leq u \leq [\sqrt{p}]$, so there are $[\sqrt{p} + 1]^2 > \sqrt{p}^2 = p$ integers of the form $au - v$, with u, v in this range. Since there are only p congruence classes, two of these must be congruent modulo p, say, $au_1 - v_1 \equiv au_2 - v_2 \pmod p$. Then $a(u_1 - v_1) \equiv v_1 - v_2 \pmod p$. Let $x =$

$u_1 - v_1$ and $y = v_1 - v_2$, then $|x|, |y| < \sqrt{p}$ as desired.

13.3.15. If $p = 2$ the theorem is obvious. Else, $p = 4k + 1$, whence -1 is a quadratic residue modulo p, say $a^2 \equiv -1 \pmod{p}$. Let x and y be as in Thue's Lemma. Then $x^2 < p$ and $y^2 < p$ and $-x^2 \equiv (ax)^2 \equiv y^2 \pmod{p}$. Thus $p \mid x^2 + y^2 < 2p$; therefore $p = x^2 + y^2$ as desired.

13.3.16. Since $3^3 = 27 > 23$, only $0^3, 1^3$, and 2^3 can appear in the sum, and 2^3 can appear at most twice. Therefore the smallest possibility is $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$; nine cubes.

13.3.17. The left sum runs over every pair of integers $i < j$, for $1 \le i < j \le 4$, so there are six terms. Each integer subscript 1, 2, 3, and 4 appears in exactly three pairs, so

$$\sum_{1\le i<j\le 4} [(x_i + x_j)^4 + (x_i - x_j)^4] = \sum_{1\le i<j\le 4} (2x_i^4 + 12x_i^2x_j^2 + 2x_j^4)$$

$$= \sum_{k=1}^{4} 6x_k^4 + \sum_{1\le i<j\le 4} 12x_i^2x_j^2 = 6\left(\sum_{k=1}^{4} x_k^2\right)^2.$$

13.3.18. If n is a positive integer, then $n = \sum_{k=1}^{4} x_k^2$, for some x_k's. From Exercise 17,

$$6n^2 = 6\left(\sum_{k=1}^{4} x_k^2\right)^2 = \sum_{1\le i<j\le 4} [(x_i + x_j)^4 + (x_i - x_j)^4].$$

Since there are 6 terms in the last sum, it represents the sum of twelve 4th powers.

13.3.19. If m is positive, then $m = \sum_{k=1}^{4} x_k^2$, for some x_k's. Then $6m = 6\sum_{k=1}^{4} x_k^2 = \sum_{k=1}^{4} 6x_k^2$. Each term of the last sum is the sum of twelve fourth powers by Exercise 18. Therefore $6m$ is the sum of forty-eight fourth powers.

13.3.20. Check $81 \equiv 3, 16 \equiv 4, 17 \equiv 5 \pmod{6}$. Also, $0 = 0^4, 1 = 1^4, 2 = 1^4 + 1^4, 81 = 3^4, 16 = 2^4$, and $17 = 2^4 + 1^4$. If $n > 81$, then write $n = 6m + k$ where $k = 0, 1, 2, 81, 16$, or 17. From Exercise 19, $6m$ is the sum of forty-eight 4th powers, and each k-value is the sum of two 4th powers, so $n = 6m + k$ is the sum of fifty 4th powers.

13.3.21. For $n = 1, 2, \ldots, 50$, $n = \sum_1^n 1^4$. For $n = 51, 52, \ldots, 81$, $n - 48 = n - 3(2^4) = \sum_1^{n-48} 1^4$, so $n = 2^4 + 2^4 + 2^4 + \sum_1^{n-48} 1^4$ is the sum of $(n - 45)$ 4th powers, and $n = 45 \le 36 \le 50$. This result, coupled with the result from Exercise 20, shows that all positive integers can be written as the sum of 50 or fewer 4th powers. That is, $g(4) \le 50$.

13.3.22. The cubic residues modulo 9 are 0, 1, and -1. Therefore, the only possible residues for the sum of three cubes modulo 9 are ± 3, ± 2, ± 1, and 0, which excludes $\pm 4 \pmod{9}$.

13.3.23. The only quartic residues modulo 16 are 0 and 1. Therefore, the sum of fewer than 15 fourth powers must have a least nonnegative residue between 0 and 14 $\pmod{16}$, which excludes any integer congruent to 15 $\pmod{16}$.

13.3.24. Suppose that $n = 31 \cdot 16^m$, with $m \ge 1$, is the sum of 15 fourth powers, say $n = \sum_{i=1}^{15} x_i^4$. If an x_i is even then $x_i^4 \equiv 0 \pmod{16}$, and if an x_i is odd, then $x_i^4 \equiv 1 \pmod{16}$, so the least nonnegative residue of $\sum_{i=1}^{15} x_i^4$ counts the number of odd x_i's. But $n \equiv 0 \pmod{16}$, so there are no odd numbers among the x_i's. Then $n/16 = 31 \cdot 16^{m-1} = \sum_{i=1}^{15} (x_i/2)^4$ is also the sum of 15 fourth powers. By the method of descent, this implies that 31 is the sum of 15 fourth powers, which is a contradiction.

13.4. Pell's Equation

13.4.1. a. Clearly $|x| \le 2$. Checking all possibilities gives $(\pm 2, 0)$ and $(\pm 1, \pm 1)$ for solutions.

b. Clearly $|x| < 3$. Checking all possibilities gives no solution.

c. Clearly $|x| < 4, |y| \leq 2$. Checking all possibilities gives the solutions $(\pm 1, \pm 2)$.

13.4.2. a. We have $x^2 - y^2 = (x-y)(x+y) = 8 = 1 \cdot 8 = 2 \cdot 4$. The system $x - y = 1; x + y = 8$ has no integer solution. The system $x - y = 2; x + y = 4$ has the solution $x = 3, y = 1$. Then the solutions are $(\pm 3, \pm 1)$.

b. We have $x^2 - 4y^2 = (x - 2y)(x + 2y) = 40 = 40 \cdot 1 = 2 \cdot 20 = 4 \cdot 10 = 4 \cdot 8$. The system $x - 2y = 1; x + 2y = 40$ has no solution. The system $x - 2y = 2; x + 2y = 20$ has no solution. The system $x - 2y = 4; x + 2y = 10$ has no solution. The system $x - 2y = 5; x + 2y = 40$ has no solution. Therefore the equation has no solution.

c. We have $4x^2 - 9y^2 = (2x - 3y)(2x + 3y) = 100 = 1 \cdot 100 = 2 \cdot 50 = 4 \cdot 25 = 5 \cdot 20 = 10 \cdot 10$. Then$2x - 3y = 1; 2x + 3y = 100$ has no solution, but $2x - 3y = 2; 2x + 3y = 50$ has solution $x = 13, y = 8$. Also $2x - 3y = 4; 2x + 3y = 25$ has no solution, and $2x - 3y = 5; 2x + 3y = 20$ has no solution, but $2x - 3y = 10; 2x + 3y = 10$ has solution $x = 5, y = 0$. Therefore all the solutions are given by $(\pm 13, \pm 8)$ and $(\pm 5, 0)$.

13.4.3. We have $\sqrt{31} = [5; \overline{1, 1, 3, 5, 3, 1, 1, 10}]$, which has period 8. The first few convergents are 5/1, 6/1, 11/2, 39/7, ... For part (a), there are solutions by Theorem 13.11. For part (b), there are no solutions by Theorem 13.11. Trying the convergents p/q in the equation with $x = p, y = q$ gives us the values $-6, 5, -3, 2, \ldots$, so we have solutions for parts (c), (d), and (e). Then for part (f), reduce modulo 4 to get $x^2 + y^2 \equiv 3 \pmod 4$ which has no solution.

13.4.4. a. We have $\sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$ which has period 5. Theorem 13.11 gives the first solution as $p_4 = 70, q_4 = 13$.

b. Using the continued fraction expansion from part (a), Theorem 13.11 gives the first solution as $p_9 = 9801, q_9 = 1820$.

13.4.5. We have $\sqrt{37} = [6; \overline{12}]$ of period 1. Theorem 13.11 gives the first 3 solutions as $x = 73, y = 12$; $x = 10657, y = 1752$; $x = 1555849, y = 255780$.

13.4.6. By Theorem 13.11 there is a solution if and only if the period of the continued fraction for $\sqrt{d}$ has odd period. Table E.5 in the text gives us that only (a), (b), (e), (g), and (h) have odd period. The rest have no solution.

13.4.7. We have $x_1 = 1766319049, y_1 = 226153980$. We apply Theorem 13.12 to get $x_2 + y_2\sqrt{61} = (x_1 + y_1\sqrt{61})^2$, which gives $x_2 = 6239765965720528801, y_2 = 798920165762330040$. We used *MAPLE* to do these calculations.

13.4.8. The last paragraph of the proof to Theorem 12.15 shows that $|p_k - \sqrt{d}q_k| < 1/q_{k+1} < 1/q_k$. Hence we have $|p_k^2 - dq_k^2| = |p_k - \sqrt{d}q_k| \cdot |p_k + \sqrt{d}q_k| < 1/q_k|p_k - \sqrt{d}q_k + 2\sqrt{d}q_k| \leq 1/q_k(|p_k - \sqrt{d}q_k| + 2\sqrt{d}q_k) < 1/q_k(1 + 2\sqrt{d}) \leq 1 + 2\sqrt{d}$, as desired.

13.4.9. Reduce modulo p to get $x^2 \equiv -1 \pmod p$. Since -1 is a quadratic nonresidue modulo p if $p = 4k + 3$, there is no solution.

13.4.10. a. We evaluate $(Xr \pm dYs)^2 - d(Xs \pm Yr)^2 = X^2r^2 \pm 2XYdrs + d^2Y^2 - dX^2s^2 \mp 2dXYsr - dY^2r^2 = X^2(r^2 - ds^2) + dY^2(ds^2 - r^2) = X^2 - dY^2 = n$.

b. Theorem 13.12 gives infinitely many solutions to $x^2 - dy^2 = 1$. If there is one solution to $x^2 - dy^2 = n$, then the construction in part (a) gives infinitely many.

13.4.11. Following the hint, we solve $a^2 - 2b^2 = \pm 1$. By Theorem 13.10, we find that every convergent p_k/q_k of $\sqrt{2}$ is a solution. Note that $p_1 = 0, p_1 = 3, p_k = 2p_{k-1} + 2_{k-2}$, $q_0 = 1$, $q_1 = 1$, and $q_k = 2q_{k-1} + q_{k-2}$.

Then solving $s-t=a, t=b$ yields $s=a+b=p_k+q_k$ and $t=q_k$, whence $x=p_k^2+2p_kq+k$ and $y=2p_kq_k+2q_k^2$. The first few solutions are $p_0=1, q_0=1$ corresponding to $x=1^2+2\cdot 1\cdot 1=3$ and $y=2\cdot 1\cdot 1+2\cdot 1^2=4$; $p_1=3, q_1=2$ corresponding to $x=3^2+2\cdot 3\cdot 2=21$ and $y=2\cdot 3\cdot 2+2\cdot 2^2=20$; $p_2=7, q_2=5$ corresponding to $x=7^2+2\cdot 7\cdot 5=119$ and $y=2\cdot 7\cdot 5+2\cdot 5^2=120$; $p_3=17, q_3=12$ corresponding to $x=17^2+2\cdot 17\cdot 12=697$ and $y=2\cdot 17\cdot 12+2\cdot 12^2=696$.

13.4.12. Since $x^4=2y^4+1$, x must be odd. So $x^2-1\equiv 0 \pmod 4$, $x^2+1\equiv 2 \pmod 4$, and $\gcd((x^2-1)/4, (x^2-1)/2)=1$. Then $(y^2/2)^2=(x^4-1)/8=((x^2-1)/4)((x^2-1)/2)$. Since these last two factors are relatively prime, we must have that x^2-1 is a perfect square. Hence $x=\pm 1$ which gives $y=0$ as the only solutions.

13.4.13. Suppose there is a solution (x,y). Then x must be odd. Note that $(x^2+1)^2=x^4+2x^2+1=2y^2+2x^2$ and $(x^2-1)^2=x^4-2x^2+1=2y^2-2x^2$. Multiplying these two equations together yields $(x^4-1)^2=4(y^4-x^4)$, or since $x^4\equiv 1 \pmod 4$, $((x^4-1)/2)^2=y^4-x^4$. But this is a violation of Exercise 4 in Section 13.2.

13.4.14. Making the appropriate substitutions, we have $x^2-2y^2=(2n+1)^2-8m^2=(2n+1)^2-8n(n+1)/2=4n^2+4n+1-4n^2-4n=1$ as desired. We must have $|x|\geq 3$, and we find that $x=3, y=1$ is a solution, so this is the smallest positive solution. By Theorem 13.12, all positive solutions are given by $x_k+y_k\sqrt{8}=(x_1+y_1\sqrt{8})^k$, and we find that the smallest 5 solutions. First, $(x,y)=(3,1)$ which corresponds to $n=1, m=1$, and we check that $t_1=1=m^2$. Second, $(x,y)=(17,6)$ which gives $(n,m)=(8,6)$ and $t_8=8(8+1)/2=36=6^2$. Third, $(x,y)=(99,35)$ which gives $(n,m)=(49,35)$, and $t_{49}=49(49+1)/2=1225=35^2$. Fourth, $(x,y)=(577,204)$ which gives $(n,m)=(228,204)$ and $t_{577}=577(577+1)/2=41616=204^2$. Fifth, $(x,y)=(3363,1189)$ which give $(n,m)=(1681,1189)$ and $t_{1681}=1681(1681+1)=1413721=1189^2$.

CHAPTER 14

The Gaussian Integers

14.1. Gaussian Integers and Gaussian Primes

14.1.1. a. First, $(2+i)(2+i) = 4+2i+2i+i^2 = 4+4i-1 = 3+4i$. Then we have $(2+i)^2(3+i) = (3+4i)(3+i) = 9+12i+3i+4i^2 = 9+15i-4 = 5+15i$.

b. First, $(2-3i)(2-3i) = 4-6i-6i+9i^2 = 4-12i-9 = -5-12i$. Then we have $(2-3i)^3 = (-5-12i)(2-3i) = -10-24i+15i+36i^2 = -10-9i-36 = -46-9i$.

c. First, $(-i+3)(-i+3) = i^2-3i-3i+9 = -1-6i+9 = 8-6i$. Next, $-i(-i+3) = i^2-3i = -1-3i$. Finally, we have $-i(-i+3)^3 = (8-6i)(-1-3i) = -8+6i-24i+18i^2 = -8-18i-18 = -26-18i$.

14.1.2. a. First we compute $(-1+i)(1+i) = -1+i-i+i^2 = -2$. Then we have $(-1+i)^3(1+i)^3 = (-2)^3 = -8$.

b. First, $(3-i)(3-i) = 9-6i+i^2 = 8-6i$, so that $(3+2i)(3-i)^2 = (3+2i)(8-6i) = 24+16i-18i-12i^2 = 36-2i$.

c. By the Binomial Theorem, we have $(5-i)^3 = 5^3-3\cdot 5^2 i+3\cdot 5i^2-i^3 = 125-75i-15+i = 110-74i$. Also $(2+i)(2+i) = 4+4i+i^2 = 3+4i$. Therefore, $(2+i)^2(5-i)^3 = (3+4i)(110-74i) = 330+440i-222i-296i^2 = 626+218i$.

14.1.3. a. We evaluate the fraction $\dfrac{\beta}{\alpha} = \dfrac{5+5i}{2-i} = \dfrac{(5+5i)(2+i)}{(2-i)(2+i)} = \dfrac{5+15i}{5} = 1+3i$, which is a Gaussian integer. Therefore, α divides β, since $\alpha(1+3i) = \beta$.

b. We evaluate the fraction $\dfrac{8}{1-i} = \dfrac{8(1+i)}{(1-i)(1+i)} = \dfrac{8+i}{2} = 4+4i$, which is a Gaussian integer. Therefore $8 = (1-i)(4+4i)$ and so α divides β.

c. Since $N(\alpha) = N(5) = 25$ and $N(\beta) = N(2+3i) = 4+9 = 13$, we observe that $25 \nmid 13$. Therefore, α can not divide β.

d. We evaluate the fraction $\dfrac{26}{3+2i} = \dfrac{26(3-2i)}{(3+2i)(3-2i)} = \dfrac{78-52i}{13} = 6-4i$, which is a Gaussian integer. Therefore, α, divides β.

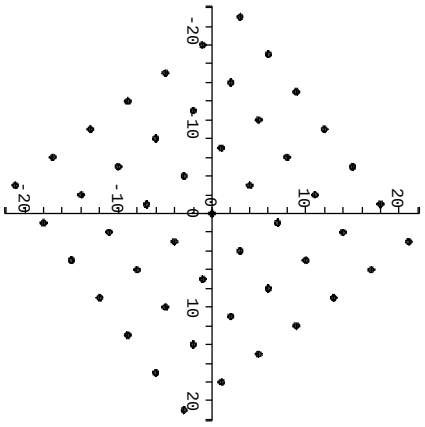
14.1.4. a. Since $N(\alpha) = N(3) = 9$ and $N(\beta) = N(4+7i) = 16+49 = 65$, we observe that $9 \nmid 65$, and so α can not divide β.

b. We evaluate the fraction $\dfrac{15}{2+i} = \dfrac{15(2-i)}{(2+i)(2-i)} = \dfrac{30-15i}{5} = 6-3i$, which is a Gaussian integer. Therefore, α divides β.

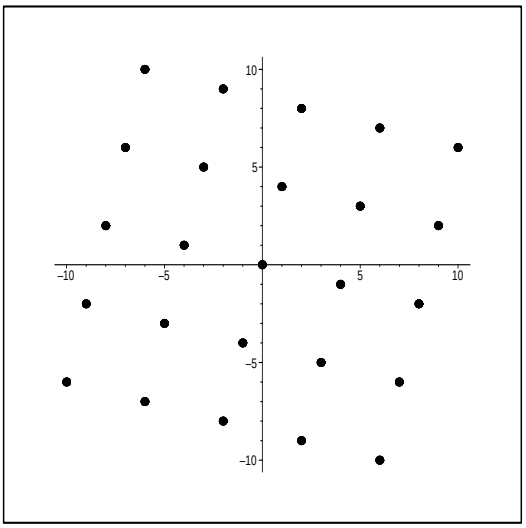
c. We evaluate the fraction $\dfrac{30+6i}{5+3i} = 6\dfrac{(5+i)(5-3i)}{(5+3i)(5-3i)} = 6\dfrac{28-10i}{34}$, which is not a Gaussian integer. Therefore, α does not divide β.

d. We evaluate the fraction $\dfrac{274}{11+4i} = \dfrac{274(11-4i)}{(11+4i)(11-4i)} = \dfrac{274(11-4i)}{137} = 2(11-4i) = 22-8i$, which is a Gaussian integer. Therefore, α divides β.

14.1.5. Since a Gaussian integer must be of the form $a+bi$ with a and b rational integers, then for a Gaussian integer α to be divisible by $4+3i$, we must have $\alpha = (4+3i)(a+bi) = (4a-3b) + (4b+3a)i$ and this gives us a formula for all Gaussian integers divisible by $4+3i$ in terms of rational integers a and b. To the right is a display of the pattern of this set in the plane.



14.1.6. Since a Gaussian integer must be of the form $a+bi$ with a and b rational integers, then for a Gaussian integer α to be divisible by $4-i$, we must have $\alpha = (4-i)(a+bi) = (4a+b)+(4b-a)i$ and this gives us a formula for all Gaussian integers divisible by $4-i$ in terms of rational integers a and b. To the right is a display of the pattern of this set in the plane.



14.1.7. Since $\alpha|\beta$ and $\beta|\gamma$, there exist Gaussian integers μ and ν such that $\mu\alpha = \beta$ and $\nu\beta = \gamma$. Since the product of Gaussian integers is again a Gaussian integer, we have that $\nu\mu$ is also a Gaussian integer. Then $\gamma = \nu\beta = \nu\mu\alpha$ and so $\alpha|\gamma$.

14.1.8. Since $\gamma \mid \alpha$ and $\gamma \mid \beta$ there exist Gaussian integers ρ and σ such that $\alpha = \rho\gamma$ and $\beta = \sigma\gamma$. Then we have $\mu\alpha + \nu\beta = \mu\rho\gamma + \nu\sigma\gamma = (\mu\rho + \nu\sigma)\gamma$. Since $(\mu\rho + \nu\sigma)$ is a Gaussian integer, we have $\gamma \mid (\mu\alpha + \nu\beta)$.

14.1.9. Consider the equation $x^5 = x$ or $x^5 - x = 0$. The left side factors over the Gaussian integers as $x(x-1)(x+1)(x-i)(x+i) = 0$, so the solutions of the equation are $0, 1, -1, i$, and $-i$. Since this includes all of the units for the Gaussian integers, this proves the result.

14.1.10. If $\overline{\alpha}$ is an associate of $\alpha = a+bi$ then we must have $\overline{\alpha} = \epsilon\alpha$ where ϵ is a unit, so there are 4 cases to consider. If $\epsilon = 1$, we have $a-bi = a+bi$ and so $b = 0$ and $\alpha = a$ is a rational integer. If $\epsilon = -1$, we have $a-bi = -a-bi$ and so $a = 0$ and $\alpha = bi$ is a pure imaginary number. If $\epsilon = i$, we have $a-bi = i(a+bi) = -b+ai$ from which we deduce $a = -b$, so α is of the form $a-ai = a(1-i)$. If $\epsilon = -i$, we have $a-bi = -i(a+bi) = b-ai$ from which we deduce $a = b$, so α is of the form $a+ai = a(1+i)$. Therefore if α is an associate of its conjugate it must be of the one of the forms $a, ai, a(1-i), a(1+i)$, where a is a

rational integer.

14.1.11. Since $\alpha|\beta$ and $\beta|\alpha$, there exist Gaussian integers μ and ν such that $\alpha\mu = \beta$ and $\beta\nu = \alpha$. Then $\alpha = \alpha\mu\nu$. Taking norms of both sides yields $N(\alpha) = N(\alpha\mu\nu) = N(\alpha)N(\mu\nu)$ by Theorem 14.1. So that $N(\mu)N(\nu) = 1$. Since μ and ν are Gaussian integers their norms must be nonnegative rational integers. Therefore $N(\mu) = N(\nu) = 1$, and so μ and ν are units, and hence, α and β are associates.

14.1.12. Since $\alpha \mid \beta$, there exists a Gaussian integer γ such that $\alpha\gamma = \beta$. From Theorem 14.1 (ii) we have $N(\beta) = N(\alpha\gamma) = N(\alpha)N(\gamma)$, which shows that $N(\alpha) \mid N(\beta)$.

14.1.13. Note that $N(1+2i) = N(2+i) = 5$, so the condition on norms holds, but $(1+2i)/(2+i) = 4/5+3/5i$, so this is a counterexample.

14.1.14. Since $\alpha \mid \beta$, there exists a Gaussian integer γ such that $\beta = \alpha\gamma$. Taking conjugates of this equation, we get $\overline{\beta} = \overline{\alpha\gamma} = \overline{\alpha}\,\overline{\gamma}$, which shows that $\overline{\alpha} \mid \overline{\beta}$.

14.1.15. First we show existence. If $a > 0$ and $b \geq 0$ we're done. If $a \leq 0$ and $b > 0$ then we multiply by $-i$ to get $-i\alpha = b - ai = c + di$ which has $c > 0$ and $d \geq 0$. If $a < 0$ and $b \leq 0$ then we multiply by -1 to get $-\alpha = -a - bi = c + di$ which has $c > 0$ and $d \geq 0$. If $a \geq 0$ and $b < 0$ then we multiply by i to get $i\alpha = -b + ai = c + di$ which has $c > 0$ and $d \geq 0$. (We have covered the quadrants in the plane in counterclockwise order.) Having found the associate $c + di$ in the first quadrant, we observe that it is unique, since if we multiply by any unit other than one we get, respectively $-c - di$, which has $-c < 0$, $-d + ci$, which has $-d \leq 0$, or $d - ci$, which has $-c < 0$.

14.1.16. a. First we divide $\alpha = 14 + 17i$ by $\beta = 2 + 3i$ to get $\alpha/\beta = 79/13 - 8/13i$. Rounding to the nearest rational integer we get $\gamma = [79/13 + 1/2] + [-8/13 + 1/2]i = 6 - i$. Then we compute $\rho = \alpha - \beta\gamma = (14+17i)-(2+3i)(6-i) = -1+i$. Finally, we note that $N(\rho) = (-1)^2+1^2 = 2 < N(\beta) = 2^2+3^2 = 13$.

b. We have $\alpha/\beta = (7 - 19i)/(3 - 4i) = 97/25 - 29/25i$. Rounding to the nearest integers in each part yields $\gamma = 4 - i$. Then we compute $\rho = \alpha - \beta\gamma = (7 - 19i) - (3 - 4i)(4 - i) = -1$. Finally, we note that $N(\rho) = (-1)^2 = 1 < N(\beta) = 3^2 + 4^2 = 25$.

c. We have $\alpha/\beta = 33/(5 + i) = 165/26 - 33/26i$. Rounding to the nearest integer in each part yields $\gamma = 6 - i$. Then we compute $\rho = \alpha - \beta\gamma = 33 - (5 + i)(6 - i) = 2 - i$. Finally, we note that $N(\rho) = 2^2 + (-1)^2 = 5 < N(\beta) = 5^2 + 1^2 = 26$.

14.1.17. a. We have $\alpha/\beta = (24 - 9i)/(3 + 3i) = 5/2 - 11/2i$. Rounding to the nearest integer in each part, and going up in each case, since we have half integers, yields $\gamma = 3 - 5i$. Then $\rho = \alpha - \beta\gamma = -3i$. Then $N(\rho) = 3^2 + 0^2 = 9 < N(\beta) = 3^2 + 3^2 = 18$.

b. We have $\alpha/\beta = (18 + 15i)/(3 + 4i) = 114/25 - 27/25i$. Rounding to the nearest integer in each part yields $\gamma = 5 - i$. Then we compute $\rho = \alpha - \beta\gamma = -1 - 2i$, so that $N(\rho) = (-1)^2 + (-2)^2 = 5 < N(\beta) = 25$.

c. We have $\alpha/\beta = 87i/(11 - 2i) = -174/125 + 957/125i$. Rounding to the nearest integer in each part yields $\gamma = -1 + 8i$. Then we compute $\rho = \alpha - \beta\gamma = -5 - 3i$, so that $N(\rho) = 5^2 + 3^2 = 34 < N(\beta) = 11^2 + 2^2 = 125$.

14.1.18. a. We have $\alpha/\beta = (14 + 17i)/(2 + 3i) = 79/13 - 8/13i$. Instead of rounding $-8/13$ to the nearest integer, we choose to round it to 0 which yields $\gamma = 6$. Then we compute $\rho = \alpha - \beta\gamma = (14 + 17i) - (2 + 3i)(6) = 2 - i$. Finally, we note that $N(\rho) = 2^2 + (-1)^2 = 5 < N(\beta) = 2^2 + 3^2 = 13$.

b. We have $\alpha/\beta = (7-19i)/(3-4i) = 97/25-29/25i$. Instead of rounding $97/25$ to the nearest integer, we round it to 3, which yields $\gamma = 3-i$. Then we compute $\rho = \alpha-\beta\gamma = (7-19i)-(3-4i)(3-i) = 2-4i$. Finally, we note that $N(\rho) = 2^2 + (-4)^2 = 20 < N(\beta) = 3^2 + 4^2 = 25$.

c. We have $\alpha/\beta = 33/(5+i) = 165/26 - 33/26i$. Instead of rounding $165/26$ to the nearest integer, we round it to 7, which yields $\gamma = 7 - i$. Then we compute $\rho = \alpha - \beta\gamma = 33 - (5+i)(7-i) = -3 - 2i$. Finally, we note that $N(\rho) = (-3)^2 + (-2)^2 = 13 < N(\beta) = 5^2 + 1^2 = 26$.

14.1.19. a. We have $\alpha/\beta = (24 - 9i)/(3 + 3i) = 5/2 - 11/2i$. Instead of rounding up in each part, we round $5/2$ down to 2, which yields $\gamma = 2 - 5i$. Then $\rho = \alpha - \beta\gamma = 3$. Then $N(\rho) = 3^2 + 0^2 = 9 < N(\beta) = 3^2 + 3^2 = 18$.

b. We have $\alpha/\beta = (18 + 15i)/(3 + 4i) = 114/25 - 27/25i$. Instead of rounding $114/25$ to the nearest integer, we round it down to 4, which yields $\gamma = 4 - i$. Then we compute $\rho = \alpha - \beta\gamma = 2 + 2i$, so that $N(\rho) = 2^2 + 2^2 = 8 < N(\beta) = 25$.

c. We have $\alpha/\beta = 87i/(11 - 2i) = -174/125 + 957/125i$. Instead of rounding $957/125$ to the nearest integer, we round it down to 7, which yields $\gamma = -1 + 7i$. Then we compute $\rho = \alpha - \beta\gamma = -3 + 8i$, so that $N(\rho) = (-3)^2 + 8^2 = 73 < N(\beta) = 11^2 + 2^2 = 125$.

14.1.20. Suppose that $\alpha/\beta = x + yi$. Since $\beta \nmid \alpha$ we know that $x + yi$ is not a Gaussian integer, therefore it lies in the interior of a unit square with vertices Gaussian integers. (One of these vertices is the Gaussian integer $s + ti$ in the proof of Theorem 14.6.) The diagonals of this square divide it into 4 triangular regions and $x + yi$ must lie in one of these regions. If it lies on the boundary between regions, then we may choose either region. Having determined the triangular region in which $x + yi$ lies, we see that two of the vertices of the triangle are Gaussian lattice points, call them γ_1 and γ_2. Note that circles of radius 1 centered at these lattice points contain the entire triangle. Therefore the distance from the lattice points to $x + yi$ is less than 1. Define $\rho_1 = \alpha - \beta\gamma_1$. Then $N(\rho_1) = N(\alpha - \beta\gamma_1) = N((\alpha/\beta - \gamma_1)\beta) = N(x + yi - \gamma_1)N(\beta)$. Since $N(x + yi - \gamma_1)$ is just the distance from γ_1 to $x + iy$, we know it is less than 1, so we have the last expression $< N(\beta)$ as desired. If we define $\rho_2 = \alpha - \beta\gamma_2$, the same calculation holds, giving us two pairs of Gaussian integers meeting the conditions.

14.1.21. If $\beta|\alpha$ then there is only one pair $\gamma = \alpha/\beta$ and $\rho = 0$. If not, then the complex number α/β can be plotted in the complex plane and lies in a unit square whose vertices are lattice points. If $\alpha = \beta\gamma + \rho$, then $\alpha/\beta - \gamma = \rho/\beta$. Then taking absolute values, we see that $|\alpha/\beta - \gamma| = |\rho/\beta| < 1$. We conclude that the possible values for γ are those Gaussian integers inside a unit circle centered at α/β, each of which generates a unique ρ.

14.1.22. Suppose $\alpha = r + si$ is an algebraic integer. Then it is a root of a monic polynomial $f(x)$ with integer coefficients. We may assume $f(x)$ has smallest positive degree of all such polynomials. If $f(x) = x + b$, then $f(\alpha) = r + si + b$ so that $s = 0$ and $r = b$, which are both integers. So assume that $\deg(f) \geq 2$. Note that $f(x)$ is necessarily irreducible over the integers, since if $f(x) = g(x)h(x)$ is a nontrivial factorization of f, then $g(\alpha)h(\alpha) = 0$ and so α satisfies one of g or h which contradicts the minimality of f.

Note that α is a root of $g(x) = (x - \alpha)(x - \overline{\alpha}) = (x^2 - 2rx + r^2 + s^2)$. If we divide $f(x)$ by $g(x)$ we get $f(x) = q(x)g(x) + r(x)$, with $\deg(r) < \deg(g) = 2$ or $r(x) = 0$. Then we have $f(\alpha) = q(\alpha)g(\alpha) + r(\alpha)$, so that $r(\alpha) = 0$. But α can not be the root of a polynomial of degree 1 or 0, so $r(x) = 0$ and we have $f(x) = q(x)g(x)$, where $q(x)$ and $g(x)$ have rational coefficients. We can factor out any common factors of the coefficients of q and g and write $f(x) = (a/b)q_1(x)g_1(x)$, where q_1 and g_1 are primitive integer polynomials and $(a, b) = 1$. But by Gauss' Lemma, (see the solution to Exercise 43 part (a) in Section 2) q_1g_1 is primitive, so no prime factor of b can divide all of the coefficients. Therefore $b = 1$, and since $f(x)$ is monic, we have $a = 1$. Further, since f is irreducible, we must have $q_1 = 1$ and so $f(x) = g(x) = x^2 - 2rx + r^2 + s^2$ and we know that $2r = b$ and $r^2 + s^2 = c$ for some integers b and c. Then $r = b/2$ and $s^2 = (4c - b^2)/4$ for some integers b and c. So $s = e/2$ for some integer e. Substituting these expressions in for r and s, we have $(b/2)^2 + (e/2)^2 = c$, or, upon multiplication by 4, $b^2 + e^2 = 4c \equiv 0 \pmod 4$ which has solutions only when b and e are even. Therefore r and s are rational integers.

14.1.23. If a and b are both even then the Gaussian integer is divisible by 2. Since $(1 + i)(1 - i) = 2$, then $1 + i$ is a divisor of 2 which is in turn a divisor of $a + bi$. If a and b are both odd we may write $a + bi = (1 + i) + (a - 1) + (b - 1)i$, and $a - 1$ and $b - 1$ are both even. Since both of theses Gaussian integers are

multiples of $1+i$, so is there sum. If a is odd and b is even, then $a-1+bi$ is a multiple of $1+i$ and hence $(a+bi)-(a-1+bi)=1$ is a multiple of $1+i$ if $a+bi$ is, a contradiction. A similar argument shows that if a is even and b is odd then $1+i$ does not divide $a+bi$.

14.1.24. If $\pi = a+bi$ is a Gaussian prime, then $N(\pi) = a^2+b^2$. There are no solutions to $a^2+b^2 \equiv 3 \pmod 4$ and the only solutions to $a^2+b^2 \equiv 0 \pmod 4$ require both a and b to be even, in which case $2 \mid \pi$ and since 2 is not a Gaussian prime, this can not be the case. Therefore the only possibilities are that $N(\pi) \equiv 1$ or $2 \pmod 4$. Then note that $N(1+i) = 1^2+1^2 \equiv 2 \pmod 4$ and $N(1+2i) = 1^2+2^2 \equiv 1 \pmod 4$. Since $1+i$ and $1+2i$ are Gaussian primes, this shows that both cases can happen.

14.1.25. Let $\alpha = a+bi$, and suppose α^2+1 is a Gaussian prime. Since we can factor $\alpha^2+1 = (\alpha+i)(\alpha-i) = (a+(b+1)i)(a+(b-1)i)$, we must have one of these factors a unit. One way is for $b=0$, so that $a=0$ and then the first factor is i. But then the second factor is $-i$ and $\alpha^2+1=0$ which is not prime. The only other way is for $b=\pm 1$, which forces $a=\pm 1$ and leads to $\alpha^2+1 = \pm 1 \pm 2i$. Since $N(\pm 1 \pm 2i) = 5$ is prime, we know that $\pm 1 \pm 2i$ are Gaussian primes, and these four are the only ones of this form.

14.1.26. Suppose $\gamma \mid (b+ai)$. Then $\overline{\gamma} \mid (b-ai)$, and hence $\overline{\gamma}$ divides its associate $\overline{\gamma} \mid i(b-ai) = a+bi$. Since $a+bi$ is a Gaussian prime, $\overline{\gamma}$ is either an associate of $a+bi$ or unit. Hence, γ is either an associate of $b+ai$ or a unit. Since γ was chosen as an arbitrary divisor of $b+ai$, this shows that $b+ai$ is also prime.

14.1.27. Suppose $7 = (a+bi)(c+di)$ where $a+bi$ and $c+di$ are nonunit Gaussian integers. Taking norms of both sides yields $49 = (a^2+b^2)(c^2+d^2)$. Since $a+bi$ and $c+di$ are not units, we have that the factors on the right are not equal to 1, so we must have $a^2+b^2 = 7$, a contradiction, since 7 is not the sum of 2 squares.

14.1.28. Suppose $p \equiv 3 \pmod 4$ is a rational prime, and that $p = (a+bi)(c+di)$ in the Gaussian integers, where neither factor is a unit. Then using part ii of Theorem 14.1, we have $p^2 = N(p) = N(a+bi)N(c+di)$. Since neither of these last factors is a unit, their norms can not be 1, so we must have $N(a+bi) = a^2+b^2 = p \equiv 3 \pmod 4$, which is impossible. Therefore p has no such factorization, and is a Gaussian prime.

14.1.29. Since α in not a unit or a prime, it has a nontrivial factors $\alpha = \beta\gamma$ with β and γ nonunits, so that $1 < N(\beta)$ and $1 < N(\gamma)$. Then $N(\alpha) = N(\beta)N(\gamma)$. If $N(\beta) > \sqrt{N(\alpha)}$ then $N(\gamma) = N(\alpha)/N(\beta) < N(\alpha)/\sqrt{N(\alpha)} = \sqrt{N(\alpha)}$. So if β doesn't satisfy the conditions, then γ does.

14.1.30. If α is a Gaussian integer which is not prime, then it has nontrivial divisors $\alpha = \beta\gamma$ where β and γ are not units. Then $N(\alpha) = N(\beta)N(\gamma)$ where $1 < N(\beta) \le N(\gamma)$. Then $N(\beta) \le \sqrt{N(\alpha)}$. So if π is a Gaussian prime dividing β, then $N(\pi) \le N(\beta) \le \sqrt{N(\alpha)}$. Therefore, we know that every composite Gaussian integer α is divisible by a Gaussian prime π with $N(\pi) \le \sqrt{N(\alpha)}$.

Observe that if $\pi = a+bi$ is a Gaussian prime, then so are its associates and their conjugates. So it suffices to find the primes in the $1/8$ plane in the 1st Quadrant on or below the line $y=x$.

To find all Gaussian primes with norms less than a specified limit M, we plot the Gaussian integers in the 1st Quadrant, on or below the line $y=x$, and inside the circle $x^2+y^2 = M$, since these are the Gaussian integers with norm less than M. Since 0, 1 and i are not primes, they are not considered. The next closest Gaussian integer to the origin in the region is $1+i$, so it must be prime and we circle it. We then cross out all other multiples of $1+i$ in the region and note that they form a pattern of vertices of squares. Since $1+i$ is a multiple of its own conjugate, we are done with this step.

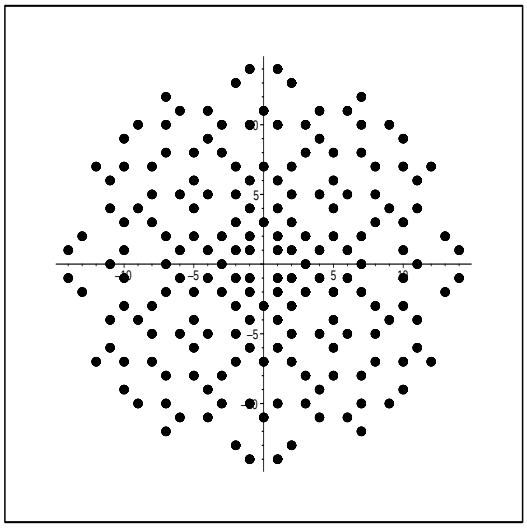
The next closest Gaussian integer to the origin which is not crossed out is $2+i$, so it is prime and we circle it and cross out all other multiples. Again, the multiples form a pattern of vertices of squares, so an easy way to determine the multiples is to see that $2+i$ and $4+2i$ must from one side of a square. Then $3-i$ and 5 must be the other two vertices. Since 5 and $4+2i$ are in the region, they are crossed out. By repeating the pattern of this square throughout the region, we find all multiples of $2+i$. We must also consider all multiples of its conjugate $2-i$, which forms a different lattice squares. We cross these out also.

The next closest multiple to the origin which is not crossed out is 3, so it is prime and we circle it. The square with vertices 0, 3, $3i$, and $3+3i$ establishes the pattern to find all multiples of 3 and we cross these out.

We continue in this fashion until every Gaussian integer in our region of norm less than $\sqrt{M}$ is either circled or crossed out. Then all Gaussian integers in the region which are either circled or not crossed out are Gaussian primes. We then take their associates and conjugates to get the complete set of all Gaussian primes up to the specified norm of M.

14.1.31. Following the procedure in Exercise 30, we note that $1+i$ is a Gaussian prime. Its multiples in the 1st Quadrant on or below the line $y = x$ are those Gaussian integers $a + bi$ where a and b are both even or both odd, so we cross these out. The closest integer to the origin not crossed out is $2 + i$, so we circle it and cross out its multiples. The new numbers crossed out with norm less than 10 are 5, $6 + 3i$, $9 + 2i$, and $7 + 6i$. We also cross out multiples of its conjugate $2 - i$, which eliminates $4 + 3i$, $8 + i$ and $7 + 4i$. The next closest integer not crossed out is 3 and the only multiple not crossed out is 9, which we cross out. The next closest number to the origin which is not crossed out is $3 + 2i$, but its norm is 13, which is greater than $\sqrt{100}$ so we are done with the sieving process. This leaves the following numbers as Gaussian primes with norm less than 100: 3, 7, $1 + i$, $2 + i$, $4 + i$, $6 + i$, $3 + 2i$, $5 + 2i$, $7 + 2i$, $8 + 3i$, $5 + 4i$, $9 + 4i$, $6 + 5i$, and $8 + 5i$, plus their conjugates and associates.

14.1.32.



14.1.33. a. Note that $\alpha - \alpha = 0 = 0 \cdot \mu$, so $\mu | \alpha - \alpha$. Thus, $\alpha \equiv \alpha \pmod{\mu}$.

b. Since $\alpha \equiv \beta \pmod{\mu}$, we have $\mu | \alpha - \beta$, so there exists a Gaussian integer γ such that $\mu\gamma = \alpha - \beta$. But then $\mu(-\gamma) = \beta - \alpha$, so $\mu | \beta - \alpha$. Therefore $\beta \equiv \alpha \pmod{\mu}$.

c. Since $\alpha \equiv \beta \pmod{\mu}$ and $\beta \equiv \gamma \pmod{\mu}$, there exist Gaussian integers δ and ϵ such that $\mu\delta = \alpha - \beta$ and $\mu\epsilon = \beta - \gamma$. Then $\alpha - \gamma = \alpha - \beta + \beta - \gamma = \mu\delta + \mu\epsilon = \mu(\delta + \epsilon)$. Therefore $\alpha \equiv \gamma \pmod{\mu}$.

14.1.34. a. Since $\alpha \equiv \beta \pmod{\mu}$ and $\gamma \equiv \delta \pmod{\mu}$, we have $\mu | (\alpha - \beta)$ and $\mu | (\gamma - \delta)$. Then $\mu | ((\alpha - \beta) + (\gamma - \delta)) = ((\alpha + \gamma) - (\beta + \delta))$. Therefore, $\alpha + \gamma \equiv \beta + \delta \pmod{\mu}$.

b. Since $\alpha \equiv \beta \pmod{\mu}$ and $\gamma \equiv \delta \pmod{\mu}$, we have $\mu | (\alpha - \beta)$ and $\mu | (\gamma - \delta)$. Then $\mu | ((\alpha - \beta) - (\gamma - \delta)) = ((\alpha - \gamma) - (\beta - \delta))$. Therefore, $\alpha - \gamma \equiv \beta - \delta \pmod{\mu}$.

c. Since $\alpha \equiv \beta \pmod{\mu}$ and $\gamma \equiv \delta \pmod{\mu}$, we have $\mu | (\alpha - \beta)$ and $\mu | (\gamma - \delta)$. Note that $\alpha\gamma - \beta\delta = \alpha\gamma - \alpha\delta + \alpha\delta - \beta\delta = \alpha(\gamma - \delta) + (\alpha - \beta)\delta$, which is a linear combination of multiples of μ, so $\mu | \alpha\gamma - \beta\delta$ and hence $\alpha\gamma \equiv \beta\delta \pmod{\mu}$.

14.1.35. If $\alpha = a_1 + ib_1$ and $\beta = a_2 + ib_2$ let $p = (a_1 + b_1)(a_2 + b_2)$. Then the real part of $\alpha\beta$ is given by the two multiplications $R = a_1a_2 - b_1b_2$ and the imaginary part is given by $p - R$ which requires only one more multiplication. The second way in the hint goes as follows. Let $m_1 = b_2(a_1 + b_1)$, $m_2 = a_2(a_1 - b_1)$, and $m_3 = b_1(a_2 - b_2)$. These are the three multiplications. Then the real part of $\alpha\beta$ is given by $m_2 + m_3$ and

the imaginary part by $m_1 + m_3$.

14.1.36. Let $z = a+bi$. We compute $z - \{z\} = a+bi-\{a+bi\} = (a+bi)-(\{a\}+\{b\}i) = (a-\{a\})+(b-\{b\})i$. Since $\{a\}$ is the closest integer to a, we have $a - \{a\} \leq 1/2$ and likewise $b - \{b\} \leq 1/2$. Therefore $N(z - \{z\}) = (a-\{a\})^2+(b-\{b\})^2 \leq (1/2)^2+(1/2)^2 = 1/2$. Suppose $\gamma = c+di$ is a Gaussian integer closer to z but different from $\{z\}$. Then $N(z-\gamma) = N((a+bi)-(c+di)) = N((a-c)+(b-d)i) = (a-c)^2+(b-d)^2 \geq (a-\{a\})^2+(b-\{b\})^2$ since $\{a\}$ and $\{b\}$ are the integers nearest a and b. But this last expression is $N(z-\{z\})$, so this shows that the distance from γ to z is at least as great as the distance from $\{z\}$ to z.

14.1.37. a. We have $G_0 = 0+i$, $G_1 = 1+i$, $G_2 = 1+2i$, $G_3 = 2+3i$, $G_4 = 3+5i$, $G_5 = 5+8i$.

b. Using the definition of G_k and the properties of the Fibonacci sequence we have $G_k = f_k + if_{k+1} = (f_{k-1}+f_{k-2})+(f_k+f_{k-1})i = (f_{k-1}+f_ki)+(f_{k-2}+f_{k-1}i) = G_{k-1}+G_{k-2}$.

14.1.38. Note that $N(G_k) = N(f_k+if_{k+1}) = f_k^2+f_{k+1}^2$. We seek to show that this last expression is equal to f_{2k+1} for all nonnegative integers k. We proceed by induction on k. For $k=1$ we have $f_3 = 1 = f_2^2+f_1^2 = 1^2+1^2$. And when $k=2$ we have $f_5 = 5 = 2^2+1^2 = f_3^2+f_2^2$, so the basis steps hold for mathematical induction. Now assume, for the strong form of induction, that the identity holds for all values of k. Then $f_{2k-3} = f_{k-1}^2+f_{k-2}^2$ and $f_{2k-1} = f_k^2+f_{k-1}^2$. Now we calculate $f_{2k+1} = f_{2k}+f_{2k-1} = f_{2k-1}+f_{2k-2}+f_{2k-1} = 2f_{2k-1}+(f_{2k-1}-f_{2k-3}) = 3f_{2k-1}-f_{2k-3}$. Now substituting in the induction hypothesis, makes this last expression equal to $3(f_k^2+f_{k-1}^2)-f_{k-1}^2-f_{k-2}^2 = 3f_k^2+2f_{k-2}^2-(f_k-f_{k-1})^2 = 2f_k^2+f_{k-1}^2+2f_kf_{k-1} = 2f_k^2+(f_{k+1}-f_k)^2+2f_k(f_{k+1}-f_k) = f_{k+1}^2+f_k^2$, which completes the induction step.

14.1.39. We proceed by induction. For the basis step note that $G_2G_1-G_3G_0 = (1+2i)(1+i)-(2+3i)(i) = 2+i$, so the basis step holds. Now assume the identity holds for values less than n. We compute, using the identity in Exercise 37, $G_{n+2}G_{n+1}-G_{n+3}G_n = (G_{n+1}+G_n)G_{n+1}-(G_{n+2}+G_{n+1})G_n = G_{n+1}^2-G_{n+2}G_n = G_{n+1}^2-(G_{n+1}+G_n)G_n = G_{n+1}^2-G_n^2-G_{n+1}G_n = (G_{n+1}+G_n)(G_{n+1}-G_n)-G_{n+1}G_n = G_{n+2}G_{n-1}-G_{n+1}G_n = -(-1)^{n-1}(2+i) = (-1)^n(2+i)$, which completes the induction step.

14.1.40. Let $\beta = -1+i$ and note that $N(\beta) = 2$. Let α be a Gaussian integer. By Exercise 23, either α or $\alpha-1$ is divisible by $1+i$ and hence by β, its associate. Let $\alpha_0 = \alpha$. Then there exists $a_0 = 0$ or 1 such that $\beta|(\alpha_0-a_0)$, so there exists a Gaussian integer α_1 such that $\alpha_1\beta = \alpha_0-a_0$.

We seek to show that if $|\alpha_0| \geq \sqrt{6}$, then $N(\alpha_1) < N(\alpha_0)$. If $a_0 = 0$ then $N(\alpha_1)N(\beta) = N(\alpha_1)2 = N(\alpha_0)$, so that $N(\alpha_1) < N(\alpha_0)$. If $a_0 = 1$, then note that the lines $y = x$ and $y = (x+1)/\sqrt{2}$ intersect when $x = \sqrt{2}+1 < \sqrt{6}$. By the Triangle Inequality we have $|\alpha_1||\beta| = |\alpha_0-1| \leq |\alpha_0|+1$, so $|\alpha_1| \leq (|\alpha_0|+1)/\sqrt{2} < |\alpha_0|$ by our observation in the previous sentence and the assumption that $|\alpha_0| \geq \sqrt{6}$.

Given that $|\alpha_0| \geq \sqrt{6}$, we produce the equation

$$\alpha_1\beta = \alpha_0 - a_0, a_0 = 0 \text{ or } 1.$$

We repeat the process on α_1 to get

$$\alpha_2\beta = \alpha_2 - a_2, a_2 = 0 \text{ or } 1.$$

And continue in this fashion generation a sequence of α_j's such that $N(\alpha_0) > N(\alpha_1) > N(\alpha_2) > \cdots$. Since this is a decreasing series of positive integers, eventually the norms must decrease to be less than 6. There are 21 Gaussian integers with norm less than 6 and we need to deal with each of these cases, to show that the process terminates with $\alpha_{n+1} = 0$. If $\alpha_k = 2+i$, then we note that $\alpha_k = 3+i-1 = (-1+2i)\beta-1$, so we take $\alpha_{k+1} = -1+2i$ and $a_{k+1} = 1$. Note that the norm did not decrease in this step. But now $\alpha_{k+1} = 2i-1 = (1-i)\beta-1$, so we take $\alpha_{k+2} = 1-i$. Then $\alpha_{k+2} = -1\beta$ so $\alpha_{k+3} = -1$ and we can take $\alpha_{k+4} = 0$. This chain accounts for the Gaussian integers $2+i, -1+2i, 1-i$, and -1. The other 16 integers are dealt with similarly. So the above sequence of equations continues:

$$\alpha_3\beta = \alpha_2 - a_2, a_2 = 0 \text{ or } 1.$$

$$\vdots$$

$$\alpha_n\beta = \alpha_{n-1} - a_{n-1}, a_0 = 0 \text{ or } 1.$$

$$\alpha_{n+1} = 0.$$

Then we have $\alpha_0 = \alpha_1\beta + a_0 = (\alpha_2\beta + a_1)\beta + a_0 = \alpha_2\beta^2 + a_1\beta + a_0 = \cdots = a_n\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_1\beta + a_0$, as desired.

14.1.41. Since the coefficients of the polynomial are real, the other root is $r - si$, and over the complex numbers the polynomial must factor as $(z - (r + si))(z - (r - si)) = z^2 - 2rz + r^2 + s^2$. The z-coefficients, $a = 2r$ and $b = r^2 + s^2$ are integers. Then $r = a/2$ and $s^2 = (4b - a^2)/4$, which shows that $s = c/2$ for some integer c. Multiplying by 4 we have $a^2 + c^2 \equiv 0 \pmod 4$ which can be true only if both a and c are even, hence r and s are integers and $r + si$ is a Gaussian integer.

14.1.42. From Exercise 23, we know that the Gaussian prime $1 + i$ divides a Gaussian integer $c + di$ if and only if c and d have the same parity. If $\pi = 1 + i$, then the surrounding 4 Gaussian integers are $2 + i$, i, $1 + 2i$, and 1, of which only $2 + i$ and $1 + 2i$ are prime. Similar arguments follow if π is one of the associates of $1 + i$.

If $\pi = a + bi$ is not an associate of $1 + i$, then since it is prime, it is not divisible by $1 + i$ and so a and b must have opposite parity. But then all of $(a + 1) + bi$, $(a - 1) + bi$, $a + (b + 1)i$, and $a + (b - 1)i$ must have real and imaginary parts of the same parity, and therefore are divisible by $1 + i$. Since one of them is prime, we conclude that one of them is an associate of $1 + i$. Hence, π must be of one of the forms $\pm 1 \pm 2i$ or $\pm 2 \pm i$.

14.1.43. Let $\beta = 1 + 2i$ so that $N(\beta) = 5$. From the proof of the Division algorithm, we have for a Gaussian integer α, that there exist Gaussian integers γ and ρ such that $\alpha = \gamma\beta + \rho$ with $N(\rho) \leq N(\beta)/2 = 5/2$. Therefore the only possible remainders upon division by $1 + 2i$ are $0, 1, i, 1 + i$ and their associates. Further, if $\alpha = \beta\gamma + (1 + i) = \beta(\gamma + 1) + (1 + i) - (1 + 2i) = \beta(\gamma + 1) - i$. So we may take the entire set of remainders to be $0, 1, -1, i$ and $-i$. Consider dividing each of the Gaussian primes $\pi_1, \ldots, \pi_4$, by β. If any two left the same remainder ρ, then β divides the difference between the two primes. But all these differences are either 2 or $1 \pm i$, which are not divisible by β. Further, since these are all prime, none of the remainders are 0. Therefore, the remainders are exactly the set $1, -1, i$ and $-i$. Now divide $a + bi$ by β and let the remainder be ρ. If ρ is not zero, then it is one of $1, -1, i$ or $-i$. But then one of $\pi_1, \ldots, \pi_4$ leaves the same remainder upon division by β, say π_k. Then β divides $\pi_k - (a + bi)$ which is a unit, a contradiction. Therefore $\rho = 0$. Therefore $1 + 2i$ divides $a + bi$. A similar argument shows that $1 - 2i$ also divides $a + bi$. Therefore the product of these primes $(1 - 2i)(1 + 2i) = 5$ also divides $a + bi$, and hence each of the components.

14.1.44. Let $S = a + bi : a = 1, 2, \ldots, m, b = 1, 2 \ldots, n$ and let P be the product of the elements of S. Then if $c + di \in S$, we have $(c + di)|(P + c + di)$, and so $P + c + di$ is not a Gaussian prime. So the block of Gaussian integers with diagonal running from $P + 1 + i$ to $P + m + ni$ contains no Gaussian primes.

14.1.45. Taking norms of the equation $\alpha\beta\gamma = 1$ shows that all three numbers must be units in the Gaussian integers, which restricts our choices to $1, -1, i$ and $-i$. Choosing three of these in the equation $\alpha + \beta + \gamma = 1$, we have the possible solutions, up to permutation, $(1, 1, -1)$, $(1, i, -i)$, but only the second solution works in the first equation, leaving $(1, i, -i)$ as the only solution.

14.1.46. Let $\pi = a + bi$. Note that $(1 + i)|4$. If $N(\pi) \neq 2$ then $(1 + i) \nmid \pi$ so by Exercise 23, a and b must have opposite parity. If $\pi \equiv c + di \pmod 4$ then c and d must have opposite parity also, otherwise, $\pi \equiv c + di \equiv 0 \pmod{1 + i}$ a contradiction. Further, we can subtract multiples of 4 and $4i$ from π so as to guarantee that c and d are between 0 and 3, inclusive. If $c + di = 1$, then the associates of π are $\pi \equiv 1 \pmod 4$, $i\pi \equiv i \pmod 4$, $-\pi \equiv -1 \equiv 3 \pmod 4$, and $-i\pi \equiv -i \equiv 3i \pmod 4$. We see that if π were congruent to any of 1, i, 3, or $3i$ then exactly one of its associates would be congruent to $1 \pmod 4$. Similarly, if $\pi \equiv c + di \equiv 3 + 2i \pmod 4$, then its associates are $\pi \equiv 3 + 2i \pmod 4$, $i\pi \equiv -2 + 3i \equiv 2 + 3i \pmod 4$, $-\pi \equiv -3 - 2i \equiv 1 + 2i \pmod 4$ and $-i\pi \equiv 2 - 3i \equiv 2 + i \pmod 4$. We see that if π were congruent to any of $3 + 2i$, $2 + 3i$, $1 + 2i$, or $2 + i$, then exactly one of its associates would be congruent to $3 + 2i$. Since the 8 congruence classes represented are all of the classes relatively prime to 4, there are no other cases.

14.2. Greatest Common Divisors and Unique Factorization

14.2.1. Certainly $1|\pi_1$ and $1|\pi_2$. Suppose $\delta|\pi_1$ and $\delta|\pi_2$ Since π_1 and π_2 are Gaussian primes, δ must be either a unit or an associate of the primes. But since π_1 and π_2 are not associates, then they can not have an associate in common, so δ is a unit and so $\delta|1$. Therefore 1 satisfies the definition of a greatest common divisor for π_1 and π_2.

14.2.2. Certainly $1|\epsilon$ and $1|\alpha$. Suppose $\delta|\epsilon$ and $\delta|\alpha$. Then there exists a Gaussian integer μ such that $\delta\mu = \epsilon$ and so $N(\delta)N(\mu) = N(\epsilon) = 1$, since ϵ is a unit. But then $N(\delta)$ is a positive rational integer which divides 1, so $N(\delta) = 1$ and therefore we know δ is a unit and we conclude that $\delta|1$. Hence 1 is a greatest common divisor of ϵ and α.

14.2.3. Since γ is a greatest common divisor of α and β, we have $\gamma|\alpha$ and $\gamma|\beta$, so there exist Gaussian integers μ and ν such that $\mu\gamma = \alpha$ and $\nu\gamma = \beta$. So that $\overline{\mu\gamma} = \overline{\mu} \cdot \overline{\gamma} = \overline{\alpha}$ and $\overline{\nu\gamma} = \overline{\nu} \cdot \overline{\gamma} = \overline{\beta}$ so that $\overline{\gamma}$ is a common divisor of $\overline{\alpha}$ and $\overline{\beta}$. Further if $\delta|\overline{\alpha}$ and $\delta|\overline{\beta}$ then $\overline{\delta}|\alpha$ and $\overline{\delta}|\beta$ and so $\overline{\delta}|\gamma$ by the definition of greatest common divisor. But then $\overline{\overline{\delta}}|\overline{\gamma}$ and $\overline{\overline{\delta}} = \delta$, which shows that $\overline{\gamma}$ is a greatest common divisor for $\overline{\alpha}$ and $\overline{\beta}$.

14.2.4. a. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be Gaussian integers. A *greatest common divisor* of $\alpha_1, \alpha_2, \ldots, \alpha_n$, is a Gaussian integer γ with the two properties: (i) $\gamma|\alpha_j$ for every $j = 1, \ldots, n$ and (ii) if $\delta|\alpha_j$ for every $j = 1, \ldots, n$, then $\delta|\gamma$.

b. Let δ be a greatest common divisor of α, β, and γ as defined in part (a). Then $\delta|\alpha$ and $\delta|\beta$, so if σ is a greatest common divisor of α and β then $\delta|\sigma$. So δ is a common divisor of γ and σ. Let τ be another common divisor of γ and σ. Then since σ divides α and β, so does τ. Therefore τ divides α, β, and γ, and so must divide δ, by definition of greatest common divisor. This shows that δ is a greatest common divisor of γ and σ also.

14.2.5. Let $\epsilon\gamma$, where ϵ is a unit, be an associate of γ. Since $\gamma|\alpha$ there is a Gaussian integer μ such that $\mu\gamma = \alpha$. Since ϵ is a unit, $1/\epsilon$ is also a Gaussian integer. Then $(1/\epsilon)\mu(\epsilon\gamma) = \alpha$, so $\epsilon\gamma|\alpha$. Similarly, $\epsilon\gamma|\beta$. If $\delta|\alpha$ and $\delta|\beta$ then $\delta|\gamma$ by definition of greatest common divisor, so there exists a Gaussian integer ν such that $\nu\delta = \gamma$. Then $\epsilon\nu\delta = \epsilon\gamma$, and since $\epsilon\nu$ is a Gaussian integer, we have $\delta|\epsilon\gamma$, so $\epsilon\gamma$ satisfies the definition of a greatest common divisor.

14.2.6. Let δ be a greatest common divisor of α and β. Say $\alpha = \mu\delta$ and $\beta = \nu\delta$. Then $N(\alpha) = N(\mu)N(\delta)$ and $N(\beta) = N(\nu)N(\delta)$. Since $N(\alpha)$ and $N(\beta)$ are relatively prime, we must have $N(\delta) = 1$, which shows that δ must be a unit and therefore α and β are relatively prime Gaussian integers.

14.2.7. Good examples are the factors of rational primes which factor in the Gaussian integers, such as $13 = (3 - 2i)(3 + 2i)$. Then $\gcd(3 + 2i, 3 - 2i) = 1$, but $N(3 + 2i) = N(3 - 2i) = 13$.

14.2.8. Since γ divides α and β, there exist Gaussian integers μ and ν such that $\alpha = \mu\gamma$ and $\beta = \nu\gamma$. Then $N(\alpha) = N(\mu)N(\gamma)$ and $N(\beta) = N(\nu)N(\gamma)$, so we see that $N(\gamma)$ is a common divisor of $N(\alpha)$ and $N(\beta)$. Therefore $N(\gamma)$ must divide $(N(\alpha), N(\beta))$.

14.2.9. Since a and b are relatively prime rational integers, there exist rational integers m and n such that $am + bn = 1$. Let δ be a greatest common divisor of the Gaussian integers a and b. Then δ divides $am + bn = 1$. Therefore δ is a unit in the Gaussian integers and hence a and b are relatively prime Gaussian integers.

14.2.10. Let the prime factorization of $\gamma = \pi_1\pi_2\cdots\pi_k$. Then the unique prime factorization of γ^n is $\gamma^n = \pi_1^n\pi_2^n\cdots\pi_k^n = \alpha\beta$. For each Gaussian prime π_j, we have $\pi_j|\alpha\beta$ and so either $\pi_j|\alpha$ or $\pi_j|\beta$ but not both, since α and β are relatively prime. Therefore either $\pi_j^n|\alpha$ or $\pi_j^n|\beta$. So, after re-indexing if necessary, there is an index r such that $\pi_1^n\cdots\pi_r^n|\alpha$ and $\pi_{r+1}^n\cdots\pi_k^n|\beta$. And since $N(\gamma) = N(\alpha)N(\beta) = N(\pi_1^n\cdots\pi_r^n)N(\pi_{r+1}^n\cdots\pi_k^n)$, we see that $N(\alpha) = N(\pi_1^n\cdots\pi_r^n)$, and so α and $\pi_1^n\cdots\pi_r^n$ are associates. Therefore $\alpha = \epsilon\pi_1^n\cdots\pi_r^n = \epsilon(\pi_1\pi_2\cdots\pi_r)^n = \epsilon\delta^n$ where epsilon is a unit.

14.2.11. a. We have $44+18i=(12-16i)(1+2i)+10i$. Then $12-16i=(10i)(-2-i)+(2+4i)$. Then $10i=(2+4i)(2+i)+0$. Since the last nonzero remainder is $2+4i$, this is a greatest common divisor.

b. From the equations in part (a) we have $2+4i=(12-16i)-(10i)(-2-i)=(12-16i)-((44+18i)-(12-16i)(1+2i))(-2-i)=(2+i)(44+18i)+(1+(1+2i)(-2-i))(12-16i)=(2+i)(44+18i)+(1-5i)(12-16i)$. So we take $\mu=2+i$ and $\nu=1-5i$.

14.2.12. a. We have $(2-11i)/(7+8i)=-74/113-93i/113$ and the nearest Gaussian integer to this quotient is $-1-i$. Then we compute $(2-11i)-(7+8i)(-1-i)=(1+4i)$ to get the remainder in the division algorithm. Now we divide $(7+8i)/(1+4i)=39/17-20i/17$ the nearest integer to which is $2-i$. Then we compute $(7+8i)-(1+4i)(2-i)=(1+i)$ to get the next remainder. Now we divide $(1+4i)/(1+i)=5/2+3i/2$, the nearest integer (rounding up) to which is $3+2i$. Then we compute $(1+4i)-(1+i)(3+2i)=-i$ which is a unit, so we deduce that $2-11i$ and $7+8i$ are relatively prime.

b. We start with the last equation in part (a) and replace every remainder with its equivalent expression as needed in the other equations given. $-i=(1+4i)-(1+i)(3+2i)=((2-11i)-(7+8i)(-1-i))-((7+8i)-(1+4i)(2-i))(3+2i)=(2-11i)-(2+i)(7+8i)+(8+i)(1+4i)=(2-11i)-(2+i)(7+8i)+(8+i)((2-11i)-(7+8i)(-1-i))=(9+i)(2-11i)+(5+8i)(7+8i)$. Now if we multiply through by i we have $1=(-1+9i)(2-11i)+(-8+5i)(7+8i)$, so we may take $\mu=-1+9i$ and $\nu=-8+5i$.

14.2.13. We proceed by induction. We have $G_0=i$ and $G_1=1+i$. Since G_0 is a unit, these are relatively prime and this completes the basis step. Assume we know that G_k and G_{k-1} are relatively prime. Suppose $\delta|G_k$ and $\delta|G_{k+1}$. Then $\delta|(G_{k+1}-G_k)=(G_k+G_{k-1}-G_k)=G_{k-1}$, so δ is a common divisor of G_k and G_{k-1} which are relatively prime. Hence $\delta|1$ and so 1 is a greatest common divisor of G_{k+1} and G_k.

14.2.14. It takes k divisions. We prove this by induction on k. Note that for $k=1$, we have $G_2=1\cdot G_2+G_0$ and since $G_0=1$, we know the greatest common divisor. Now suppose that it takes k divisions to find (G_{k+1},G_k). We perform the Euclidean algorithm on (G_{k+2},G_{k+1}) to get $G_{k+2}=1\cdot G_{k+1}+G_k$ for the first step. The second step is $G_{k+1}=G_k+G_{k-1}$, but this is the first step for finding (G_{k+1},G_k), which takes k steps. Therefore finding (G_{k+2},G_{k+1}) takes only one additional step, that is, $k+1$ steps. This completes the induction.

14.2.15. Let k be the smallest rational integer such that $N(\alpha)<2^k$. Dividing $\beta=\rho_0$ by $\alpha=\rho_1$ in the first step of the Euclidean Algorithm gives us $\beta=\gamma_2\alpha+\rho_2$ with $N(\rho_2)<N(\alpha)<2^{k-1}$. The next step of the Euclidean Algorithm, gives us $\alpha=\gamma_3\rho_2+\rho_3$ with $N(\rho_3)<N(\rho_2)<2^{k-2}$. Continuing with the algorithm shows us that $N(\rho_k)<2^{k-(k-1)}=2$, so that the Euclidean Algorithm must terminate in no more than $k=[\log_2 N(\alpha)]+1$ steps. And thus we have $k=O(\log_2(N(\alpha))$.

14.2.16. a. We compute $N(9+i)=82=2\cdot 41$. Since $1+i$ and its associates have norm 2 and since $5\pm 4i$ and their associates have norm 41, we try these and discover that $9+i=-i(1+i)(4+5i)$.

b. Since $N(1+i)=2$, we try factorizations using its associates and find $4=-(1+i)^4$.

c. We compute $N(22+7i)=533=13\cdot 41$. Since $N(2\pm 3i)=13$ and $N(4\pm 5i)=41$, we try the associates of these numbers and discover that $22+7i=-i(2+3i)(4+5i)$.

d. Note that $210+2100i=210(1+10i)=2\cdot 3\cdot 5\cdot 7(1+10i)$. Note that $N(1+10i)=101$, which is a rational prime, and so $1+10i$ is a Gaussian prime. Also, we know that 3 and 7 are Gaussian primes. It remains to factor 2 and 5. We find that $210+2100i=-1(1+i)^2(1+2i)(2+i)(3)(7)(1+10i)$.

14.2.17. a. We compute $N(7+6i)=85=5\cdot 17$. Since $1\pm 2i$ and their associates have norm 5 and $1\pm 4i$ and their associates have norm 17, we try these and discover that $7+6i=(-1)(1-2i)(1-4i)$.

b. We compute $N(3-13i)=178=2\cdot 89$. Only $1+i$ has norm 2 and it divides $3-13i$ only once, leaving $-5-8i$ which has norm 89, which is a rational prime. Therefore $5+8i$ is a Gaussian prime

and we have $3 - 13i = (-1)(1+i)(5+8i)$.

c. By Exercise 7 in Section 1, we know 7 is a Gaussian prime and since $4 = 2^2 = (i(1+i)^2)^2 = -(1+i)^4$, we have $28 = (-1)(1+i)^4(7)$.

d. We have $400i = 16 \cdot 25i = (i(1+i)^2)^4(5^2)i = (1+i)^8((1+2i)(1-2i))^2 i = i(1+i)^8(1+2i)^2(1-2i)^2$.

14.2.18. When $k = 1$ and 6 we have $N(1+6i) = N(6+i) = 37$, which is prime, so $1+6i$ and $6+i$ are Gaussian primes. When $k = 2$ and 5 we have $N(2+5i) = N(5+2i) = 29$, which is prime, so $2+5i$ and $5+2i$ are Gaussian primes. If $k = 3$ or 4 we have $N(3+4i) = N(4+3i) = 25$, so we seek factorizations involving $2 \pm i$ and its associates. We find that $3+4i = (2+i)^2$ and $4+3i = i(2-i)^2$. Finally, when $k = 7$, we have $k + (7-k)i = 7$, which is a Gaussian prime.

14.2.19. a. We find that $10 = -i(1+i)^2(1+2i)(1-2i)$, so a divisor of 10 must have one of the three Gaussian primes to a power less than or equal to the power to which it appears in this factorization. So the possible number of factors, ignoring associates is $(2+1)(1+1)(1+1) = 12$. Since there are 4 units, when we count associates, there are a total of $4 \cdot 12 = 48$ divisors of 10.

b. We have $128 + 256i = i(1+i)^{14}(1+2i)$, so the number of divisors is $4(14+1)(1+1) = 120$.

c. We have $27000 = i(1+i)^6(1+2i)^3(1-2i)^3(3)^3$, so the number of divisors is $4(6+1)(3+1)(3+1)(3+1) = 1792$.

d. We have $5040 + 40320i = (1+i)^8(1+2i)(1-2i)^2(3)^2(7)(-3+2i)$, so the number of divisors is $4(8+1)(1+1)(2+1)(2+1)(1+1)(1+1) = 2592$.

14.2.20. a. We find $198 = -i(1+i)^2(3)^2(11)$. So a divisor of 198 must have one of these Gaussian primes to a power less than or equal to the power to which it appears in this factorization. So the possible number of factors, ignoring associates is $(2+1)(2+1)(1+1) = 18$. Since there are 4 units, when we count associates, there are a total of $4 \cdot 18 = 72$ divisors of 198.

b. We have $128 + 256i = i(1+i)^{14}(1+2i)$, so the number of divisors is $4(14+1)(1+1) = 120$.

c. We have $169000 = (1+i)^6(1+2i)^3(2+i)^3(3+2i)^2(2+3i)^2$, so the number of divisors is $4(6+1)(3+1)(3+1)(2+1)(2+1) = 4032$.

d. We have $4004 + 8008i = i(1+i)^4(1+2i)(3+2i)(2+3i)(7)(11)$, so the number of divisors is $4(4+1)(1+1)(1+1)(1+1)(1+1)(1+1) = 640$.

14.2.21. Assume n and $a+bi$ are relatively prime. Then there exist Gaussian integers μ and ν such that $\mu n + \nu(a+bi) = 1$. If we take conjugates of both sides and recall that the conjugate of a rational integer is itself, we have $\overline{\mu} n + \overline{\nu}(a-bi) = 1$, so n is also relatively prime to $a-bi$. Since $a-bi$ is an associate of $b+ai$ (multiply by i), we have the result. The converse is true by symmetry.

14.2.22. Let α be a Gaussian integer with unique prime factorization, up to associates, $\alpha = \rho_1\rho_2\cdots\rho_t$, given by Theorem 14.10. By Exercise 15 in Section 14.1, each Gaussian prime ρ_k has exactly one associate $\pi_k = r_k + s_k i$ such that $r > 0$ and $s \geq 0$. Let $\rho_k = \epsilon_k \pi_k$ for $k = 1, 2, \ldots, t$. Then $\alpha = \epsilon_1\pi_1\epsilon_2\pi_2\cdots\epsilon_t\pi_t = (\epsilon_1\cdots\epsilon_t)\pi_1\cdots\pi_t$. Let $\epsilon = \epsilon_1\cdots\epsilon_t$. Then ϵ is also a unit and we have $\alpha = \epsilon\pi_1\cdots\pi_t$, where each π_k satisfies our criteria. After we gather like primes into powers, we have, after renumbering $\alpha = \epsilon\pi_k^{e_k}\cdots\pi_s^{e_s}$. The uniqueness of this expression follows from the uniqueness of the factorization given by Theorem 14.10 and the uniqueness of the associate given by Exercise 15 in Section 14.1.

14.2.23. Suppose that $\pi_1, \pi_2, \ldots, \pi_k$ are all of the Gaussian primes and form the Gaussian integer $Q = \pi_1\pi_2\cdots\pi_k + 1$. From Theorem 14.10, we know that Q has a unique factorization into Gaussian primes, and hence is divisible by some Gaussian prime ρ. Then $\rho | Q$ and $\rho | \pi_1\pi_2\cdots\pi_k$, so ρ divides their difference, which is 1, a contradiction, unless ρ is a prime different from $\pi_1, \pi_2, \ldots, \pi_k$, proving that we did not have all the Gaussian primes.

14.2.24. a. A Gaussian integer β is an inverse for α modulo μ if $\alpha\beta \equiv 1 \pmod{\mu}$.

b. If α and μ are relatively prime, then we can use the Euclidean algorithm to find Gaussian integers β and γ such that $\alpha\beta + \mu\gamma = 1$. Then $\mu\gamma = 1 - \alpha\beta$, so $\mu \mid 1 - \alpha\beta$. Therefore $\alpha\beta \equiv 1 \pmod{\mu}$ and so β is an inverse for α modulo μ.

14.2.25. Since $2+3i$ and $1+2i$ are necessarily relatively prime, we perform the Euclidean algorithm to express 1 as a linear combination of the two numbers to get $1 = i(2+3i) - 2i(1+2i)$. Then we have that $-2i$ is an inverse for $1+2i \pmod{2+3i}$.

14.2.26. We perform the Euclidean algorithm on 4 and $5+2i$. We have $5+2i = 1(4)+(1+2i)$. Then $4 = (1-2i)(1+2i)-1$, so that $1 = (1-2i)(1+2i)-4 = (1-2i)((5+2i)-4)-4 = (1-2i)(5+2i)+(-2+2i)(4)$. Then $(-2+2i)4 \equiv 1 \pmod{5+2i}$ and so $-2+2i$ is an inverse for 4 modulo $5+2i$.

14.2.27. Since α and μ are relatively prime, there exist Gaussian integers σ and τ such that $\sigma\alpha + \tau\mu = 1$. If we multiply through by β we get $\beta\sigma\alpha + \beta\tau\mu = \beta$, so that we know $\alpha(\beta\sigma) \equiv \beta \pmod{\mu}$ and thus $x \equiv \beta\sigma \pmod{\mu}$ is the solution.

14.2.28. a. Using the Euclidean algorithm we have $4-i = (1-i)(2+i)+1$, so that $1 = (4-i)+(-1+i)(2+i)$, so that $(2+i)(-1+i) \equiv 1 \pmod{4-i}$. Multiplying through by 3 gives us $(2+i)(-3+3i) \equiv 3 \pmod{4-i}$ and so $x \equiv -3+3i \equiv 1+2i \pmod{4-i}$.

b. In Exercise 26 we found that $-2+2i$ is an inverse for 4 modulo $5+2i$. Therefore $x \equiv (-2+2i)(-3+4i) \equiv -2-14i \equiv -1-2i \pmod{5+2i}$.

c. Since $3-2i = (1-i)2+1$, we see that $-1+i$ is an inverse for 2 modulo $3-2i$. Then $x \equiv (-1+i)(5) \equiv -5+5i \equiv 1+i \pmod{3-2i}$.

14.2.29. a. From the Euclidean algorithm we get $1 = (-4)3 + (1)13$. We multiply by $(2+i)$ to get $2+i = (-4)(2+i)3 + 13(2+i)$, so that we see $x \equiv -8-4i \equiv 5-4i \pmod{13}$ is the solution.

b. From the Euclidean algorithm we get $1 = (-1-2i)(5) + (2+2i)(4+i)$. Then we must have $x \equiv (-1-2i)(3-2i) \equiv -7-4i \equiv 1-2i \pmod{4+i}$.

c. From the Euclidean algorithm we get $1 = (1-i)(3+i)+i(2+3i)$. Then we must have $x \equiv 4(1-i) \equiv 3i \pmod{2+3i}$.

14.2.30. a. Since 9 is an inverse for 5 modulo 11, we have $x \equiv 9(2-3i) \equiv -2(2-3i) \equiv -4+6i \pmod{11}$.

b. Using the Euclidean algorithm, we find $1 = (3+2i)(-1-2i) + (2i)4$, so that $2i$ is an inverse for 4 modulo $3+2i$. Then $x \equiv (2i)(7+i) \equiv -2+14i \equiv 1+3i \pmod{3+2i}$.

c. We have $1 = (4-7i) + (1+i)(2+5i)$, so $1+i$ is an inverse for $2+5i$ modulo $4-7i$. Then $x \equiv (1+i)3 \equiv 3+3i \pmod{4-7i}$.

14.2.31. Statement: Let $\mu_1, \mu_2, \ldots, \mu_r$ be pairwise relatively prime Gaussian integers and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be Gaussian integers. Then the system of congruences $x \equiv \alpha_i \pmod{\mu_i}, i = 1, \ldots, r$ has a unique solution modulo $M = \mu_1\mu_2\cdots\mu_r$.

Proof: To construct a solution, for each $k = 1, \ldots, r$, let $M_k = M/\mu_k$. Then M_k and μ_k are relatively prime, since μ_k is relatively prime to all of the factors of M_k. Then from Exercise 24, we know M_k has an inverse λ_k modulo μ_k, so that $M_k\lambda_k \equiv 1 \pmod{\mu_k}$. Now let $x = \alpha_1 M_1\lambda_1 + \cdots + \alpha_r M_r\lambda_r$. We will show x is the solution to the system.

Since $\mu_k | M_j$ whenever $j \neq k$, we have $\alpha_j M_j\lambda_k \equiv 0 \pmod{\mu_k}$ whenever $j \neq k$. Therefore $x \equiv \alpha_k M_k\lambda_k \pmod{\mu_k}$ Also, since λ_k is an inverse for M_k modulo μ_k, we have $x \equiv \alpha_k \pmod{\mu_k}$ for every k, as desired.

Now suppose there is another solution y to the system. Then $x \equiv \alpha_k \equiv y \pmod{\mu_k}$ and so $\mu_k|(x-y)$ for every k. Since the μ_k are pairwise relatively prime, no Gaussian prime appears in more than one of their prime factorizations. Therefore, if a Gaussian prime power $\pi^e|(x-y)$ then it divides exactly one of the μ_k's. Therefore, the product M of the μ_k's also divides $x-y$ and so $x \equiv y \pmod{M}$ showing that x is unique modulo M.

14.2.32. Using Exercise 31, we let $M = (2+3i)(1+4i) = -10+11i$, so that $M_1 = 1+4i$ and $M_2 = 2+3i$. From the Euclidean algorithm, we have $1 = 2(1+4i) + (-2-i)(2+3i)$, so $\lambda_1 = 2$ is an inverse for M_1 modulo $2+3i$ and $\lambda_2 = (-2-i)$ is an inverse for M_2 modulo $1+4i$. Then the solution to the system is $x = 2(1+4i)2 + 3(2+3i)(-2-i) = 1-8i \pmod{-10+11i}$.

14.2.33. Using Exercise 31, we let $M = (2+5i)(3-4i) = 26+7i$, so that $M_1 = 3-4i$ and $M_2 = 2+5i$. An inverse for M_1 modulo $2+5i$ is $\lambda_1 = -1+2i$. An inverse for M_2 modulo $3-4i$ is $\lambda_2 = -2$. Then the solution is $x = (1+3i)(3-4i)(-1+2i) + (2-i)(2+5i)(-2) = -43+9i \equiv 9+23i \pmod{26+7i}$.

14.2.34. We seek a solution to the system of congruences $x \equiv 1 \pmod{11}$, $x \equiv 2 \pmod{4+3i}$, $x \equiv 3 \pmod{1+7i}$. Note that $4+3i = -i(1+2i)^2$ and $1+7i = -i(1+i)(1+2i)^2$, so the moduli are not relatively prime. Indeed, $1+7i = (1+i)(4+3i)$, so if x is a solution to the system, then $(1+7i) \mid (x-3)$. But then $(4+3i) \mid (x-3)$, so $x \equiv 3 \pmod{4+3i}$, a contradiction. Therefore, there are no solutions to the system.

14.2.35. a. Using the construction in the solution to Exercise 37, we note that $N(1-i) = 2$ and $(1,1) = 1 = d$, so that $S = \{0,1\}$ which is a complete residue system.

b. Using the construction in Exercise 37, we note that $N(2) = 4$ and $(2,0) = 2 = d$, so that $S = \{0,1,i,1+i\}$, which is a complete residue system.

c. Using the construction in Exercise 37, we note that $N(2+3i) = 13$ and $(2,3) = 1 = d$, so that $S = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$. Reducing each of these modulo $2+3i$ gives us $\{0,1,2,2i,-1-i,-i,1-i,-1+i,i,1+i,-2i,-2,-1\}$ for a complete residue system.

14.2.36. a. Using the construction in Exercise 37, we note that $N(1+2i) = 5$ and $(1,2) = 1 = d$, so $S = \{p+qi \mid 0 \le p < 5, 0 \le q < 1\} = \{0,1,2,3,4\}$. Reducing each of these modulo $1+2i$ gives us $\{0,1,i,1+i,2i\}$ for a complete residue system.

b. Using the construction in Exercise 37, we note that $N(3) = 9$, and $(3,0) = 3 = d$, so $S = \{0,1,2,i,1+i,2+i,2i,1+2i,2+2i\}$. Reducing each of these modulo 3 gives us $\{0,1,-1,i,1+i,-1+i,-i,1-i,-1-i\}$ for a complete residue system.

c. Using the construction in Exercise 37, we note that $N(4+i) = 17$ and $(4,1) = 1 = d$, so that $S = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$. Reducing each of these modulo $4+i$ gives us $\{0,1,2,-1-i,-i,1-i,2-i,-1-2i,-2i,2i,1+2i,-2+i,-1+i,i,1+i,-2,-1\}$ for a complete residue system.

14.2.37. Let $\alpha = a+bi$ and $d = \gcd(a,b)$. We assert that the set $S = \{p+qi | 0 \le p < N(\alpha)/d, 0 \le q < d\}$ is a complete residue system. Note that this represents a rectangle of lattice points in the plane. We create two multiples of α. First, $N(\alpha)/d = \alpha(\overline{\alpha}/d)$ is a real number and a multiple of α. Second, there exist rational integers r and s such that $ra+sb = d$. So we have the multiple of α given by $v = (s+ir)\alpha = (s+ir)(a+bi) = (as-br)+di$. Now it is clear that any Gaussian integer is congruent modulo α to an integer in the rectangle S, since first we can add or subtract multiples of v until the imaginary part is between 0 and $d-1$ and then add and subtract multiples of $N(\alpha)/d$ until the real part is between 0 and $N(\alpha)/d-1$. It remains to show the elements of S are incongruent to each other modulo α. Suppose β and γ are in S and congruent to each other modulo α. Then the imaginary part of $\beta-\gamma$ must be divisible by d, but since these must lie in the interval from 0 to $d-1$, they must be equal. Therefore the difference between β and γ is real and divisibly by α, hence by $\overline{\alpha}$ and hence by $\alpha\overline{\alpha}/d = N(\alpha)/d$, which proves they are equal. Since S has $N(\alpha)$ elements, we are done.

14.2.38. a. From Exercise 37, we find a complete residue system modulo $-1+3i$ to be $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Also, we have $-1 + 3i = (1 + i)(1 + 2i)$ as a product of primes. Since $1 + i$ divides 2, we know that no even number is relatively prime to $-1 + 3i$, so we remove those, which leaves us with the set $\{1, 3, 5, 7, 9\}$. Factoring each of these into Gaussian primes gives us $\{1, -i, 2 - i, i, -1\}$ respectively. Note that the 3rd element $2 - i$ is an associate of $1 + 2i$ which divides $-1 + 3i$, so it is deleted also. A reduced residue system, then, is $\{1, -i, i, -1\}$

b. From Exercise 37, we find a complete residue system modulo 2 to be $S = \{0, 1, i, 1 + i\}$. Since $(1 + i, 2) = 1 + i$, and the other elements of S are units, a reduced residue system modulo 2 is $\{0, 1, i\}$.

c. From Exercise 37, we find a complete residue system modulo $5 - i$ to be $S = \{0, 1, 2, \ldots, 25\}$. Also, we have $5 - i = (1 + i)(2 - 3i)$, and since $1 + i$ divides every even integer, we delete those. Reducing the remaining (odd) integers modulo $5 - i$ gives us $\{1, -2 + i, i, 2 + i, -1 + 2i, 1 + 2i, 3 + 2i, -1 - 2i, 1 - 2i, -2i, -i, 2 - i, -1\}$. The seventh entry is not relatively prime to $5 + i$, so we delete it. Since all the rest these have norm less than $N(2 - 3i) = 13$, and since $(2 - 3i)$ is prime, we know that these remaining integers are all relatively prime to $5 - i$, and so a reduced residue system is $\{1, -2 + i, i, 2 + i, -1 + 2i, 1 + 2i, -1 - 2i, 1 - 2i, -2i, -i, 2 - i, -1\}$.

14.2.39. a. From Exercise 37, we find a complete residue system modulo $2+2i$ to be $S = \{0, i, 2i, 3i, 1, 1+i, 1+2i, 1+3i\}$. Also, we have $2 + 2i = -i(1 + i)^3$, so every element in S with the same parity in real and imaginary parts is not relatively prime to $2 + 2i$. Deleting these gives us $\{i, 3i, 1, 1 + 2i\}$. Reducing modulo $2 + 2i$ gives us $\{i, -i, 1, -1\}$ for a reduced residue system.

b. From Exercise 37, we find a complete residue system modulo 4 to be $S = \{0, i, 2i, 3i, 1, 1 + i, 1 + 2i, 1 + 3i, 2, 2 + i, 2 + 2i, 2 + 3i, 3, 3 + i, 3 + 2i, 3 + 3i\}$. Also, we have $4 = -(1 + i)^4$, so every element in S with the same parity in real and imaginary parts is not relatively prime to 4. Deleting these gives us $\{i, 3i, 1, 1 + 2i, 2 + i, 2 + 3i, 3, 3 + 2i\}$. Reducing modulo 4 gives us $\{i, -i, 1, 1 + 2i, 2 + i, 2 - i, -1, -1 + 2i\}$ for a reduced residue system.

c. From Exercise 37, we find a complete residue system modulo $4+2i$ to be $S = \{0, i, 2i, 3i, 4i, 5i, 6i, 7i, 8i, 9i, 1, 1 + i, 1 + 2i, 1 + 3i, 1 + 4i, 1 + 5i, 1 + 6i, 1 + 7i, 1 + 8i, 1 + 9i\}$. Also, we have $4 + 2i = (1 + i)^2(1 - 2i)$, so every element in S with the same parity in real and imaginary parts is not relatively prime to $4 + 2i$. Deleting these gives us $\{i, 3i, 5i, 7i, 9i, 1, 1 + 2i, 1 + 4i, 1 + 6i, 1 + 8i\}$. Reducing modulo $4 + 2i$ gives us $\{i, 2 - i, 2 + i, -2 + i, -i, 1, 1 + 2i, -1 - 2i, -1, -1 + 2i\}$. Note that $2 + i$ and $-1 + 2i$ are associates of $1 - 2i$ which is a prime divisor of $4 + 2i$, so we delete them, leaving $\{i, 2 - i, -2 + i, -i, 1, 1 + 2i, -1 - 2i, -1\}$ for a reduced residue system.

14.2.40. If $\pi = p$ is a rational prime, then $N(\pi) = p^2$ and $d = (p, 0) = p$ in the solution to Exercise 37, so $S = \{a + bi \mid 0 \leq a < p, 0 \leq b < p\}$ is a complete residue system modulo π. Let $a + bi \in S$ and suppose $p \mid a + bi$. Then $\overline{p} = p \mid a - bi$, so that $p \mid (a + bi) + (a - bi) = 2a$. Therefore $p \mid a$, and similarly $p \mid b$. Since $0 \leq a < p$ and $0 \leq b < p$, we must have $a = b = 0$ as the only multiple of π in S. Therefore a reduced residue system has $|S| - 1 = p^2 - 1 = N(\pi) - 1$ elements. If π is not a rational prime, then $\pi = p + qi$ where p and q are rational integers. Then $N(\pi) = p^2 + q^2$ and $(p, q) = 1 = d$ in the construction from Exercise 37, so a complete residue system modulo π is $S = \{a + bi \mid 0 \leq a < p^2 + q^2, 0 \leq b < 1\} = \{0, 1, 2, \ldots, p^2 + q^2 - 1\}$. Suppose π divides an element a of S. Then $\overline{\pi} \mid \overline{a}$, so that $p - qi \mid a$. Since $(\pi, \overline{\pi}) = 1$, we must have $\pi\overline{\pi} = p^2 + q^2 \mid a$. But $a < p^2 + q^2$, so $a = 0$ is the only element of S not relatively prime to π. Therefore there are $|S| - 1 = N(\pi) - 1$ elements in a reduced residue system modulo π.

14.2.41. From the properties of the norm function and Exercise 37, we know that there are $N(\pi^e) = N(\pi)^e$ residue classes modulo π^e. Let $\pi = r + si$, and $d = \gcd(r, s)$. Also, by Exercise 37, a complete residue system modulo π^e is given by the rectangle $S = \{p + qi | 0 \leq p < N(\pi^e)/d, 0 \leq q < d\}$, while a complete residue system modulo π is given by the rectangle $T = \{p + qi | 0 \leq p < N(\pi)/d, 0 \leq q < d\}$. Note that in T there is exactly one element not relatively prime to π, and that there are $N(\pi)^{e-1}$ copies of T, congruent modulo π, inside of S. Therefore, there are exactly $N(\pi)^{e-1}$ elements in S not relatively prime to π. Thus there are $N(\pi)^e - N(\pi)^{e-1}$ elements in a reduced residue system modulo π^e.

14.2.42. a. Suppose $\alpha = r + s\sqrt{-3}$ is an algebraic integer. Then it is a root of a monic polynomial $f(x)$ with integer coefficients. We may assume $f(x)$ has smallest positive degree of all such polynomials. If $f(x) = x + b$, then $f(\alpha) = r + s\sqrt{-3} + b$ so that $s = 0$ and $r = b$, which are both integers. So assume that $\deg(f) \geq 2$. Note that $f(x)$ is necessarily irreducible over the integers, since if $f(x) = g(x)h(x)$ is a nontrivial factorization of f, then $g(\alpha)h(\alpha) = 0$ and so α satisfies one of g or h which contradicts the minimality of f.

Note that α is a root of $g(x) = (x - \alpha)(x - \overline{\alpha}) = (x^2 - 2rx + r^2 + 3s^2)$. If we divide $f(x)$ by $g(x)$ we get $f(x) = q(x)g(x) + r(x)$, with $\deg(r) < \deg(g) = 2$ or $r(x) = 0$. Then we have $f(\alpha) = q(\alpha)g(\alpha) + r(\alpha)$, so that $r(\alpha) = 0$. But α can not be the root of a polynomial of degree 1 or 0, so $r(x) = 0$ and we have $f(x) = q(x)g(x)$, where $q(x)$ and $g(x)$ have rational coefficients. We can factor out any common factors of the coefficients of q and g and write $f(x) = (a/b)q_1(x)g_1(x)$, where q_1 and g_1 are primitive integer polynomials and $(a, b) = 1$. But by Gauss' Lemma, (see the solution to Exercise 43 part (a)) q_1g_1 is primitive, so no prime factor of b can divide all of the coefficients. Therefore $b = 1$, and since $f(x)$ is monic, we have $a = 1$. Further, since f is irreducible, we must have $q_1 = 1$ and so $f(x) = g(x) = x^2 - 2rx + r^2 + 3s^2$ and we know that $2r = b$ and $r^2 + 3s^2 = c$ for some integers b and c. Then $r = b/2$ and $3s^2 = (4c - b^2)/4$ for some integers b and c. So $s = e/2$ for some integer e. (5 can not appear in the denominator of s, else when we square it, the single factor of 5 in the expression leaves a remaining factor in the denominator, which does not appear on the right side of the equation.) We check that if n and m have opposite parity, then $f(x)$ will not have integer coefficients. Therefore n and m have the same parity and α must be of the form $a + b\omega$.

b. Let $\alpha = a + b\sqrt{-3}$ and $\beta = c + d\sqrt{-3}$. Then $\alpha + \beta = (a + c) + (b + d)\sqrt{-3}$ and $\alpha - \beta = (a - c) + (b - d)\sqrt{-3}$, and $\alpha\beta = (ac - 3bd) + (ad + bc)\sqrt{-3}$. Since the rational integers are closed under addition, subtraction and multiplication, all of the results are again of the form $p + q\sqrt{-3}$ with p and q rational integers.

c. First we check that $\omega^2 = (1 - 2\sqrt{-3} - 3)/4 = (-1 - \sqrt{-3})/2 = \overline{\omega}$. Also note that $-1 - \omega = -1 - (-1/2 + \sqrt{-3}/2) = -1/2 - \sqrt{-3}/2 = \omega^2$. By part (a), we have $\alpha = a + b\omega$ for some integers a and b. Then $\overline{\alpha} = a + b\overline{\omega} = a + b\omega^2 = a + b(-1 - \omega) = (a - b) - b\omega$, which is an Eisenstein integer, since $a - 1$ and b are rational integers.

d. Note that $\omega^3 = 1$ and recall from part (b) that $\overline{\omega} = \omega^2 = -1 - \omega$. Then we compute $\alpha\overline{\alpha} = (a + b\omega)(a + b\overline{\omega}) = a^2 + ab(\omega + \overline{\omega}) + b^2\omega\overline{\omega} = a^2 + ab(\omega + (-1 - \omega)) + b^2\omega \cdot \omega^2 = a^2 - ab + b^2 = N(\alpha)$.

e. First, we seek rational integers a and b such that $1 + 5\omega = (1 + 2\omega)(a + b\omega) = (a - 2b) + (2a - b)\omega$ where we have used the fact that $\omega^2 = -1 - \omega$. Then we have $a - 2b = 1$ and $2a - b = 5$. We solve this system to discover that $a = 3$ and $b = 1$, which makes $3 + \omega$ an Eisenstein integer and so $1 + 2\omega$ divides $1 + 5\omega$. Next, we seek rational integers a and b such that $9 + 8\omega = (3 + \omega)(a + b\omega) = (3a - b) + (a + 2b)\omega$, where we again used the fact that $\omega^2 = -1 - \omega$. Then we have $3a - b = 9$ and $a + 2b = 8$. Solving this system shows that $b = -15/7$ is forced, which makes $a + b\omega$ not an Eisenstein integer, and so $3 + \omega$ does not divide $9 + 8\omega$.

f. We check that for the norm defined in part (d), we have, for Eisenstein integers α and β, $N(\alpha\beta) = N(\alpha)N(\beta)$. Let $\alpha = a + b\omega \neq 0$. Then $N(\alpha) = a^2 - ab + b^2 = (a - b/2)^2 + 3b^2/4$, which shows that $N(\alpha)$ is non-negative. We conclude that if ϵ is a Eisenstein unit, then $N(\epsilon) = 1$. If $\epsilon = e + f\omega$, the identity above gives us $N(\epsilon) = (e - f/2)^2 + 3f^2/4 = 1$, so that $|f| < 2$ else $N(\epsilon)$ is too large. Then $(e - f/2)^2 = 1 - 3f^2/4 \leq 1$, so $|e| < 2$ also. This gives us 9 possibilities as $e, f = -1, 0$, and 1. Note that $N(1 - \omega) = N(-1 + \omega) = 3$ and $N(0 + 0\omega) = 0$, so none of these three are units. The other six are $1, -1, \omega, -\omega, 1 + \omega = -\omega^2$, and $-1 - \omega = \omega^2$. The norms of all six of these are equal to 1, so they are all units.

g. As in part (f), we check that, for an Eisenstein integer γ, if $\gamma = \alpha\beta$, then $N(\gamma) = N(\alpha)N(\beta)$, so if $N(\gamma)$ is a rational prime, then one of $N(\alpha)$, $N(\beta)$ equals 1 and implies that one of α, β is a unit, and hence γ is an Eisenstein prime. Note that $N(1 + 2\omega) = 1 - 2 + 2^2 = 3$, which is a rational prime. Therefore $1 + 2\omega$ is an Eisenstein prime. Likewise $N(3 - 2\omega) = 3^2 - 3(-2) + 4 = 19$ is a rational

prime, and so $3-2\omega$ is an Eisenstein prime. Next, note that $N(5+4\omega)=21=3\cdot 7$, so we suspect that $1+2\omega$ might be a factor of $5+4\omega$. We consider $(1+2\omega)(a+b\omega)=(a-2b)+(2a-b)\omega=5+4\omega$. Then we must have $a-2b=5$ and $2a-b=4$ which implies $a=1$ and $b=-2$. We check that $(1+2\omega)(1-2\omega)=5+4\omega$, which is therefore not an Eisenstein prime. Next $N(-7-2\omega)=39=3\cdot 13$, so we suspect that $1+2\omega$ is a factor. We consider $(1+2\omega)(a+b\omega)=(a-2b)+(2a-b)\omega=-7+-2\omega$. Then we must have $a-2b=-7$ and $2a-b=-2$ which implies $a=1$ and $b=4$. We check that $(1+2\omega)(1+4\omega)=-7-2\omega$, which is therefore not an Eisenstein prime.

h. Note that $\alpha/\beta=\alpha\overline{\beta}/\beta\overline{\beta}=r+s\omega$, where, since $\beta\overline{\beta}=N(\beta)$ is an integer, we know that r and s are rational numbers. Then we can find integers m and n such that $|r-m|\leq 1/2$ and $|s-n|\leq 1/2$. Set $\gamma=m+n\omega$ and $\rho=\alpha-\gamma\beta$. If $\rho=0$ we are done. If not, note that $N(\rho)=N(\beta(\alpha/\beta-\gamma))=N(\beta)N(\alpha/\beta-\gamma)=N(\beta)N((r-m)+(s-n)\omega)=N(\beta)((r-m)^2-(r-m)(s-n)+(s-n)^2)\leq N(\beta)(1/4+1/4+1/4)=N(\beta)(3/4)$. Thus $N(\rho)<N(\beta)$ as desired.

i. Theorem 14.9 holds for Eisenstein integers and follows from part (h). Likewise, the proofs of Lemmas 14.1 and 14.2 go through unchanged, except for noting that each Eisenstein prime πhas exactly 12 divisors, $\pm 1,\pm\omega,\pm\omega^2,\pm\pi,\pm\pi\omega$ and $\pm\pi\omega^2$. Then the proof of Theorem 14.10 goes through verbatim.

j. Since $N(1+2\omega)=3$, we suspect it might divide 6, and we find that $6=-2(1+2\omega)^2$. Since 2 and $1+2\omega$ are primes (see part (g)), this is the prime factorization for 6. Since $N(5+9\omega)=61$, which is a rational prime, we have, by the argument in part (g) that $5+9\omega$ is an Eisenstein prime, so it is already factored. Note that $114=6\cdot 19$. Since $N(3-2\omega)=19$, we know it is prime. We try dividing 19 by $3-2\omega$ and find $19=(3-2\omega)(5+2\omega)$. And since $N(5+2\omega)=19$, which is prime, we have the prime factorization for 19. Then from our work above, we have $114=6\cdot 19=-2(1+2\omega)^2(3-2\omega)(5+2\omega)$. Since $37+74\omega=37(1+2\omega)$ we try to find an Eisenstein prime with norm 37. We find $N(3+7\omega)=37$ and upon division, that $37+74\omega=(1+2\omega)(3+7\omega)(-4-7\omega)$.

14.2.43. a. A polynomial is called *primitive* if the greatest common divisor of its coefficients is 1. We require a result from algebra called Gauss' Lemma, which states that the product of primitive polynomials is primitive. To prove this, suppose $f(x)=a_0+a_1x+\cdots+a_nx^n$ and $g(x)=b_0+b_1x+\cdots+b_mx^m$ are primitive integer polynomials. Let p be any prime. Let a_j be the first coefficient of $f(x)$ which p doesn't divide. Likewise, let b_k be the first coefficient of $g(x)$ which p doesn't divide. Then $f(x)g(x)=c_0+c_1x+\cdots+c_{j+k}+\cdots+c_{n+m}x^{n+m}$, where $c_{j+k}=b_0a^{j+k}+b_1a_{j+k-1}+\cdots+b_ka_j+\cdots+b_{j+k}a_0$. Since every term is divisible by p except b_ka_j, we see that c_{j+k} is not divisible by p. We conclude that no prime can divide all the coefficients of $f(x)g(x)$ and so it is primitive.

Now suppose $\alpha=r+s\sqrt{-5}$ is an algebraic integer. Then it is a root of a monic polynomial $f(x)$ with integer coefficients. We may assume $f(x)$ has smallest positive degree of all such polynomials. If $f(x)=x+b$, then $f(\alpha)=r+s\sqrt{-5}+b$ so that $s=0$ and $r=b$, which are both integers. So assume that $\deg(f)\geq 2$. Note that $f(x)$ is necessarily irreducible over the integers, since if $f(x)=g(x)h(x)$ is a nontrivial factorization of f, then $g(\alpha)h(\alpha)=0$ and so α satisfies one of g or h which contradicts the minimality of f.

Note that α is a root of $g(x)=(x-\alpha)(x-\overline{\alpha})=(x^2-2rx+r^2+5s^2)$. If we divide $f(x)$ by $g(x)$ we get $f(x)=q(x)g(x)+r(x)$, with $\deg(r)<\deg(g)=2$ or $r(x)=0$. Then we have $f(\alpha)=q(\alpha)g(\alpha)+r(\alpha)$, so that $r(\alpha)=0$. But α can not be the root of a polynomial of degree 1 or 0, so $r(x)=0$ and we have $f(x)=q(x)g(x)$, where $q(x)$ and $g(x)$ have rational coefficients. We can factor out any common factors of the coefficients of q and g and write $f(x)=(a/b)q_1(x)g_1(x)$, where q_1 and g_1 are primitive integer polynomials and $(a,b)=1$. But by Gauss' Lemma, q_1g_1 is primitive, so no prime factor of b can divide all of the coefficients. Therefore $b=1$, and since $f(x)$ is monic, we have $a=1$. Further, since f is irreducible, we must have $q_1=1$ and so $f(x)=g(x)=x^2-2rx+r^2+5s^2$ and we know that $2r$ and r^2+5s^2 are integers. Then $r=b/2$ and $5s^2=(4c-b^2)/4$ for some integers b and c. So $s=e/2$ for some integer e. (5 can not appear in the denominator of s, else when we square it, the single factor of 5 in the expression leaves a remaining factor in the denominator, which does not appear on the right side of the equation.) Substituting these expressions in for r and s, we have

$(b/2)^2 + 5(e/2)^2 = c$, or, upon multiplication by 4, $b^2 + 5e^2 = 4c \equiv 0 \pmod 4$ which has solutions only when b and e are even. Therefore r and s are rational integers.

b. Let $\alpha = a + b\sqrt{-5}$ and $\beta = c + d\sqrt{-5}$. Then $\alpha + \beta = (a + c) + (b + d)\sqrt{-5}$ and $\alpha - \beta = (a - c) + (b - d)\sqrt{-5}$, and $\alpha\beta = (ac - 5bd) + (ad + bc)\sqrt{-5}$. Since the rational integers are closed under addition, subtraction and multiplication, all of the results are again of the form $p + q\sqrt{-5}$ with p and q rational integers.

c. First we seek rational integers a and b such that $(2 + 3\sqrt{-5})(a + b\sqrt{-5}) = -9 + 11\sqrt{-5}$. Multiplying out the left side yields $(2a - 15b) + (3a + 2b)\sqrt{-5} = -9 + 11\sqrt{-5}$. So we must have $2a - 15b = -9$ and $3a + 2b = 11$. Solving this system of equations gives us $a = 3$ and $b = 1$. Since these are rational integers, we have $(2 + 3\sqrt{-5})(3 + 1\sqrt{-5}) = -9 + 11\sqrt{-5}$.

Next, we seek rational integers a and b such that $(1 + 4\sqrt{-5})(a + b\sqrt{-5}) = (a - 20b) + (4a + b)\sqrt{-5} = 8 + 13\sqrt{-5}$. We must have $a - 20b = 8$ and $4a + b = 13$, but this system leads to $b = -19/81$, which is not an integer, so we conclude that $1 + 4\sqrt{-5}$ does not divide $8 + 13\sqrt{-5}$.

d. Let $\alpha = a + b\sqrt{-5}$ and $\beta = c + d\sqrt{-5}$. Then $N(\alpha)N(\beta) = (a^2 + 5b^2)(c^2 + 5d^2) = a^2c^2 + 5a^2d^2 + 5b^2c^2 + 25b^2d^2$. On the other hand, $\alpha\beta = (ac - 5bd) + (ad + bc)\sqrt{-5}$ and $N(ac - 5bd) + (ad + bc)\sqrt{-5}) = (ac - 5bd)^2 + 5(ad + bc)^2 = a^2c^2 - 10acbd + 25b^2d^2 + 5(a^2d^2 + 2adbc + b^2c^2) = a^2c^2 + 5a^2d^2 + 5b^2c^2 + 25b^2d^2$, which is equal to the expression above, proving the assertion.

e. If ϵ is a unit in $\mathbb{Z}[\sqrt{-5}]$, then there exists an η such that $\epsilon\eta = 1$. From part (d) we have $N(\epsilon\eta) = N(\epsilon)N(\eta) = N(1) = 1$, so $N(\epsilon) = 1$. Suppose $\epsilon = a + b\sqrt{-5}$, then $N(\epsilon) = a^2 + 5b^2 = 1$, which shows that $b = 0$, and hence $a^2 = 1$, so that we know $a = \pm 1$. Therefore the only units are 1 and -1.

f. If an integer α in $\mathbb{Z}[\sqrt{-5}]$ is not a unit and not prime, then it must have two non unit divisors β and γ such that $N(\beta)N(\gamma) = N(\alpha)$. To see that 2 is prime, note that a divisor $\beta = a + b\sqrt{-5}$ has norm $a^2 + 5b^2$, while $N(2) = 4$, which forces $b = 0$. If β is not a unit, then $a = \pm 2$, but then this forces γ to be a unit, hence 2 is prime. To see that 3 is prime, we seek divisors of $N(3) = 9$ among $a^2 + 5b^2$. We see that b can be only 0 or ± 1 or else the norm is too large. And if $b = \pm 1$, then the only possible divisor is 9 itself, forcing the other divisor to be a unit. If $b = 0$ then $a = \pm 3$, and hence 3 is prime. To see that $1 \pm \sqrt{-5}$ is prime, note that its norm is 6. A divisor $a + bi$ can have b take on the values 0 and ± 1 else the norm is too large. If $b = 0$, then $a^2|6$ a contradiction, so $b = \pm 1$. But then $(a^2 + 5)|6$ forcing $a = \pm 1$. But $N(\pm 1 \pm \sqrt{-5}) = 6$ so the other divisor is a unit, and so $1 \pm \sqrt{5}$ is also prime. Note then that $2 \cdot 3 = 6$ and $(1 - \sqrt{-5})(1 + \sqrt{-5}) = 6$, so that we do not have unique factorization into primes in $\mathbb{Z}[\sqrt{-5}]$.

g. Suppose γ and ρ exist. Note first that $(7 - 2\sqrt{-5})/(1 + \sqrt{-5}) = -1/2 - 3/2\sqrt{-5}$, so $\rho \neq 0$. Let $\gamma = a + b\sqrt{-5}$ and $\rho = c + d\sqrt{-5}$. Then from $7 - 2\sqrt{-5} = (1 + \sqrt{-5})(a + b\sqrt{-5}) + (c + d\sqrt{-5}) = (a - 5b + c) + (a + b + d)\sqrt{-5}$ we get $7 = a - 5b + c$ and $-2 = a + b + d$. If we subtract the second equation from the first we have $9 = -6b + c - d$ or $c - d = 6b + 9$. Therefore, $3|c - d$, and since $\rho \neq 0$, $c - d \neq 0$, so $|c - d| \geq 3$. We consider $N(\rho) = c^2 + 5d^2$. If $d = 0$, then $N(\rho) \geq c^2 \geq 3^2 > 6$. If $d = \pm 1$, then $|c| \geq 2$ and $N(\rho) = c^2 + 5d^2 \geq 4 + 5 > 6$. If $|d| \geq 2$, then $N(\rho) \geq 5d^2 \geq 5 \cdot 2^2 = 20 > 6$, so in every case the norm of ρ is greater than 6. So no such γ and ρ exist, and there is no analog for the division algorithm in $\mathbb{Z}[\sqrt{-5}]$.

h. Suppose $\mu = a + b\sqrt{-5}$ and $\nu = c + d\sqrt{-5}$ is a solution to the equation. Then $3(a + b\sqrt{-5}) + (1 + \sqrt{-5})(c + d\sqrt{-5}) = (3a + c - 5d) + (3b + c + d)\sqrt{-5} = 1$. So we must have $3a + c - 5d = 1$ and $3b + c + d = 0$. If we subtract the second equation from the first, we get $3a - 3b - 6d = 1$ which implies that $3|1$, an absurdity. Therefore no such solution exists.

14.3. Gaussian Integers and Sums of Squares

14.3.1. a. Since the prime factorization for 5 is 5^1 and $5 \equiv 1 \pmod 4$, we have, by Theorem 14.13, that the number of ways to write 5 as the sum of two squares is $4(1+1) = 8$.

b. The prime factorization of 20 is $2^2 5$, and $5 \equiv 1 \pmod 4$. So by Theorem 14.13, the number of ways to write 20 as the sum of two squares is $4(1+1) = 8$.

c. We have $120 = 2^3 5 \cdot 3$, where $5 \equiv 1 \pmod 4$ but $3 \equiv 3 \pmod 4$. So by Theorem 14.3, there is no way to write 120 as the sum of two squares.

d. We have $1000 = 2^3 5^3$, so the number of ways to write 1000 as the sum of two squares is $4(3+1) = 16$.

14.3.2. a. We have $16 = 2^4$, so $e_i = f_i = 0$ for all i. Then by Theorem 14.13, we see that there are 4 ways to write 16 as the sum of two squares.

b. We have $99 = 3^2 11$, and $11 \equiv 3 \pmod 4$. Since 11 appears to an odd exponent, it is impossible to write 99 as the sum of two squares.

c. We have $650 = 2 \cdot 5^2 \cdot 13$, and $5 \equiv 13 \equiv 1 \pmod 4$, so there are $4(2+1)(1+1) = 24$ ways to write 650 as the sum of 2 squares.

d. We have $1001000 = 2^3 5^3 7 \cdot 11 \cdot 13$. Since $7 \equiv 11 \equiv 3 \pmod 4$ and both primes occur to odd powers, it is impossible to write 1001000 as the sum of two squares.

14.3.3. We first check that a greatest common divisor δ of α and β divides γ, otherwise no solution exists. If a solution exists, we use the Euclidean algorithm and back substitution to express δ as a linear combination of α and β: $\alpha\mu + \beta\nu = \delta$. Since δ divides γ there is a Gaussian integer η such that $\delta\eta = \gamma$. If we multiply the last equation by η we have $\alpha\mu\eta + \beta\nu\eta = \delta\eta = \gamma$, so we may take $x_0 = \mu\eta$ and $y_0 = \nu\eta$ as a solution. The set of all solutions is given by $x = x_0 + \beta\tau/\delta$, $y = y_0 - \alpha\tau/\delta$, where τ ranges over the Gaussian integers.

14.3.4. a. We perform the Euclidean algorithm on $3+2i$ and 5 to get $3+2i = 5 + (-2+2i)$ and $5 = -(1+i)(-2+2i) + 1$ and so we find that a greatest common divisor of $3+2i$ and 5 is 1, which divides $7i$. Then using back-substitution, we have $1 = 5 + (1+i)(-2+2i) = 5 + (1+i)((3+2i) - 5) = (3+2i)(1+i) - 5(i)$. Multiplying through by $7i$ gives us $7i = (3+2i)(-7+7i) - 5(-7)$, so we can take $x_0 = -7+7i$ and $y_0 = -7$ as a solution to the equation. Then the set of all solutions is given by $x = (-7+7i) + 5\tau$, $y = -7 - (3+2i)\tau$, where τ ranges over the Gaussian integers. Here we have followed the method outlined in the solution to Exercise 3.

b. Note that $(2+i)(2-i) = 5$, and so $2-i$ is a greatest common divisor of 5 and itself. But $2-i$ does not divide 3, so there are no solutions to this equation.

14.3.5. a. We find that a greatest common divisor of $3+4i$ and $3-i$ is $2+i$. Then we compute $7i/(2+i) = 7/5 + 14/5i$, which is not a Gaussian integer. Therefore there are no solutions to the diophantine equation.

b. We find that a greatest common divisor of $7+i$ and $7-i$ is $1+i$ which does not divide 1. Therefore the diophantine equation has no solutions.

14.3.6. a. First note that x and y must have opposite parity. If x is odd and y even, then we have $x^2 + 1 \equiv 0 \pmod 8$, which has no solutions. Therefore x is even and y is odd. Let γ be a greatest common divisor of $x-i$ and $x+i$. Then $\gamma \mid ((x+i) - (x-i)) = 2i$, but the only prime divisors of $2i$ are the associates of $1+i$, whose multiples are exactly those Gaussian integers in which the real and imaginary parts have the same parity. Since $x+i$ and $x-i$ are not of this form, we know γ is a unit, and hence $x+i$ and $x-i$ are relatively prime.

b. Since $x+i$ and $x-i$ are relatively prime and $(x-i)(x+i) = x^2+1 = y^3$, we can apply Exercise 10 of Section 14.2 and we have $x+i = \eta\delta^3$ for some unit η and some Gaussian integer δ. Note that $1^3 = 1$, $(-1)^3 = -1$, $i^3 = -i$ and $(-i)^3 = i$, so that every unit has a cube root in the Gaussian integers and we can write $\eta = \epsilon^3$ for some unit ϵ. So we have $x+i = (\epsilon\delta)^3$. Let $\epsilon\delta = r+si$ and write $x+i = (r+si)^3 = r^3+3r^2si-3rs^2-s^3i$. Equating real and imaginary parts give us $x = r^3-3rs^2$ and $1 = 3r^2s-s^3$.

c. We have $1 = 3r^2s-s^3 = s(3r^2-s^2)$, so $s \mid 1$ and we know that $s = \pm 1$. If $s = 1$, we have $1 = 3r^2-1$ or $3r^2 = 2$, which is impossible. If $s = -1$, the equation reduces to $3r^2 = 0$, and so $r = 0$. Then from the other equation we have $x = r^3-3rs^2 = 0$, which forces $y = 1$, and this is the only solution.

14.3.7. Suppose x, y, z is a primitive Pythagorean triple with y even, so that x and z are necessarily odd. Then $z^2 = x^2+y^2 = (x+iy)(x-iy)$ in the Gaussian integers. If a rational prime p divides $x+iy$, then it must divide both x and y, which contradicts the fact that the triple is primitive. Therefore, the only Gaussian primes which divide $x+iy$ are of the form $m+in$ with $n \neq 0$. Also, if $1+i|x+iy$, then we have the conjugate relationship $1-i|x-iy$, which implies that $2 = (1-i)(1+i)$ divides z^2, which is odd, a contradiction. Therefore we conclude that $1+i$ does not divide $x+iy$, and hence neither does 2. Suppose δ is a common divisor of $x+iy$ and $x-iy$. Then δ divides the sum $2x$ and the difference $2iy$. Since we know that 2 is not a common factor, δ must divide both x and y, which we know are relatively prime. Hence δ is a unit and $x+iy$ and $x-iy$ are also relatively prime. Then we know that every prime which divides $x+iy$ is of the form $\pi = u+iv$ and so $\overline{\pi} = u-iv$ divides $x-iy$. Since their product equals a square, each factor is a square. Thus $x+iy = (m+in)^2$ and $x-iy = (m-in)^2$ for some Gaussian integer $m+in$ and its conjugate. But then $x+iy = m^2-n^2+2mni$ so $x = m^2-n^2$ and $y = 2mn$. And $z^2 = (m+ni)^2(m-ni)^2 = (m^2+n^2)^2$, so $z = m^2+n^2$. Further, if m and n were both odd or both even, we would have z even, a contradiction, so we may conclude that m and n have opposite parity. Finally, having found m and n which work, if $m < n$ we can multiply by i and reverse their roles to get $m > n$. The converse is exactly as in Section 13.1.

14.3.8. If p is a prime of the form $4k+3$ which appears in the factorization of z to an odd power, then it also appears in the factorization of z^3 to an odd power. Therefore z^3 can be written as a sum of two squares if and only if z can. Suppose z satisfies the hypotheses of Theorem 14.13 so it can be written as $z = a^2+b^2$. Then $z = (a+bi)(a-bi)$ and $z^3 = (a+bi)^3(a-bi)^3$. Likewise z^3 satisfies the hypotheses of Theorem 14.13 and so it can be written as $z^3 = x^2+y^2 = (x+yi)(x-yi)$. Since $(a+bi)^3 = (a^3-3ab^2)+(3a^2b-b^3)i$, we can set $x = a^3-3ab^2$ and $y = 3a^2b-b^3$, so that $z = a^2+b^2$. This investigation shows that if we choose any integers a and b, then a solution of the diophantine equation is given by the last three equations. Further, by our construction, all solutions must arise in this fashion.

14.3.9. By Lemma 14.3, there is a unique rational prime p such that $\pi|p$. Let $\alpha = a+bi$ and consider 3 cases.

Case 1: If $p = 2$, then π is an associate of $1+i$ and $N(\pi)-1 = 1$. Since there are only two congruence classes modulo $1+i$ and since α and $1+i$ are relatively prime, we have $\alpha^{N(\pi)-1} = \alpha \equiv 1 \pmod{1+i}$.

Case 2: If $p \equiv 3 \pmod 4$, then $\pi = p$ and $N(\pi)-1 = p^2-1$. Also $(-i)^p = -i$. By the Binomial theorem, we have $\alpha^p = (a+bi)^p \equiv a^p+(bi)^p \equiv -ib^p \equiv a-bi \equiv \alpha \pmod p$, using Fermat's little theorem. Similarly $\overline{\alpha}^p \equiv \alpha \pmod p$, so that $\alpha^{p^2} \equiv \overline{\alpha}^p \equiv \alpha \pmod p$ and since $p = \pi$ and α and π are relatively prime, we have $\alpha^{N(\pi)-1} \equiv 1 \pmod p$.

Case 3: If $p \equiv 1 \pmod 4$, then $\pi\overline{\pi} = p$, $i^p = i$, and $N(\pi)-1 = p-1$. By the Binomial theorem, we have $\alpha^p = (a+bi)^p \equiv a^p+(bi)^p \equiv a+bi \equiv \alpha \pmod p$, using Fermat's little theorem. Cancelling an α gives us $\alpha^{p-1} \equiv 1 \pmod p$, and since $\pi|p$ we have $\alpha^{N(\pi)-1} \equiv 1 \pmod \pi$, which concludes the proof.

14.3.10. Let $r = \phi(\gamma)$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a reduced residue system modulo γ. We assert that the set $\{\alpha\alpha_1, \alpha\alpha_2, \ldots, \alpha\alpha_r\}$ is also a reduced residue system modulo γ. To see this, first note that since both α and α_k are relatively prime to γ, for any k, so is $\alpha\alpha_k$. Second, suppose $\alpha\alpha_j \equiv \alpha\alpha_k \pmod\gamma$ for some j and k. By Exercise 24 of Section 14.2, α must have an inverse modulo γ, and we have $\alpha_k \equiv \alpha_j \pmod\gamma$, which shows that $\alpha_j = \alpha_k$. This proves our assertion. Then we must have $\alpha_1\alpha_2\cdots\alpha_r \equiv (\alpha\alpha_1)(\alpha\alpha_2)\cdots(\alpha\alpha_r) \equiv \alpha^r(\alpha_1\alpha_2\cdots\alpha_r) \pmod\gamma$. Since each α_k has an inverse modulo γ, we can cancel them, and we are left with $\alpha^r \equiv 1 \pmod\gamma$, which is the result.

14.3.11. Let π be a Gaussian prime. If $\alpha^2 \equiv 1 \pmod{\pi}$, then $\pi | \alpha^2 - 1 = (\alpha - 1)(\alpha + 1)$, so that either $\alpha \equiv 1$ or $\alpha \equiv -1 \pmod{\pi}$. Therefore only 1 and -1 can be their own inverses modulo π. Now let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r = -1$ be a reduced residue system modulo π. For each α_k, $k = 2, 3, \ldots, r-1$, there is a multiplicative inverse modulo π α'_k such that $\alpha_k \alpha'_k \equiv 1 \pmod{\pi}$. If we group all such pairs in the reduced residue system together, then the product is easy to evaluate: $\alpha_1 \alpha_2 \cdots \alpha_r = 1(\alpha_2 \alpha'_2)(\alpha_3 \alpha'_3) \cdots (\alpha_{r-1})(\alpha'_{r-1})(-1) \equiv -1 \pmod{\pi}$, which proves the theorem.

14.3.12. a. Suppose that $2 = \alpha\beta$ is a nontrivial factorization in the Eisenstein integers. Then we have $4 = N(2) = N(\alpha)N(\beta)$, and since neither factor is a unit, we must have $N(\alpha) = 2$. Let $\alpha = a + b\omega$, so that $N(a + b\omega) = a^2 - ab + b^2 = 2$. We can complete the square in a in this last equation to get $(a - b/2)^2 + 3b^2/4 = 2$, from which we see that if $|b| \geq 2$, then the left side of the equation is at least 3. Therefore $b = 1$ or 0. If $b = 1$, we can solve the equation for a and we get $a = (1 \pm \sqrt{5})/2$, which is not a rational integer. Therefore $b = 0$ and $N(\alpha) = N(a) = a^2 = 2$. But there are no solutions to this last equation, and we conclude that 2 is an Eisenstein prime.

b. Let p be a rational prime with $p \equiv 2 \pmod{3}$, and suppose $p = \pi\rho$ is a nontrivial factorization in the Eisenstein integers. Then we have $p^2 = N(p) = N(\pi)N(\rho)$, and since neither factor is a unit, we conclude that $N(\pi) = p$. Let $\pi = a + b\omega$, so that $N(a + b\omega) = a^2 - ab + b^2 = p$. If $a \equiv -b \pmod{3}$, then this equation becomes $p = a^2 - ab + b^2 \equiv a^2 + a^2 + a^2 \equiv 3a^2 \equiv 0 \pmod{3}$, a contradiction, since $3 \nmid p$. Therefore $a + b \not\equiv 0 \pmod{3}$ and so $a + b$ has an inverse modulo 3. Then we can write $p \equiv a^2 - ab + b^2 \equiv (a+b)^{-1}(a+b)(a^2 - ab + b^2) \equiv (a^3 + b^3)(a+b)^{-1} \equiv (a+b)(a+b)^{-1} \equiv 1 \pmod{3}$, where we have used Fermat's little theorem to write $a^3 + b^3 \equiv a + b \pmod{3}$. But this contradicts the fact that $p \equiv 2 \pmod{3}$, and so we conclude that p is an Eisenstein prime.

c. Note that if a rational prime p divides an Eisenstein integer $a + b\omega$, then we have $p(c + d\omega) = a + b\omega$ for some integers c and d. This implies that $a = pc$ and $b = pd$. That is, if a rational prime divides an Eisenstein integer, then it divides the respective parts. Since p odd and of the form $3k + 1$, we know that $p \equiv 1 \pmod{6}$ and then from Exercise 3 in Section 11.2, we see that -3 is a quadratic residue modulo p. So there is a rational integer u such that $p \mid u^2 + 3 = (u - \sqrt{-3})(u + \sqrt{-3}) = (u - 1 - 2\omega)(u + 1 + 2\omega)$. If p were an Eisenstein prime, then p would have to divide one of these factors, and hence, by our comment above, p would have to divide 2, which it can not. Therefore p is not an Eisenstein prime, and some Eisenstein integer $c + d\omega$ divides p nontrivially. Then $N(c + d\omega) \mid N(p) = p^2$, and since the division is nontrivial, we must have $N(c + d\omega) = c^2 - cd + d^2 = p$. We note that $p = N(c + d\omega) = (c + d\omega)(c + d\omega^2)$, which gives us a factorization for p. It remains to check that these factors are not associates. If they were associates, then when we divide one by the other, we would get a unit. But $(c + d\omega)/(c + d\omega^2) = (c + d\omega)(c + d\omega)/((c + d\omega)(c + d\omega^2)) = (c^2 - d^2 + (2cd - d^2)\omega)/p$, so that $p \mid c^2 - d^2$ and so $c \equiv \pm d \mod p$. But also $p \mid 2cd - d^2$, so $0 \equiv 2cd - d^2 \equiv \pm 2d^2 - d^2 \mod p$, from which we conclude $p \mid d$ and so $p \mid c$. But then $p = N(c + d\omega) = N(p(a + b\omega)) = p^2 N(a + b\omega) > p$, a contradiction. Therefore $c + d\omega$ and $c + d\omega^2$ are not associates.

APPENDIX A

Axioms for the Set of Integers

A.0.1. a. By the commutative law, $a(b+c) = (b+c)a$. Now, using the distributive law, $a(b+c) = (b+c)a = ba + ca = ab + ac$.

b. By the distributive law, $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2$. By the law of commutativity, this is equal to $a^2 + 2ab + b^2$.

c. From the commutative law of addition, $a + (b+c) = a + (c+b)$. This is equal to $(a+c) + b$ by associativity. With a final application of commutativity, we see that $a + (b+c) = (c+a) + b$.

d. Using the definition of subtraction and additive commutativity, $(b-a) + (c-b) + (a-c) = (-a+b) + (-b+c) + (-c+a)$. By associativity, this is equal to $-a + (b-b) + (c-c) + a$. Using the definition of an additive inverse, this is 0.

A.0.2. a. We have $(-1)a + 1a = (-1+1)a = 0 = -a + a = -a + 1a$. Now cancel the $1a$'s from the beginning and end of this equation.

b. Note that $a(-b) + ab = a(-b+b) = 0 = (ab) - (ab)$. Now cancel the ab's.

c. Using part (b), $(-a)(-b) + (-a)b = -a(-b+b) = 0 = ab + (-a)b$. Now cancel the $(-a)b$'s.

d. We compute $-(a+b) = -1(a+b) = (a+b)(-1) = a(-1) + b(-1) = -1a - 1b = (-a) + (-b)$.

A.0.3. By the definition of the inverse of an element, $0 + (-0) = 0$. But since 0 is an identity element, we have $0 + (-0) = -0$. It follows that $-0 = 0$.

A.0.4. Suppose that $ab = 0$. Suppose further that $b \neq 0$. We also have $0b = 0$ by Example 1.1. Hence $ab = 0b$. By the cancellation law it follows that $a = 0$. Hence either $a = 0$ or $b = 0$.

A.0.5. Let x be a positive integer. Since $x = x - 0$ is positive, $x > 0$. Now let $x > 0$. Then $x - 0 = x$ is positive.

A.0.6. a. We have $(b+c) - (a+c) = b - a$, which is positive since $a < b$. Therefore, $a + c < b + c$.

b. If $a = 0$, the $a^2 = 0$. If $a > 0$, then $a^2 > 0$ by the closure of the positive integers. If $a < 0$, then by the trichotomy law, $-a$ is a positive integer. Thus $a^2 = (-a)(-a) > 0$ by the closure of the positive integers.

c. We have $ac - bc = (a-b)c$. By part (a) of Exercise 2, $(a-b)c$ is positive since both $a - b$ and c are negative. Thus, $bc < ac$.

d. By part (b), $c^2 > 0$. Thus $c^3 < 0$ since $0 - c^3 = (-c)c^2$ is positive.

A.0.7. We have $a - c = a + (-b+b) - c = (a-b) + (b-c)$, which is positive from our hypothesis and the closure of the positive integers.

A.0.8. Suppose that there are positive integers less than 1. By the well ordering property there is a least such integer, say a. Since $a < 1$ and $a > 0$, Example 1.2 shows that $a^2 = aa < 1a = a$. Since $a^2 > 0$, it follows

that a^2 is a positive integer less than a, which is a contradiction.

APPENDIX B

Binomial Coefficients

B.0.1. a. We have $\binom{100}{0} = 100!/(0!100!) = 1$.

b. We have $\binom{50}{1} = 50!/(1!49!) = 50$.

c. We have $\binom{20}{3} = 20!/(3!17!) = 1140$.

d. We have $\binom{11}{5} = 11!/(5!6!) = 462$.

e. We have $\binom{10}{7} = 10!/(7!3!) = 120$.

f. We have $\binom{70}{70} = 70!/(70!0!) = 1$.

B.0.2. We have $\binom{9}{3} = 84$, $\binom{9}{4} = 126$, $\binom{10}{4} = 210$, and $84 + 126 = 210$.

B.0.3. a. We compute $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

b. We compute $(x+y)^{10} = x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$.

c. We compute $(m-n)^7 = m^7 - 7m^6n + 21m^5n^2 - 35m^4n^3 + 35m^3n^4 - 21m^2n^5 + 7mn^6 - n^7$.

d. We compute $(2a+3b)^4 = 16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$.

e. We compute $(3x-4y)^5 = 243x^5 - 1620x^4y + 4320x^3y^2 - 5760x^2y^3 + 3840xy^4 - 1024y^5$.

f. We compute $(5x+7)^8 = 390625x^8 + 4375000x^7 + 21437500x^6 + 60025000x^5 + 105043750x^4 + 117649000x^3 + 82354300x^2 + 32941720x + 5764801$.

B.0.4. The coefficient of $x^{99}y^{101}$ in $(2x+3y)^{200}$ is $\binom{200}{99}2^{99}3^{101} = \frac{200!}{99!101!}2^{99}3^{101}$.

B.0.5. On the one hand, $[1+(-1)]^n = 0^n = 0$. On the other hand, by the binomial theorem, $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1+(-1))^n$.

B.0.6. We have $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ and $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$. Adding these two equations gives $2\left(\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right) = 2^n$. Hence $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$. It follows immediately that $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$.

B.0.7. We have $\binom{n}{r}\binom{r}{k} = n!/(r!(n-r)!)\cdot r!/(k!(r-k)!) = n!(n-k)!/(k!(n-k)!(n-r)!(n-k-n+r)!) = \binom{n}{k}\binom{n-k}{n-r}$.

B.0.8. When $n = [m/2]$, $\binom{m}{n}$ is at a maximum. To see this, consider the ratio $\binom{m}{k}/\binom{m}{k-1} = (m!/(k!(m-k)!))/(m!/((k-1)!(m-k+1)!)) = (m-k+1)/k$. Therefore, $\binom{m}{k} \geq \binom{m}{k-1}$ if and only if $m-k+1 \geq k$, that is if $n \leq (m+1)/2 = [m/2]$. Thus, the value of $\binom{m}{n}$ increases as n increases to $[m/2]$, and then decreases.

B.0.9. We use Exercise 44 in Section 1.3. $\alpha^2 = \alpha+1$ and $\beta^2 = \beta+1$, since they are roots of $x^2 - x - 1 = 0$. Then we have $f_{2n} = (\alpha^{2n} - \beta^{2n})/\sqrt{5} = (1/\sqrt{5})((\alpha+1)^n - (\beta+1)^n) = (1/\sqrt{5})\left(\sum_{j=0}^{n}\binom{n}{j}\alpha^j - \sum_{j=0}^{n}\binom{n}{j}\beta^j\right) = (1/\sqrt{5})\sum_{j=0}^{n}\binom{n}{j}(\alpha^j - \beta^j) = \sum_{j=1}^{n}\binom{n}{j}f_j$ since the first term is zero in the second to last sum.

B.0.10. We proceed by induction. When $k = 1$, this is clear. For the inductive step, we assume that $\binom{x}{k} = x!/(k!(x-k)!)$. Then $\binom{x}{k+1} = (x-k)/(k+1)\binom{x}{k} = (x-k)/(k+1)\cdot x!/(k!(x-k)!) = x!/((k+1)!(x-k-1)!)$.

B.0.11. Using Exercise 10, $\binom{x}{n} + \binom{x}{n+1} = x!/(n!(x-n)!) + x!/((n+1)!(x-n-1)!) = (x!(n+1))/((n+1)!(x-n)!) + (x!(x-n))/((n+1)!(x-n)!) = (x!(x-n+n+1))/((n+1)!(x-n)!) = (x+1)!/((n+1)!(x-n)!) = \binom{x+1}{n+1}$.

B.0.12. An extremely short combinatorial proof of the binomial theorem can be given. The coefficient of $x^k y^{n-k}$ in $(x+y)^n$ is the number of ways to choose x k times from the n factors $(x+y)$, and consequently, y $n-k$ times. This equals the number of subsets with k elements of a set with n elements. (Here the elements in the subsets are the terms where x is chosen, and the n elements are the n terms.) Hence the coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$. It follows that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

B.0.13. Let S be a set of n copies of $x+y$. Consider the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$. Choosing the x from each element of a $k-$element subset of S, we notice that the coefficient of $x^k y^{n-k}$ is the number of $k-$element subset of S, $\binom{n}{k}$.

B.0.14. The number of elements that have either property P_1 or property P_2 is $n(P_1) + n(P_2) - n(P_1P_2)$ since an element with one, but not both, of these properties, is counted once by the sum $n(P_1) + n(P_2)$ but not by the term $n(P_1P_2)$ and an element with both of these properties is counted twice by the sum $n(P_1) + n(P_2)$, and the overcount is removed since it is counted once again by $n(P_1P_2)$. Hence the number of elements possessing neither property is $n - [n(P_1) + n(P_2) - n(P_1P_2)]$.

B.0.15. By counting elements with exactly 0, 1, 2, and 3 properties, we see that only elements with 0 properties are counted in $n - [n(P_1) + n(P_2) + n(P_3)] + [n(P_1, P_2) + n(P_1, P_3) + n(P_2, P_3)] - [n(P_1, P_2, P_3)]$, and those only once.

B.0.16. The hint follows from Exercise 12. Using this, if $k \geq 1$, then an element with k properties isn't counted. If $k = 0$, then it is clearly counted once.

B.0.17. A term of the sum is of the form $a x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ where $k_1 + k_2 + \cdots + k_m = n$ and $a = \frac{n!}{k_1! k_2! \cdots k_m!}$.

B.0.18. Using the formula from Exercise 17 we have $x^7 + 7\,x^6\,y + 21\,x^5\,y^2 + 35\,x^4\,y^3 + 35\,x^3\,y^4 + 21\,x^2\,y^5 + 7\,x\,y^6 + y^7 + 7\,x^6\,z + 42\,x^5\,y\,z + 105\,x^4\,y^2\,z + 140\,x^3\,y^3\,z + 105\,x^2\,y^4\,z + 42\,x\,y^5\,z + 7\,y^6\,z + 21\,x^5\,z^2 + 105\,x^4\,y\,z^2 + 210\,x^3\,y^2\,z^2 + 210\,x^2\,y^3\,z^2 + 105\,x\,y^4\,z^2 + 21\,y^5\,z^2 + 35\,x^4\,z^3 + 140\,x^3\,y\,z^3 + 210\,x^2\,y^2\,z^3 + 140\,x\,y^3\,z^3 + 35\,y^4\,z^3 + 35\,x^3\,z^4 + 105\,x^2\,y\,z^4 + 105\,x\,y^2\,z^4 + 35\,y^3\,z^4 + 21\,x^2\,z^5 + 42\,x\,y\,z^5 + 21\,y^2\,z^5 + 7\,x\,z^6 + 7\,y\,z^6 + z^7$

B.0.19. From Exercise 17 it follows that the coefficient is $\frac{12!}{3!4!5!} 2^3 (-3^4) 5^5 = 27720 \cdot 8 \cdot 81 \cdot 3125 = 56,133,000,000$.